Def. A Higgs bundle on $X$ is a hol. v.b. $E$ over $X$ with a map 
$\phi: E \to E \otimes \Omega^1_X$ (usually we also assume that $\phi \theta: E \to E \otimes \Omega^1_X$ is 0).
We set $D'' = \overline{\nabla} + \Theta$ ($\overline{\nabla}$ defines the hol. structure on $E$).

If $h$ is an metric on $E$, then we get an operator $\partial_\Theta$ defined in such a way that $\partial_\Theta = \partial_\Theta + \overline{\nabla}$ is the metric (or Chern, or unitary) connection on $E$ (i.e., $\partial_\Theta$ is the unique connection compatible with the hol. structure and with the metric).

Then we can also define $\Theta_\partial$ by requiring that $(\Theta u, v)_h = (u, \Theta_\partial v)_h$.

Then we set $D'_\partial = \partial_\Theta + \Theta_\partial$ and $D'_\partial = D'_\partial + D''$ (this is equivalent to the splitting of a connection in its $(1,0)$ and $(0,1)$ parts).

Then $D'_\partial$ is a connection on $E$. This is the replacement of the usual metric (or Chern) connection on $E$, but now it depends also on the Higgs field $\Theta$!

We denote by $F_{\partial\Theta}$ the curvature of $D'_\partial$ (and by $F_{\partial\Theta}^\perp$ the trace-free part of $F_{\partial\Theta}$).

The degree of a v.b. $E$ is defined as usual (i.e., \[ \frac{1}{4} \int_{\Omega} \nabla \wedge \Theta \wedge \Theta^{m-1} \])

but if $X$ is not compact, we have to assume a bounded (in $c_1(E) \wedge \Theta^{m-1}$) condition on $\Theta_{\partial\Theta}$ such that the integral converges!

A sub-Higgs-sheaf (obtained by) is a subsheaf $F_{\partial\Theta}$ of $E$ such that $\partial_{\Theta} F_{\partial\Theta}$ sends $\Phi$ to $\Phi \otimes \Omega^1_X \subset E \otimes \Omega^1_X$.

(Semi) stability is defined as usual, but considering only sub-Higgs-sha.

Now, the thing is that if $(E, \Theta)$ is a Higgs bundle on $X$ (and the integral above converges, so that we can define the degree of $E$), then there is a metric $h$ on $E$ that satisfies the HE-condition (also called Hermite-Yang-Mills).

(Note that in this thing it is not assumed that $\Theta_{\partial\Theta} = 0$.)

If we also assume that $\Theta_{\partial\Theta} = 0$, that $\text{Tr}(F_{\partial\Theta}) = 0$ (i.e., $c_1(E, \Theta) = 0$) and that $c_2(E, \Theta) \cdot [\Phi]^{m-2} = 0$, then the connection $D'_\partial$ on $E$ is flat.

(In this way we can construct flat connections on $E$.)
Higgs bundles vs. Flat bundles

Any establishment of correspondence between Higgs bundles \((E, \Phi)\) and Flat bundles \((E, \mathcal{D})\).

**Strategy:**

1. To a Higgs bundle \((E, \Phi)\) we associate a connection \(\mathcal{D}'\) (it depends on the choice of a Hermitian metric \(h\) on \(E\)).

2. To a flat bundle \((F, \mathcal{D})\) we associate an operator \(D''\) (it depends on the Hermitian metric \(h\)), and the operator \(D''\) encodes the structure of a Higgs bundle. (and the hol. structure).

3. Show that, under suitable hypotheses, it is possible to choose the Hermitian metric \(h\) such that \((E, D_h)\) is a flat bundle and \((F, D''_h)\) is a Higgs bundle.

We start with a Higgs bundle \((E, \Phi)\). Then \(E\) is a hol. bundle, and the hol. structure is given by an operator \(\overline{\partial}_E : A^0(E) \to \Lambda^1(E)\).

The Higgs field \(\Phi : E \to \Omega^1_x\) is a map \(\Phi : A^0(E) \to A^1,0(E)\).

We combine these to form an operator \(D'' : \overline{\partial}_E + \Phi : A^0(E) \to A^1(E)\). Note that \(D''\) is not a connection!

