ARITHMETIC MONODROMY

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Classical Case

Let $\Delta$ denote the disk around 0 in $\mathbb{C}$. Let $X$ be a smooth complex variety. Consider a semi-stable degeneration $\pi : X \to \Delta$: i.e. a holomorphic, proper and flat map of relative dimension $n$ such that $X_t = \pi^{-1}(t)$ is a proper smooth complex variety for $t \neq 0$, such that the fiber $X_0$ is a divisor with normal crossing (NCD) and smooth components.

We may define limit cohomology $H^m_{\lim}$ for $X_0$ with unipotent monodromy operator $N$ and a weights filtration.
As vector spaces $H^m(X_t) \simeq H^m_{lim}$, 
by a topological argument $H^m(X) \simeq H^m(X_0) := H^m (X$ a smooth tube around $X_0)$ hence $H_m(X) \simeq H_m(X_0) := H_m$.

Then it is possible to define the Clemens-Schmid exact sequence (respecting MHS):

$$
\ldots \rightarrow H_{2n+2-m} \overset{\alpha}{\rightarrow} H^m \rightarrow H^m_{lim} \overset{N}{\rightarrow} H^m_{lim} \overset{\beta}{\rightarrow} H_{2n-m} \overset{\alpha}{\rightarrow} \ldots
$$

Where $\alpha, \beta$ maps are the natural maps arising from Poincaré duality for $X_0$ thought as a closed in the smooth $X$, and $X_t$. 

\[ \cdots \to H_{2n+2-m} \stackrel{\alpha}{\to} H^m \to H^m_{\lim} \stackrel{N}{\to} H^m_{\lim} \stackrel{\beta}{\to} H_{2n-m} \stackrel{\alpha}{\to} \cdots \]

Prove exactness:
- topological arguments to connect a global definition of the cohomology of \( X \) with support in \( X_0 \) to a sequence involving the cohomology of the special and generic fibers.
- **Weights** argument. The sequence respects MHS and the Weights filtration on \( H^m_{\lim} \) coincides with the Monodromy one.

We recall, moreover, that the structure of the limit cohomology has been understood in the framework of the log-geometry.
Part of the Clemens Schmid exact sequence is the invariant cycles theorem. It says that the image of the cohomology of the special fiber in the "limit" cohomology (which can be seen as the nearby cycles) is exactly the set of invariants under the monodromy. This result appears as a consequence of the decomposition theorem for the constant coefficient (when the basic space is of dimension 1). Such a decomposition theorem in the classical case has been proved also for coefficients: Beilinson-Bernstein-Deligne, M. Saito’s, Drinfeld ....later Sabbah, T. Mochizuki and De Cataldo-Migliorini..
The coefficients were first indicated as semisimple perverse sheaves with geometric origin (i.e. purity), or polarized Hodge Module. But the more meaningful statement for us involves semisimple regular holonomic $\mathcal{D}$-modules or semisimple perverse sheaves. Relevant for us is the following (Sabbah-Mochizuki)

if $X$ smooth quasi projective variety and $\mathcal{F}$ a semisimple local system on $X$, and $f : X \to D$ a proper map, $D$ open disk, then if $t$ is sufficiently closed to 0 we then have an exact sequence

$$H^i(f^{-1}(0), \mathcal{F}) \to H^i(f^{-1}(t), \mathcal{F}) \xrightarrow{N} H^i(f^{-1}(t), \mathcal{F})$$

where $N$ is the monodromy around zero.
We note that the first approach by BBD involved the use of an arithmetic setting. They worked via étale $\mathbb{Q}_l$-cohomology for $l \neq p$ on varieties in $ch = p$ using pure objects under the Frobenius. Such an arithmetic setting allows also new arithmetic frameworks .....local field instead of curves. But here we want to deal with $p$–adic cohomology and we will deal with all these arithmetic variants of the geometric setting.
The arithmetic setting and results

When we speak about "arithmetic" setting we have several possibilities

1) The **geometric** one

\[ f : X \rightarrow C , \]

where \( X \) is a smooth variety of dimension \( n + 1 \) over \( C \), a smooth curve. All is defined over a field of \( ch = p, k \). The map \( f \) is proper and smooth outside a specific fiber \( s \). The fiber at \( s \) of \( f \), \( X_s \), is a NCD in \( X \).

