CONTACT NON-SQUEEZABILITY

Sheng-Fu Chiu
(joint work with Dimitri Tamarkin)

Department of Mathematics, Northwestern University

February 2014, Padova

Symplectic Non-Squeezibility

Let $B_R = \{(\mathbf{q}, \mathbf{p}) | \mathbf{q}^2 + \mathbf{p}^2 < R^2\}$ be open ball of radius R in symplectic space $\mathbb{R}^{2n}$. The celebrated non-squeezing theorem by Gromov states that:

Theorem (Gromov, 1985)

When $r < R$, there is no symplectic embedding $B_R \xrightarrow{\Phi} B_r$.

One might ask : what should an analogue in contact topology look like ?

Contact Case : which space ?

- Consider contact manifold $\mathbb{R}^{2n+1} = \{(\mathbf{q}, \mathbf{p}, z)\}$ with standard contact form $\alpha = \mathbf{p}d\mathbf{q} + dz$.
- The dilation $((\mathbf{q}, \mathbf{p}, z)) \mapsto (\lambda\mathbf{q}, \lambda\mathbf{p}, \lambda^2 z)$ always gives a contactomorphism for any $\lambda > 0$.
- Any domain can be squeezed into arbitrary small neighborhood. Size does not matter here.
- Eliashberg,Kim,and Polterovich suggest that we should consider the pre-quantization space $\mathbb{R}^{2n} \times \mathbb{S}^1$.

Contact Embedding

- However, a single contact embedding is not enough to detect the size of domain.
- $\mathbb{C}^n \times \mathbb{S}^1 \xrightarrow{\Phi_K} \mathbb{C}^n \times \mathbb{S}^1$, $(\omega, z) \mapsto (v(\omega)e^{2\pi iKz}\omega, z)$. Here $v(\omega) = 1/\sqrt{1 + K\pi|\omega|^2}$ and $K \in \mathbb{N}$.
- Then Φ_K is a contactomorphism, and takes $B_R \times \mathbb{S}^1$ onto $B_r \times \mathbb{S}^1$ when $r^2 = \frac{R^2}{1+\pi KR^2}$.
- One can even make Φ_K smoothly isotopic to the inclusion into $\mathbb{R}^{2n} \times \mathbb{S}^1$.

Contact Isotopy

Let U_1, U_2 be open domains in a contact manifold V,

Definition (Eliashberg-Kim-Polterovich)

We say U_1 can be squeezed into U_1 if there exists a compactly supported contact isotopy $\Phi_s : \bar{U}_1 \to V$, $s \in [0,1]$, such that $\Phi_0 = Id$, and

$$\Phi_1(\bar{U}_1 \subset U_2)$$

Results of Eliashberg-Kim-Polterovich, 2006

Stick to contact isotopies with compact support on $\mathbb{R}^{2n} \times \mathbb{S}^1$, we have

Theorem (Squeezing)

$n \geq 2 \Rightarrow \forall r, R < 1/\sqrt{\pi}$, $B_R \times \mathbb{S}^1$ can be squeezed into $B_r \times \mathbb{S}^1$.

Theorem (Non-Squeezing)

If $\pi r^2 \leq m \leq \pi R^2$ for some $m \in \mathbb{N}$, then $B_R \times \mathbb{S}^1$ CANNOT be squeezed into $B_r \times \mathbb{S}^1$

Results of Eliashberg-Kim-Polterovich, 2006

- Sandon(2009) applies methods of generating function to construct contact homology theory (after Viterbo and Traynor) and gives a new proof of the above non-squeezing theorem.

- The case $m < \pi r^2 < \pi R^2 < m + 1$ remains unknown in their papers.

Microlocal Sheaf Theory(after Kashiwara-Schapira)

- Ground field $\mathbb{K}$ of finite cohomological dimension.
- Y manifold, $D(Y)$ derived category of sheaves/$\mathbb{K}$
- $\mathscr{F} \in D(Y) \Rightarrow SS(\mathscr{F})$ is conic closed coisotropic in $T^\star Y$.
- It allows us to encode Hamiltonian symplectomorphisms into sheaves(sheaf quantizations).
- Before doing so, we need to conify. And we will see this fits well when passing to contact topology.