This operator satisfies the Leibnitz rule

\[
D''(\phi \sigma) = \overline{\partial}(\phi) \sigma + \phi \overline{\partial}(\sigma), \quad \forall \phi \in A^0, \quad \forall \sigma \in A^0(E)
\]

and the integrability condition \((D'' \circ D')\sigma = (\overline{\partial}_E + \Phi) \cdot (\overline{\partial}_E + \Phi)\sigma = 0\).
Hence $D'' \circ D'' = 0$ if:

\[ \begin{cases} \overline{\eta} \circ \overline{\eta} = 0 : \text{the hol. structure is integrable} \\ \eta \circ \overline{\eta} = 0 : \theta \text{ is holomorphic} \\ \eta \circ \eta = 0 : \theta \text{ is a Higgs field.} \end{cases} \]

Conversely, an operator $D'' : A^0(E) \to A^1(E)$ that satisfies the previous Leibniz rule and the integrability condition $D'' \circ D'' = 0$ defines a hol. Higgs bundle structure on $E$ (take $\eta = \overline{\eta}$ and $\theta$ the $(0,1)$ and $(1,0)$ parts of $D''$, respectively).
So a Higgs bolt \((E, S)\) is equivalent to \((E, \bar{D}'')\) with \(d'' = d_E + \bar{D}'\) [we view \((E, \bar{D}'')\) as \(C^\infty\) bolt. We call this a Higgs bolt].

Let us take a Higgs bolt \((E, D'')\). Let \(h\) be a Hermitian metric on the \(C^\infty\) bolt \(E\). We want to define a connection \(D_h\) on \(E\).

Remember that \(D'' = d_E + \bar{D}'\). Let \(\partial_h\) be the unique operator such that \(\partial_h + d_E\) is the Hermitian (or unitary, or Chern) connection, i.e., the unique connection compatible with the holomorphic and Hermitian structure.

Then we define \(\delta_h\) by requiring that \((\delta_h(\sigma), \tau) = (\delta, \delta_h(\tau))\) for \(\sigma, \tau \in A^0(E)\). Then we set \(D_h = D'' + \delta_h\), and finally we set \(D_h = D_h + D''\). It can be checked that \(D_h\) is a connection on \(E\), it is not a flat connection in general!

(We would like to choose the metric \(h\) s.t. \(D_h\) is flat.)

Now we go from flat bolts to Higgs bolts:

Let \((F, D)\) be a \(C^\infty\) flat v.b., with a Hermitian metric \(h\).

Decompose \(D = d' + d''\), operators of type \((1, 0)\) and \((0, 1)\).

Define \(\delta_h'\) and \(\delta_h''\) as the unique operators of type \((1, 0)\) and \((0, 1)\) s.t. the two connections \(d' + \delta_h''\) and \(d' + \delta_h'\) preserve the Hermitian metric \(h\). Then define

\[
\delta_h = \frac{d' + \delta_h'}{2}, \quad \bar{\delta}_h = \frac{d'' + \delta_h''}{2}, \quad \delta_h = \frac{d' - \delta_h'}{2}, \quad \bar{\delta}_h = \frac{d'' - \delta_h''}{2}.
\]

Finally set \(D_h'' = \bar{\delta}_h + \delta_h\). It can be checked that \(D_h''\) is an operator of the type needed to define a Higgs bolt (because it satisfies the right Leibnitz rule: \(D_h''(f\sigma) = \bar{\delta}(f)\sigma + fD_h''(\sigma)\)), but in general it does not define a Higgs bolt because it does not satisfy the integrability condition \(D_h'' \circ D_h'' = 0\).

(We would like to choose the metric \(h\) s.t. \(D_h'' \circ D_h'' = 0\)).

Unfortunately, the equations \(D_h'' \circ D_h = 0\) (Flat connection) and \(D_h'' \circ D_h'' = 0\) (Higgs bolt) are overdetermined so, in general, it is not possible to find a metric \(h\) s.t. these equations are satisfied!
Given a Higgs bundle \((E, D)\) with Hermitian metric \(h\), we have the connection \(D_h\) and we denote by \(F_h = D_h - D_{h}^{\dagger}\) its curvature. Analogously, given a flat bundle \((E, D)\) with Hermitian metric \(h\), we have the operator \(D_h\) and we define its pseudo-curvature \(G_h = D_h - D_{h}^{\dagger}\) (note that \(D_h^{\dagger}\) is not a connection).

Since the equations \(F_h = 0\) and \(G_h = 0\) are overdetermined, we consider the components of \(F_h\) and \(G_h\) pointing in the direction of the Kähler form.

**Def.** If \((E, D)\) is a Higgs bundle, a metric \(h\) is called Hermitian-Yang-Mills (HE) if \(\Lambda F_h = \lambda \cdot \text{Id}\), \(\lambda\) is a scalar constant (\(\lambda\) depends on the slope of \(E\)).