2) The **arithmetic** one, we consider \( \mathcal{V} \) a DVR of mixed characterestic \( K \) its fraction field and \( k \) its residual field of \( ch = p \). then

\[ f : X \rightarrow Sp\mathcal{V} , \]

where \( X \) is a proper semistable variety over \( Sp\mathcal{V} \). The fiber at \( s \) of \( f \), \( X_s \), is a NCD in \( X \) (and the generic fiber \( X_K \) is smooth).
We are going to prove the existence of a Clemens-Schmid sequence in the situation 1) together with the definition of the relevant cohomology spaces one needs coh. with support....etc...

In case 2), The invariant cycles theorem is an open problem in general. It has been proved for curves and surfaces by [Ch] Decomposition theorem : which coefficients are pure? Here we want to deal with coefficients on curves. We focus on not semisimple i.e. the decomposition theorem can "not happen" But we want a control on the obstruction to the exactness.
Clemens-Schmid sequence in case 1)

Joint work with Tsuzuki. Let’s see how to replace the objects in the classical Clemens-Schmid sequence:

- \( H^m_{\text{lim}} \Rightarrow \) log-cristalline cohomology of the log schemes \( X_s \) with log-structure induced by the log structure of \( X \) given by the NCD \( X_s \) itself.

- \( H^m = H^m(X) = H^m(X_0) \Rightarrow \) the rigid cohomology of \( X_s = X_0 \)

- \( H_m \Rightarrow \) the rigid homology of \( X_s = X_0 \). ... the dual of the compact support rigid cohomology... \( H^m_{X_s,\text{rig}}(X) \).
Theorem

In the previous hypotheses

\[ \cdots H^i_{Xs,\text{rig}}(X) \rightarrow H^i_{\text{rig}}(X_s) \xrightarrow{\gamma} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \xrightarrow{N_i} \]

\[ \xrightarrow{N_i} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K(-1) \xrightarrow{\delta} H^{i+2}_{Xs,\text{rig}}(X) \rightarrow \cdots \]

is an exact sequence.
Remark

In case 1) the equivalence of the weights filtration with the monodromy one for the log scheme $X_s$ is linked to the fact that it is a special fiber of a family in $ch = p$. Like in the étale case (Deligne, Weil II).
In case 2) always in the étale setting if the base is a local ring in equal characteristic $p$ it is ITO’s (2005) theorem. See also Scholze (2012) for a mixed characteristic case.
The ingredients

How to build it?
- "trascendental" topological argument? local to global by two definitions of the local cohomology of $X$ with support in $X_s$.

-define the cohomology of the open complement of $X_s$ as generalized log-cohomology theory hence the link with limit Hodge structure of the special fiber.

-We have also the Poincaré duality in the rigid setting

$$H^{i+2}_{X_s, \text{rig}}(X) = (H^{2n+2-i-2}_{c, \text{rig}}(X_s))^* = H^{\text{rig}}_{2n-i}(X_s).$$

Hence the first ingredients for the existence of the Clemens-Schmid sequence is at our disposal.
For the exactness?

log-crystalline cohomology of $X_s$ admits a Weights (coming from the Frobenius action) structure and monodromy operator.

Insert such a cohomology in family following the work of Shiho and moreover we will associated to such a family a differential operator with a singular regular point at $s$ and endowed with a Frobenius structure: a $F$-log isocrystal.

Re-interpretate the monodromy operator in terms of residue of the differential operator at $s$. In this differential setting the equivalence between the monodromy and weight filtration (given by Frobenius) has been proved by Crew (1998).
Some Cohomologies.

$C$ is a smooth curve over $k$ then it admits a lifting $C_V$ over $\mathcal{V}$: we indicate by $C_V$ its completion along the special fiber $C$.

$$\{s\}^\times \rightarrow (C, \{s\}) \rightarrow (C_V, N).$$

$\{s\}^\times$ is log-point given by $k$ endowed with the log-structure associated to the map $\mathbb{N} \rightarrow k$, $1 \mapsto 0$. We have a similar diagram for $(C_V, N)$.
\( \mathcal{C}_K \) is the rigid analytic space associated to the generic fiber of \( \mathcal{C}_V \). 

\((X, M)\) is \( X \) endowed with the log-structure given by the NCD fiber \( X_s \).