Adding Extra Variable $T^\star\mathbb{R}_t = \{(t,k)\}$

- $E := \{\mathbf{q} \in \mathbb{R}^n\}$ and $X = E \times \mathbb{R}_t$.
- $D(X) := D(E \times \mathbb{R}_t)$ is the derived category of sheaves of vector spaces over $\mathbb{K}$
- $D_{\leq 0}(X) = \{\mathscr{F} \in D(X) | SS(\mathscr{F}) \subset \{k \leq 0\}\}$
- We define the triangulated category $\mathscr{D}_{>0}(X)$ to be the quotient

$$\mathscr{D}_{>0}(X) = \mathscr{D}_{>0}(E \times \mathbb{R}_t) := D(X)/D_{\leq 0}(X)$$

Objects Supported on/outside Symplectic Ball

$B_R = \{(\mathbf{q},\mathbf{p}) | \mathbf{q}^2 + \mathbf{p}^2 < R^2\} \subset T^\star E$.
Let $T^\star_{k>0}(E \times \mathbb{R}_t) \xrightarrow{\gamma} T^\star E$, $(\mathbf{q},\mathbf{p},t,k) \mapsto (\mathbf{q}, \mathbf{p}/k)$

- $D_{T^*E \backslash B}(E \times \mathbb{R}_t) = \{\mathscr{F} \in \mathscr{D}_{>0}(E \times \mathbb{R}_t) | \gamma(SS(\mathscr{F})) \subset T^*E \setminus B_R\}$
- The functor $D_{T^*E \backslash B}(E \times \mathbb{R}_t) \hookrightarrow \mathscr{D}_{>0}(E \times \mathbb{R}_t)$ admits a left adjoint functor $\mathscr{D}_{>0}(E \times \mathbb{R}_t) \to D_{T^*E \backslash B}(E \times \mathbb{R}_t)$.
- Moreover, it is given by a convolution kernel $\mathbb{Q}_R \in \mathscr{D}_{>0}(E_1 \times E_2 \times \mathbb{R}_t)$.

Symplectic Projector

Let $\Delta = \{(\mathbf{q}, \mathbf{q}, t) | t \geq 0\}$, and the constant sheaf $\mathbb{K}_\Delta$ stands for the convolution kernel of the identity endofunctor.

Theorem (Projector for Symplectic Ball)

In $\mathscr{D}_{>0}(E_1 \times E_2 \times \mathbb{R}_t)$ there exists

$$\mathscr{P}_R \longrightarrow \mathbb{K}_\Delta \longrightarrow \mathscr{Q}_R \xrightarrow{+1}$$

here $\mathscr{D}_{>0}(E_1 \times \mathbb{R}_t) \to D_{T^*E \setminus B}(E_2 \times \mathbb{R}_t)$ is given by $(-) \underset{E_1}{\bullet} \mathscr{Q}_R$, and $\mathscr{D}_{>0}(E_1 \times \mathbb{R}_t) \to \mathscr{D}_B(E_2 \times \mathbb{R}_t)$ is given by $(-) \underset{E_1}{\bullet} \mathscr{P}_R$.

In particular, $\mathscr{D}_B(E \times \mathbb{R}_t) \cong \mathscr{D}_{>0}(E \times \mathbb{R}_t) / D_{T^*E \setminus B}(E \times \mathbb{R}_t)$.

