Numbers are associated with counting

The purpose of this lecture is to advocate that counting may reveal some hidden structure. That structure may be topological/geometrical algebraic
ACT 1
How do we distinguish topological shapes?

With the characteristic!

For this we triangulate the surface.
For any triangulation of a surface $S$, set

$$\chi(S) = |V| - |E| + |F|$$

For a given surface this is independent of the given triangulation!
21 vertices
63 edges
42 faces

So for a doughnut $X = 0$

Anche il mazzocchio ha $X = 0$!

For the sphere $X = 2$.

In general, for a surface with $g$ holes $X = 2 - 2g$
Counting solutions of equations

Given a two-variable polynomial with integer coefficients

\[ f(x, y) \in \mathbb{Z}[x, y] \]

one may be interested in solutions in \( \mathbb{Z} \) or in \( \mathbb{Q} \) of the equation

\[ f(x, y) = 0 \]

[for example when

\[ f(x, y) = x^n + y^n - 1 \].

To do this it is also useful to study solutions of congruences mod \( p \)

\( p \) a prime number
Recall that $\mathbb{Z}/p\mathbb{Z}$ is a field with $p$ elements. So it makes sense to count the number of solutions of $f(x, y) = 0$ in $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$.

In fact, for any $r \geq 1$, there exists a unique field $\mathbb{F}_q$ with $q = p^r$ elements.

For instance

$$\mathbb{F}_3[x]/(x^2 + 1)$$

is the field $\mathbb{F}_9$ with 9 elements.
We set

\[ N_{pr}(f) \] for the

number of solutions of

\[ f = 0 \] in \( \mathbb{F}_p \).

example: \( f = y \) (line)

\[ N_{pr}(f) = p \]

already for \( f = x^3 + y^2 - 1 \)

\( N_{pr} \) is not easy to compute:

involves Gauss sums
We can also look at the complex solutions of \( f(x,y) = 0 \): they form a surface \( X_f \) in \( \mathbb{C}^2 \cong \mathbb{R}^4 \).

We can also consider its Euler-Poincaré characteristic \( \chi(X_f) \).

**Magical Theorem:**

\[
\lim_{r \to 0} N_{pr}(f) = \chi(X_f)
\]

- for \( f = y \) yields \( 1 = 1 \)
- in fact only valid for \( p \geq \text{some } p_0 \)
- valid for (systems of) equations in any number of variables.
This follows from results of B. Dwork & A. Grothendieck

but it has no sense in the way it is stated.

To make sense of the limit one introduces the generating function

\[ Z_{f_p}(T) := \exp \left( \sum_{r \geq 1} \frac{N_{p^r}}{r} T^r \right) \]
Bernard Dwork proved in 1960 the rationality of this function. This is equivalent (by basic algebra) to:

There exist complex numbers $\alpha_i$ and $\beta_j$ (depending on $p$) such that

$$N_{pr} = \sum \alpha_i - \sum \beta_j$$

So $N_{pr}$ is an "exponential polynomial" in $r$, so one can interpolate it by as a continuous function of the variable $r$ and consider its limit as $r \to 0$. 
In fact the connexion between number of solutions of equations in finite fields and the topology of the set of complex solutions was envisioned by André Weil as early as 1949 in an influential paper.
This, and other examples which we cannot discuss here, seem to lend some support to the following conjectural statements, which are known to be true for curves, but which I have not so far been able to prove for varieties of higher dimension.

Let $V$ be a variety without singular points, of dimension $n$, defined over a finite field $k$ with $q$ elements. Let $N_r$ be the number of rational points on $V$ over the extension $k_r$ of $k$ of degree $r$. Then we have

\[ \sum_{r=1}^{\infty} N_r U^{r-1} = \frac{d}{dU} \log Z(U), \]

where $Z(U)$ is a rational function in $U$, satisfying a functional equation

\[ Z\left(\frac{1}{q^n U}\right) = \pm q^{n\chi/2} U^{n} Z(U), \]

with $\chi$ equal to the Euler-Poincaré characteristic of $V$ (intersection-number of the diagonal with itself on the product $V \times V$).

Furthermore, we have:

\[ Z(U) = \frac{P_1(U) P_3(U) \cdots P_{2n-1}(U)}{P_0(U) P_2(U) \cdots P_{2n}(U)}, \]

with $P_0(U) = 1 - U$, $P_{2n}(U) = 1 - q^n U$, and, for $1 \leq h \leq 2n - 1$:

\[ P_h(U) = \prod_{i=1}^{B_h} (1 - \alpha_{hi} U) \]

where the $\alpha_{hi}$ are algebraic integers of absolute value $q^{h/2}$.

Finally, let us call the degrees $B_h$ of the polynomials $P_h(U)$ the Betti numbers of the variety $V$; the Euler-Poincaré characteristic $\chi$ is then expressed by the usual formula $\chi = \sum_h (-1)^h B_h$. The evidence at hand seems to suggest that, if $\overline{V}$ is a variety without singular points, defined over a field $K$ of algebraic numbers, the Betti numbers of the varieties $V_p$, derived from $\overline{V}$ by reduction modulo a prime ideal $p$ in $K$, are equal to the Betti numbers of $\overline{V}$ (considered as a variety over complex numbers) in the sense of combinatorial topology, for all except at most a finite number of prime ideals $p$. For instance, consider the Grassmann variety $G_{m,r}$, the points of which are the r-dimensional linear varieties in a projective m-dimensional space, over
Interlude: two examples of fruitful interplay between combinatorics and geometry

Michel Demazure

combinatorics of convex polytopes

algebraic geometry

Ngô Bào Châu

combinatorial identity between integrals (the Fundamental Lemma)

geometry of the Hitchin fibration
ACT TWO

Through two distinct points in the plane passes exactly one line.
Through 5 points in general position in the complex plane passes exactly 1 conic
Through 8 points in general position in the complex plane there are exactly 12 rational cubics with rational parametrization of degree 3.

Through 11 points in general position exactly 620 rational quartics.

Hieronymus Georg Zeuthen

What's next?
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*Can you guess the pattern?*
3.3.1 Theorem. (Kontsevich) Let $N_d$ be the number of rational curves of degree $d$ passing through $3d - 1$ general points in the plane. Then the following recursive relation holds:

$$N_d + \sum_{d_A+d_B=d \atop d_A \geq 1, d_B \geq 1} \binom{3d-4}{3d_A-1}^2 d_A^2 N_{d_A} \cdot N_{d_B} \cdot d_A d_B = \sum_{d_A+d_B=d \atop d_A \geq 1, d_B \geq 1} \binom{3d-4}{3d_A-2} d_A N_{d_A} \cdot d_B N_{d_B} \cdot d_A d_B$$
What is the structure behind this formula?

It expresses associativity of a new structure, the quantum cohomology of the plane.