On \( X_s \) we induce the log structure of \((X, M)\) and we refer to it as \((X_s, M)\).

\[
\begin{array}{ccc}
(X_s, M) & \longrightarrow & (X, M) \\
\downarrow f_s & & \downarrow f \\
\{s\} & \longrightarrow & (\mathcal{C}, \{s\}) & \longrightarrow & (\mathcal{C}_V, N)
\end{array}
\]

Hyodo, Kato and Shiho defined the relative cohomology groups:

*relative log-crystalline cohomological groups of \((X, M)/\mathcal{C}_V,\)

\[
R^\bullet f_{(X,M)/\mathcal{C}_V,\text{crys},*}(\mathcal{O}_X)
\]
Even if their definition can be made in the integral setting, we will consider them as $K$-modules.

**Theorem**

*In the previous hypotheses, $\mathbb{R}^\bullet f_{(X,M)}/\mathcal{E}_V,\text{crys,}^\ast(\mathcal{O}_X)$ is a perfect complex of iso-coherent sheaves on $\mathcal{E}_V$; $\mathbb{R}^q f_{(X,M)}/\mathcal{E}_V,\text{crys,}^\ast(\mathcal{O}_X)$ is zero for $q \gg 0$ and for each $q$ it is a log-isocrystal endowed with Frobenius.*
we may enlarge our diagramm as

\[
\begin{array}{cccccc}
(X_s, M) & \xrightarrow{f_s} & \{s\} \times & \xrightarrow{\iota} & \mathcal{V} \times & \\
\downarrow & & \downarrow & & \downarrow & \\
(X, M) & \xrightarrow{f} & (C, \{s\}) & \xrightarrow{\varphi} & (\mathcal{E}_\mathcal{V}, N) & \\
\end{array}
\]

Note that all the squares are cartesian. The \(\iota\)'s and \(\varphi\) are exact closed immersions and, moreover \(f\) and \(f_s\) are proper and log-smooth.

And we have a base change after Shiho 2008

\[
\mathbb{R}^\bullet f_{(X,M)/\mathcal{E}_\mathcal{V},\text{crys},*}(\mathcal{O}_X)_s = H_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K
\]

Hence they have monodromy.
We need to link it to rigid cohomology: we will do it via convergent cohomology.

Shiho (2008) defined the *relative log-convergent cohomology groups of* \((X, M)/\mathcal{C}_V\). It is defined via a site but using formal schemes (enlargements) instead of the nilpotent liftings of the crystalline site.

\[
\mathbb{R}^\bullet f_{(X,M)/\mathcal{C}_V, \text{conv}^*}(\mathcal{O}_X)
\]

We then have (Shiho)

**Theorem**

*In our setting (f is proper) we have the natural iso between* \(\mathbb{R}^\bullet f_{(X,M)/\mathcal{C}_V, \text{conv}^*}(\mathcal{O}_X)\) *and* \(\mathbb{R}^\bullet f_{(X,M)/\mathcal{C}_V, \text{crys}, ^*}(\mathcal{O}_X)\) *as iso-coherent sheaves in* \(\mathcal{C}_V\).
Relative log-convergent cohomology has a base change theorem. As a corollary we have that the relative log-convergent cohomology groups of $(X, M)/\mathcal{C}_V$ are endowed with a Frobenius structure and Monodromy and they allow us to calculate the log-crystalline cohomology we are interested in.
The next step is to link

$$\mathbb{R}^\cdot f_{(X,M)/C_Y,\text{conv}^*}(\mathcal{O}_X)$$

on $C_Y$, to the \textit{relative log-analytic cohomology groups of $(X, M)/C_Y$}, and we will indicate them by

$$\mathbb{R}^\cdot f_{(X,M)/C_Y,\text{an}^*}(\mathcal{O}_X).$$

and they give sheaves on $C_K$. 
Again they are defined via embedding systems.
The compatibility of the good local setting is given by simplicial methods. Locally the situation is the following

\[
\begin{align*}
(X, M) & \longrightarrow \iota \longrightarrow (P, M') \\
\downarrow & \downarrow \\
(C, \{s\}) & \longrightarrow (C_V, N)
\end{align*}
\]

where \( \iota \) is closed immersions (not necessarily exact) and the maps \((P, M') \rightarrow (C_V, N)\) is log-smooth. For any such an embedding one takes the relative to \( C_K \) de Rham complexes on all the log-tubes of \((X, M)\) in \((P, M')\) (not the usual ones). One takes the relative to \( C_K \) de Rham complexes on all the log-tubes of \((X, M)\) in \((P, M')\) (not the usual ones).