Contact as conic Symplectic

- $X = \mathbb{R}^{2n+1} = E \times \mathbb{R}_t$
- $C_R = \{(\mathbf{q}, \mathbf{p}, z, \zeta) | \zeta > 0, (\mathbf{q}, \mathbf{p}/\zeta, z) \in \widetilde{B_R \times \mathbb{S}^1}\}$

We can lift $\{\Phi_s\}$ to conic symplectic isotopy:

$$\begin{array}{ccccc} C_R & \subset & T^*_{\zeta>0}(X) & \xrightarrow{\Phi_s} & T^*_{\zeta>0}(X) \\ \downarrow & & \downarrow & & \downarrow \\ \widetilde{B_R \times \mathbb{S}^1} & \subset & \mathbb{R}^{2n} \times \mathbb{R}_z & \xrightarrow{\Phi_s} & \mathbb{R}^{2n} \times \mathbb{R}_z \\ \downarrow & & \downarrow & & \downarrow \\ B_R \times \mathbb{S}^1 & \subset & \mathbb{R}^{2n} \times \mathbb{S}^1 & \xrightarrow{\Phi_s} & \mathbb{R}^{2n} \times \mathbb{S}^1 \end{array}$$

Objects Supported on/outside Contact Ball

- $D_{T^*_{\zeta>0}(X)\setminus C_R}(X) = \{\mathscr{F} \in \mathscr{D}_{>0}(X) | SS(\mathscr{F}) \subset T^*_{\zeta>0}(X) \setminus C_R\}$
- $D_{T^*_{\zeta>0}(X)\setminus C_R}(X) \hookrightarrow \mathscr{D}_{>0}(X)$ has a left adjoint functor

This means that $D_{T^*_{\zeta>0}(X)\setminus C_R}(X)$ admits a left semi-orthogonal complement

$$\mathscr{D}_{C_R}(X) = \mathscr{D}_{>0}(X)/D_{T^*_{\zeta>0}(X)\setminus C_R}(X)$$

Convolution v.s. Composition

The difference map $\delta : \mathbb{R}_1 \times \mathbb{R}_2 \to \mathbb{R}_t$, $(z_1, z_2) \mapsto z_2 - z_1$ gives

$$\mathscr{P}_R := \delta^{-1}\mathscr{P}_R \in D(E_1 \times \mathbb{R}_1 \times E_2 \times \mathbb{R}_2) = D(X_1 \times X_2)$$

Then for any sheaf $\mathscr{F} \in \mathscr{D}_{>0}(X_1) = \mathscr{D}_{>0}(E_1 \times \mathbb{R}_1)$, we have the following isomorphisms

$$\mathscr{F} \underset{X_1}{\circ} \mathscr{P}_R = \mathscr{F} \underset{X_1}{\circ} (\delta^{-1}\mathscr{P}_R) \cong \mathscr{F} \underset{E_1}{\bullet} \mathscr{P}_R$$

Hence $\mathscr{D}_{>0}(X)$ to $\mathscr{D}_{C_R}(X)$ is given by $(-) \circ \mathscr{P}_R$.

Contact Isotopy Invariant

Let $\mathscr{T}$ be the constant sheaf supported on the shifted diagonal, namely $\mathscr{T} := \mathbb{K}_{\{\mathbf{q}_1=\mathbf{q}_2, z_1-z_2=1\}}$ in $D(X \times X)$. Pick a positive integer N and consider the cyclic action on the space $(X \times X)^N$,

$$\sigma : (x_1, x_2, ..., x_{2N-1}, x_{2N}) \mapsto (x_{2N}, x_1, x_2, ..., x_{2N-1})$$

we can formulate an invariant :

$$\mathscr{C}_N(B_R \times \mathbb{S}^1) := Rhom_{(X\times X)^N}(\mathscr{P}_R^{\boxtimes N}; \sigma_* \mathscr{T}^{\boxtimes N})$$

Quantizing Hamiltonian Symplectomorphism

Recall that we lift $\{\Phi_s\}$ to conic symplectic isotopy:

$$\begin{array}{ccccc}
C_R & \subset & T^*_{\zeta>0}(X) & \xrightarrow{\Phi_s} & T^*_{\zeta>0}(X) \\
\downarrow & & \downarrow & & \downarrow \\
\widetilde{B_R \times \mathbb{S}^1} & \subset & \mathbb{R}^{2n} \times \mathbb{R}_z & \xrightarrow{\Phi_s} & \mathbb{R}^{2n} \times \mathbb{R}_z \\
\downarrow & & \downarrow & & \downarrow \\
B_R \times \mathbb{S}^1 & \subset & \mathbb{R}^{2n} \times \mathbb{S}^1 & \xrightarrow{\Phi_s} & \mathbb{R}^{2n} \times \mathbb{S}^1
\end{array}$$