**Def.** If \((E, D)\) is a flat bundle, a metric \(h\) is called harmonic if \(\Lambda G_h = 0\).

If a Higgs bundle \((E, D)\) has a HYM (i.e., HE)-metric \(h\), i.e., if \(\Lambda F_h = \lambda \cdot \text{Id}\), then if \(\deg(E) = c_1(E) \cdot [\hat{\omega}]^{m-1} = 0\) it follows that \(\lambda = 0\), hence the previous equation becomes \(\Lambda F_h = 0\). Now, if also \(c_2(E) \cdot [\hat{\omega}]^{m-2} = 0\) then from \(\Lambda F_h = 0\) it follows that \(F_h = 0\) (this is what we really wanted).

It turns out that, if a HYM-metric \(h\) exists, then \(F_h = 0\) if \(c_1(E) = 0\) and \(c_2(E) = 0\) (actually, \(c_1(E) \cdot [\hat{\omega}]^{m-1} = 0\) and \(c_2(E) \cdot [\hat{\omega}]^{m-2} = 0\)).

Now suppose that \((E, D)\) is a flat v.b. with a Hermitian metric \(h\). We have defined the pseudo-curvature \(G_h\). Using \(G_h\), we can define pseudo-Chern classes (is the same way as for the usual curvature).

It turns out that the pseudo-Chern classes of a flat v.b. always vanish!

A similar argument as before can be used to prove:

Thus, if there exists a Hermitian metric on \(E\) (i.e., \(\Lambda G_h = 0\)), then \(G_h = 0\).

In this case the flat bundle \((E, D)\) comes from a Higgs bundle \((E, D_h)\).
Harmonic bundles.

This result motivates the following definition:

Def. A harmonic metric is a Hermitian metric $h$ on a Higgs bundle such that $F_h = 0$.

Def. A Higgs bundle is a $C^\infty$ vector bundle provided with structure of flat bundle and Higgs bundle, which are related by a harmonic metric.

The relations with stability are clarified by the following result:

Theorem. A flat bundle $(F, \nabla)$ has a harmonic metric iff it is semistable (i.e., iff it is a direct sum of irreducible local systems).

(2) A Higgs bundle $(E, \Theta)$ has a HYM ($\mathcal{H}$E)-metric iff it is polystable (i.e., direct sum of stable Higgs bundles of the same slope).

Such a metric is harmonic iff
\[
\int_M c_1(E) \cdot [\Theta]^{n-1} = 0 \quad \text{and} \quad \int_M c_2(E) \cdot [\Theta]^{n-2} = 0.
\]

Corollary. There is a bijection between the set of semistable flat bundles and the set of polystable Higgs bundles with
\[
\int_M c_1(E) \cdot [\Theta]^{n-1} = 0 \quad \text{and} \quad \int_M c_2(E) \cdot [\Theta]^{n-2} = 0.
\]

Actually, this bijection between objects extends to an equivalence of categories.

Corollary. There is an equivalence of categories between the category of semistable flat bundles on $X$ and the category of polystable Higgs bundles with $c_1(E) \cdot [\Theta]^{n-1} = 0$ and $c_2(E) \cdot [\Theta]^{n-2} = 0$, both being equivalent to the category of harmonic bundles.

Moduli Spaces

$X$ be a smooth connected complex projective variety. $\pi : X \rightarrow X$ fixed point

let $\Gamma := \pi_* (\mathfrak{X}^X, \pi)$ - let $\Omega^X(X, \pi) := \text{Hom}(\Gamma, \text{GL}(n, \mathbb{C})) / \text{GL}(n, \mathbb{C})$

(acts by conjugation). This is the moduli space of representations of the fundamental group (simplex cells in the Betti moduli space).

The de Rham moduli space $\mathcal{M}^\text{dr}(X, \pi)$ is the moduli space of vector bundles of rank $n$ with flat connections.

The (clown) Birkhoff-Grothendieck correspondence gives a bijection, $\mathcal{M}^\text{dr}(X, \pi) \cong \mathcal{M}^\text{dr}(X, \pi)$. This is actually an analytic isomorphism.
Define the Deligne-Mumford space, $\mathcal{M}_D$ - it is a quasi-projective variety whose points parametrize direct sums of stable Higgs bundles with vanishing Chern classes.

Then the correspondence between Higgs bundles and flat bundles gives a bijection $\mathcal{M}_D \cong \mathcal{M}_{\text{DR}}$. This is a homeomorphism of topological spaces.