In particular we have \( R^\bullet f_{(X,M)/C_V,an*}(\mathcal{O}_X) \) which are only coherent sheaves on \( C_K \).
Then one has (Shiho)

\[ \text{sp} \ast \mathbb{R}^q f_{(X,M)/\mathcal{C}_V, an*}(\mathcal{O}_X) = \mathbb{R}^q f_{(X,M)/\mathcal{C}_V, \text{conv}*}(\mathcal{O}_X) \]

where \( \text{sp} : \mathcal{C}_K \to \mathcal{C}_V \). They have base change in our setting.

They are not iso-crystals on the log-convergent site (hence for any enlargement)
But Shiho (2008) was able to prove that in our situation, because $\mathcal{C}_V$ is smooth around $C$

$\mathbb{R}^q f_{(X,M)/\mathcal{C}_V, an^*}(\mathcal{O}_X)$ is endowed with a connection on $\mathcal{C}_K$.

Moreover

**Theorem**

$\mathbb{R}^q f_{(X,M)/\mathcal{C}_V, an^*}(\mathcal{O}_X)$ is a locally free sheaf on $\mathcal{C}_K$ endowed with a Frobenius structure and a logarithmic connection (the log structure given by the point $\{s\}$).

This connection coincides with the connection one can introduce via Katz-Oda methods via absolute cohomology i.e. considering the absolute cohomology of $(X, M)/\mathcal{V}$ and that one of $(X, M)$ relative to $\mathcal{C}_V$. 
For the residue at 0 of $R^q f_{(X,M)/\mathcal{E}_V, an^*}(\mathcal{O}_X)$:

$$(\mathbb{R}^q f_{(X,M)/\mathcal{E}_V, an^*}(\mathcal{O}_X))_s \simeq H^q_{\log\text{-crys}}((X_s, M)/\mathcal{V}) \otimes K.$$ 

Moreover, because $\mathbb{R}^q f_{(X,M)/\mathcal{E}_V, an^*}(\mathcal{O}_X)$ is locally free on $\mathcal{E}_K$ then it is free on $]s[\mathcal{E}_V = D(0, 1^-)$ and endowed with a log-connection with the respect to the only log-point $\{s\}$. Because it admits also a Frobenius structure compatible with the previous log-connection, the residue at 0 of such a log-connection $N_q \in \mathcal{M}_n(K)$ (if $n$ was the rank of our locally free module) is nilpotent.
$$\left( \mathbb{R}^q f_{(X,M)/\mathcal{C}_{\mathcal{V}}, \text{an}^*}(\mathcal{O}_X) |_{\mathcal{C}_{\mathcal{V}}} \right)_s \simeq H^q_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K.$$ 

Now on the right part we have a monodromy operator from Hyodo-Kato (defined via Katz-Oda type of procedure) while on the left hand part we have the residue matrix acting. We have

**Theorem**

*In the isomorphism above the two aforementioned operators are the same.*
Building up the sequence. The "topological" interpretation

In order to define the relevant cohomologies we need, we have to construct good embeddings and lifting for our geometric setting. This can be achieved only étale locally and we need to put all these data in a simplicial setting in order to use étale descent. We won’t write all the simplicial diagrams, we content ourselves to study the very local case. This is the situation:
$X$ with the log structure from the special fiber : $(X, M)$. The horizontal maps are exact closed immersion and $\tilde{Q}^{ex}$ is log-smooth over $(\mathcal{C}_V, N)$ and it is smooth around $X$ over $\mathcal{V}$.
We have a long exact sequence

$$
\cdots \to H^i_{X_s, \text{rig}}(X) \to H^i_{\text{rig}}(X) \to H^i_{\text{rig}}(X \setminus X_s) \to \cdots
$$

It can be proved that in our setting (the behavior at $\infty$ is the same for $X$ and $X \setminus X_s$)

$$
H^i_{X_s, \text{rig}}(X) = \mathbb{R}^i \Gamma(X[\tilde{\mathcal{O}}_{\text{ex}}], [\mathcal{O}]X[\tilde{\mathcal{O}}_{\text{ex}} \to j^\dagger]X \setminus X_s[\tilde{\mathcal{O}}_{\text{ex}} \Omega]X[\tilde{\mathcal{O}}_{\text{ex}}])
$$

We are now ready to interpretate the two complexes which appear in this new definition.
The first $\Omega^\bullet_{X[\varlog Q_{\text{ex}}]}$, is a complex which calculates the convergent cohomology of $X$.

The second is nothing but the analytic (convergent) cohomology of $X$ overconvergent along $X_s = X \setminus X^{\text{triv}}$ which is SNCD.

We may only observe that the immersion being exact the "usual" tubes do coincide with the log-one. So, we may write all in

$$H^i_{X_{s, \text{rig}}}(X) \simeq R^i \Gamma([X^{\log}_{\varlog Q_{\text{ex}}}, [\Omega^\bullet_{X[\varlog Q_{\text{ex}}]} \to j_!^\dagger X \setminus X_s^{\log}_{\varlog Q_{\text{ex}}} \Omega^\bullet_{X[\varlog Q_{\text{ex}}]}])$$

Then we may apply Shiho's result 2002: it says that for a smooth-log scheme $(X, M)$ endowed with a Zariski type log structure then its log convergent cohomology coincides with the convergent cohomology of $X$ overconvergent along $X \setminus X^{\text{triv}} = X_s$. 

Hence the second complex calculates the log-convergent cohomology of $X$ endowed with the log-structure coming from the NCD, $X_s$, with respect the trivial log structure on $V$ hence in $k$. Denote it by

$$H^i_{log-conv}((X, M)/K)$$

We may then re-write (to stress the link with log-differentials)

$$H^i_{X_s, rig}(X) \simeq \mathbb{R}^i \Gamma([X_{\mathbb{Q}^{ex}}, [\Omega^\bullet_{X_{\mathbb{Q}^{ex}}} \to \Omega^\bullet_{X_{\mathbb{Q}^{ex}}} \prec log >])$$

We have a long exact sequence (of $K$-vector spaces possibly of not finite dimension)

$$\cdots \to H^i_{X_s, rig}(X) \to H^i_{conv}(X) \to H^i_{log-conv}((X, M)/K) \to \cdots$$
In

\[ \mathbb{R}^i \Gamma (\mathcal{O} \times X_{\mathbb{Q}^{\text{ex}}, [\Omega]X_{\mathbb{Q}^{\text{ex}} \rightarrow \Omega]X_{\mathbb{Q}^{\text{ex}} < \log >}}) \]

Everything depends on the tube around \( X_s \) on \( \mathcal{O} \times X_{\mathbb{Q}^{\text{ex}}} \), in such a way \( H^i_{X_s, \text{rig}}(X) \) can be calculated as the derived functors of the global section functor on \( \mathcal{O} \times X_{\mathbb{Q}^{\text{ex}}} \) of

\[ [\Omega]X_{\mathbb{Q}^{\text{ex}} \rightarrow \Omega]X_{\mathbb{Q}^{\text{ex}} < \log >}] \]

(..... some Stein acyclicity.)
it would be interesting to give a name to such a cohomology in general (even if not in SNCD case)

But one could indicate it as $H_{X_s}^\bullet (\hat{X}/K)$, where $\hat{X}$ is the completion of $X$ along $X_s$.

This should have a link with Levine’s (motivic) tubular neigh.
We may interpretate the complexes which form

\[ [\Omega^\bullet_\mathcal{X}_s[\tilde{\mathcal{Q}}_{ex} \to \Omega^\bullet_\mathcal{X}_s[\tilde{\mathcal{Q}}_{ex} < \text{log}>] \]

Note that by our hypotheses \( \mathcal{X}_s \) is proper and if \( \tilde{\mathcal{Q}}_{ex} \) is smooth around \( \mathcal{X} \) then it is smooth around \( \mathcal{X}_s \) : the first complex calculates the rigid cohomology of \( \mathcal{X}_s \). While the second calculates the log-convergent cohomology with respect to the trivial log structure on the base field of \( (\mathcal{X}_s, \mathcal{M}) \) endowed with the induced log-structure from \( \mathcal{X} \) given by \( \mathcal{X}_s \).

As a corollary of these two interpretations, we obtain the following long-exact sequence:

\[ \cdots \to H_{\mathcal{X}_s, \text{rig}}^i(X) \to H_{\text{rig}}^i(\mathcal{X}_s) \to H_{\log-\text{conv}}^i((\mathcal{X}_s, \mathcal{M})/K) \to \cdots \]
We want now to see $H^i_{\log-\text{conv}}((X_s, M)/K)$ as a part of a different long exact sequence, by relative cohomology. We use again $\tilde{\mathcal{Q}}^{\text{ex}}$: it is log-smooth relative to $(\mathcal{C}_\mathcal{V}, N)$ endowed with the log structure given by a point (i.e. NCD). And not in the absolute case over $\mathcal{V}$.

We recall that $\mathcal{C}_\mathcal{V}$ is a smooth curve over $\mathcal{V}$ and $(\mathcal{C}_\mathcal{V}, N)$ is log-smooth.

\[
\begin{array}{ccc}
(X_s, M) & \rightarrow & (X, M) \rightarrow \iota \rightarrow (\tilde{\mathcal{Q}}_{\cdot}^{\text{ex}}, M'_{\cdot}) \\
\downarrow & & \downarrow \\
\{ s \} \times & \rightarrow & (\mathcal{C}, \{ s \}) \rightarrow (\mathcal{C}_\mathcal{V}, N)
\end{array}
\]

We use $\tilde{\mathcal{Q}}^{\text{ex}}$ not only to calculate the “absolute” cohomology but also for the relative log-cohomology (again all is true locally!)
Namely, we have understood \((X_s, M) \rightarrow (\tilde{\mathcal{Q}}^{\text{ex}}, M')\) as a good embedding in order to calculate the absolute log-cohomology of \((X_s, M)\). But we can use it to calculate the relative (to the pointed curve) log-analytic cohomology and its restriction to \(]s[\mathcal{C}_V\) (the open unit disk)

\[
\Omega^\bullet_{X_s[\tilde{\mathcal{Q}}^{\text{ex}}/][s[\mathcal{C}_V < \log >
\]

We may then consider the following short exact sequence

\[
0 \rightarrow \Omega^\bullet_{X_s[\tilde{\mathcal{Q}}^{\text{ex}}/][s[\mathcal{C}_V < \log > \rightarrow [\mathbf{-1}] \rightarrow \Omega^\bullet_{X_s[\tilde{\mathcal{Q}}^{\text{ex}}/][s[\mathcal{C}_V < \log > \rightarrow 0
\]

The complex in the middle is nothing but the absolute log-analytic complex associated to \((X_s, M) \rightarrow (\tilde{\mathcal{Q}}^{\text{ex}}, M')\) over the trivial log-structure on \(k\). While the other calculates the relative log-crystalline one.
The connecting map can be interpreted as a connection for the long exact sequence of the global sections. Moreover the action of the log-connection can be trivialized (having a Frobenius structure) by the action of its residue at $s$, $N_i$ (which is nilpotent) acting on a finite dimensional vector space. We then have the following long exact sequence

$$
\cdots \rightarrow H^i_{\text{log-conv}}((X_s, M)/K) \rightarrow H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \overset{N_i}{\rightarrow} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \rightarrow \cdots
$$
Merging the two long exact sequence we have found

\[ \cdots \to H^i_{\text{log-conv}}((X_s, M)/K) \to H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \xrightarrow{N_i} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \to \cdots \]

and

\[ \cdots \to H^i_{X_s,\text{rig}}(X) \to H^i_{\text{rig}}(X_s) \to H^i_{\text{log-conv}}((X_s, M)/K) \to \cdots \]

We get

\[ \cdots \to H^i_{\text{rig}}(X_s) \xrightarrow{\gamma} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K \xrightarrow{N_i} H^i_{\text{log-crys}}((X_s, M)/\mathcal{V}) \otimes K(-1) \xrightarrow{\delta} H^{i+2}_{X_s,\text{rig}}(X) \to \cdots \]

In particular because \( X_s \to X \) is a closed immersion of \( X_s \) in \( X \) which is smooth, then by Poincaré duality we have (\( \dim X = n + 1 \))

\[ H^{i+2}_{X_s,\text{rig}}(X) \cong (H^{2n+2-i-2}_{c,\text{rig}}(X_s))^* \cong H^{\text{rig}}_{2n-i}(X_s). \]
Exactness of the sequence

We want the exactness of the previous long sequence

\[ \cdots \to H^i_{\text{rig}}(X_s) \xrightarrow{\gamma} H^i_{\log-crys}((X_s, M)/\mathcal{V}) \otimes K \xrightarrow{N_i} \]

\[ H^i_{\log-crys}((X_s, M)/\mathcal{V}) \otimes K(-1) \xrightarrow{\delta} H^{i+2}_{X_s, \text{rig}}(X) \to \ldots \]

and we suppose field $k$ is finite.

We will show how such a result will follow from the fact that in $ch = p$ the monodromy filtration coincides with the weight filtration in our setting.
We have

\[(\mathbb{R}^i f(x, M)/\mathcal{C}_V, an^*(\mathcal{O}_X)|s[\mathcal{C}_V])_s \simeq H^i_{\log\text{-}crys}((X_s, M)/\mathcal{V}) \otimes K.\]

where the tube \(s[\mathcal{C}_V]\) is isomorphic to the open unit disk. Such a relative cohomology can be represented by a free module \(M_i\) of rank \(n\) on \(s[\mathcal{C}_V]\) endowed with a log-connection \(\nabla\) and a Frobenius structure compatible with the connection. Then our trivialization makes \(M_i \simeq V_i \otimes \mathcal{O}(s[\mathcal{C}_V])\) and \(\nabla = N_i \otimes dt/t\), \(N_i \in \mathcal{M}_n(K)\). I.e.

\[(M_i, \nabla) \simeq (V_i \otimes \mathcal{O}(s[\mathcal{C}_V]), N_i \otimes dt/t)\]
We then identify $V_i$ with the fiber at 0: $H^i_{\log-crys}((X_s, M)/\mathcal{V}) \otimes K$. Moreover such a fiber has also a Frobenius structure, hence on $V_i$ we have

$$FN_i = qN_i F$$

where $q$ is the cardinality of the residue field and moreover we obtain that $N_i$ is nilpotent. Associated with a nilpotent operator we have a monodromy filtration on $V_i$ (read on $H^i_{\log-crys}((X_s, M)/\mathcal{V}) \otimes K$). Moreover $F$ is a Frobenius structure on the $K$ vector space $V_i$. We have Crew’s result (1998)

**Theorem**

*The log-relative cohomology sheaves $R^i f_{(X, M)/\mathcal{V}, an^*}(\mathcal{O}_X)$ on $C \setminus \{s\}$ are pure of weight $i$. On $R^i f_{(X, M)/\mathcal{V}, an^*}(\mathcal{O}_X)$ restricted to $\mathcal{O}_s$ the monodromy is unipotent and for each $j$ the graded parts for the monodromy filtration $gr_j^M V_i$ are pure for the Frobenius action and of weight $i + j$.***
Because of this equivalence monodromy/weight filtration we have, in particular:

**Corollary**

\[ H^i_{\log-crys}(\mathcal{X}/\mathcal{V}) \otimes K \]  
the ker\(N_i\) has weights smaller than or equal to \(i\). While \(V_i/\operatorname{Im}N_i\) has weights bigger than or equal to \(i\).
Moreover we know that $H^i_{\text{rig}}(X_s)$ has weights $\leq i$ while $H^{i+1}_{X_s,\text{rig}}(X)$ has weights strictly bigger than $i$ [Ch]. Putting all together such a results we end with the exactness.
The invariant cycles sequence with coefficients for case 2)

Work with Coleman, Iovita, Di Proietto. We recall the situation from the beginning:
2) We consider $\mathcal{V}$ a DVR of mixed characteristic, $K$ its fraction field and $k$ its residual field of $\text{ch} = p$. Then

$$f : X \to \text{Sp}\mathcal{V},$$

where $X$ is a proper semistable variety over $\text{Sp}\mathcal{V}$. The fiber at $s$ of $f$, $X_s = X_0$, is a NCD in $X$ (and the generic fiber $X_K$ is smooth). We may endow $X$ with the log structure given by the special fiber $X_s$. And it induces a log structure on $X_s : (X_s, M)$. 
We may state our result [Ch,99]:

**Proposition**

*In the previous case, if $X$ is a (proper) semistable curve or a surface over $\mathcal{V}$, then we have an exact sequence:*

$$H^i_{\text{rig}}(X_s) \xrightarrow{\gamma} H^i_{dR}(X_K) \xrightarrow{N_i} H^i_{dR}(X_K)$$

**Remark:** $H^i_{\text{log-crys}}(X_s, M)/\mathcal{V}) \otimes K = H^i_{dR}(X_K)$ (Hyodo-Kato). And by Rigid GAGA $H^i_{dR}(X_K) = H^i_{dR}(X^K_{\text{rig}})$
This is the trivial coefficient case: both at the generic and special fiber. In the $p$-adic case we don’t have Riemann-Hilbert correspondance: hence we may only speak of the differential part. We don’t have Constructible sheaves or ...perverse sheaves... We need arithmetic $\mathcal{D}$-modules. Hence the first objects analogous to connections are the isocrystals.
On the special fiber the coefficients are isocrystal $E$ : "a differential module with connection which can be defined in a smooth neighborhood of $X_s$ over $\nu$". Now $X$ is not smooth, but suppose that $X$ can be seen as a closed subscheme of a smooth one $P$ over $\nu$. Then we have that the generic fiber (as a Rigid space) can be embedded in the tube of $X_s$ in $P$, $]X_s[$. Hence to an isocrystal on $X_s$, $E$, we associate a module with connection $(\mathcal{E}, \nabla)$ on $]X_s[$, which induces a module on $X_K^{\rig} : (\mathcal{E}, \nabla)_{|X_K^{\rig}}$. Hence we have a natural map

$$H^i_{\rig}(X_s, E) = H^i(]X_s[, (\mathcal{E}, \nabla)) \to H^i_{dR}(X_K^{\rig}, (\mathcal{E}, \nabla)_{|X_K^{\rig}})$$
In this sense, in the de Rham setting we have an hidden "singular" structure and Coleman and Iovita (2010) were able in case $X$ a curve to define a monodromy operator on $H^i_{dR}(X^\text{rig}_K, (\mathcal{E}, \nabla)|_{X^\text{rig}_K})$, for any $F$-isocrystal $E$.

For trivial coefficient such a monodromy coincides with the Hyodo-Kato log-crystalline one for the trivial coefficients.

We have a sequence

$$H^i_{\text{rig}}(X_s, E) \rightarrow H^i_{dR}(X^\text{rig}_K, (\mathcal{E}, \nabla)|_{X^\text{rig}_K}) \xrightarrow{N} H^i_{dR}(X^\text{rig}_K, (\mathcal{E}, \nabla)|_{X^\text{rig}_K})$$

In full generality we can prove that the first map is injective and the composition with the second is zero. It is tempting to see if the sequence is exact.
Approach via the decomposition theorem? We are not able to define a good category of "pure" isocrystals. When the invariant cycles sequence is exact.

Constant coefficients are ok. What about the case of the unipotents objects?
They are far from being irreducible (i.e. extension by trivial). We don’t expect exactness in general ......... A different landscape then the decomposition theorem .........

The unipotents objects (extension of the trivial by the trivial) in $X_s$ are related to $H^1_{\text{rig}}(X_s)$ and the unipotent diff. module on $X_K$ are related to $H^1_{dR}(X_K) = H^1_{dR}(X^\text{rig}_K)$. 
Let $E$ be a unipotent convergent $F$-isocrystal: we know it defines a differential module in a tube of $X_s : X_K \subset \mathcal{X}_s$. We have

$$H^i_{\text{rig}}(X_s, E) \to H^i_d(X^\text{rig}_K, (\mathcal{E}, \nabla)|_{X^\text{rig}_K}) \to H^i_d(X^\text{rig}_K \mathcal{S}, (\mathcal{E}, \nabla)|_{X^\text{rig}_K})$$

We suppose it is exact. Let us consider the following extension in the category of convergent $F$-isocrystals

$$0 \to E \xrightarrow{h} F \xrightarrow{g} \mathcal{O} \to 0$$

where $\mathcal{O}$ is the trivial $F$-isocrystal. Let $x \in H^1_{\text{rig}}(X_s, E)$ corresponding to the class of this extension ($x$ is then fixed by the Frobenius operator: Ch-Le Stum) Let us suppose that $x \neq 0$. 
For each of the isocrystals $E$, $F$ and $O$ we have an invariant cycles sequence (exact for $O$ and $E$). denote the first maps by $\gamma_E$, $\gamma_F$ and $\gamma_O$ respectively and the monodromy operators by $N_E$, $N_F$ and $N_O$ respectively.

We have that $H^1_{\text{rig}}(X_k, E) \otimes K$ is isomorphic via $\varphi_E^*$ to $\text{Ker}(N_E)$, and this last group contains the image of $N_E$, as this operator has square zero.
Theorem

If $\gamma^*_E(x) = N_E(y)$ for $y \in H^1_{dR}(X^\text{rig}_K, (E, \nabla)|X^\text{rig}_K)$, then if we denote by $h_{dR} : H^1_{dR}(X^\text{rig}_K, (\mathcal{E}, \nabla)|X^\text{rig}_K) \to H^1_{dR}(X^\text{rig}_K, (\mathcal{F}, \nabla)|X^\text{rig}_K)$ the map induced by $\alpha$ in the sequence, the following holds:

$$\text{Kernel}(N_\mathcal{F}) = (H^1_{\text{rig}}(X_k, F)) \oplus K_{\log - \text{crys}}(y)$$

In this way we give an example of a control of the non exactness of the sequence. We plan to find a necessary condition too...