There is a lagrangian graph in $T^*(I \times X \times X)$:

$$\Lambda = \{(s, -H_s(\Phi_s(y)), y^{\mathbf{a}}, \Phi_s(y)) | s \in I, y \in T^*_{\zeta>0}(X)^{\mathbf{a}}\}$$

Quantizing Hamiltonian Symplectomorphism

Theorem (Kashiwara-Schapira-Guillermou, 2012)

There exists an unique locally bounded sheaf $\mathscr{S}$ *in* $D(I \times X \times X)$ *s.t.*

1. $SS(\mathscr{S}) = \Lambda$
2. $\forall s \in I$, $\mathscr{S}_s \circ \mathscr{S}_s^{-1} \cong \mathscr{S}_s^{-1} \circ \mathscr{S}_s \cong \mathscr{S}_0 \cong \mathbb{K}_{\Delta_X}$. *Here* $\mathscr{S}_s^{-1} := v^{-1}\underline{Rhom}(\mathscr{S}_s; \omega_X \boxtimes \mathbb{K}_X)$, $v : (x, y) \mapsto (y, x)$.

Corollary

$\forall s \in I$, $\mathscr{C}_N(B_R \times \mathbb{S}^1) \cong \mathscr{C}_N(\Phi_s(B_R \times \mathbb{S}^1))$

Naturality

Given sequence of embeddings

$$\Phi(B_R \times \mathbb{S}^1) \overset{j}{\hookrightarrow} B_r \times \mathbb{S}^1 \overset{j_r}{\hookrightarrow} \mathbb{R}^{2n} \times \mathbb{S}^1$$

we can derive morphisms between their projectors, and then a diagram of induced morphisms:

$$\begin{array}{ccccc} \mathcal{C}_N(\Phi(B_R \times \mathbb{S}^1)) & \xleftarrow{j^*} & \mathcal{C}_N(B_r \times \mathbb{S}^1) & \xleftarrow{{j_r}^*} & \mathcal{C}_N(\mathbb{R}^{2n} \times \mathbb{S}^1) \\ \cong \downarrow & & & \swarrow {j_R}^* & \\ \mathcal{C}_N(B_R \times \mathbb{S}^1) & & & & \end{array}$$

$\mathbb{Z}/N\mathbb{Z}$- Action

The cyclic action on $(X \times X)^N$

$$\sigma^2 : (x_1, x_2, \dots, x_{2N-1}, x_{2N}) \mapsto (x_{2N-1}, x_{2N}, x_1, x_2, \dots)$$

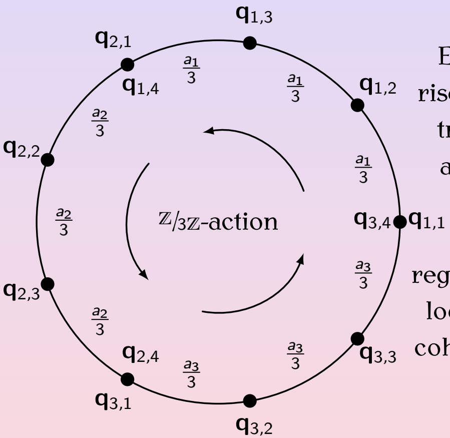
induces an $(\mathbb{Z}/N\mathbb{Z})$-action on $\mathcal{C}_N(-)$ by

$$Rhom((-)^{\boxtimes N}; \sigma_* \mathscr{T}^{\boxtimes N}) \xrightarrow{(\sigma^2)^{-1}} Rhom((\sigma^2)^{-1}(-)^{\boxtimes N}; (\sigma^2)^{-1}\sigma_* \mathscr{T}^{\boxtimes N})$$

$$\cong Rhom((-)^{\boxtimes N}; (\sigma^2)_*(\sigma^2)^{-1}\sigma_* \mathscr{T}^{\boxtimes N}) = Rhom((-)^{\boxtimes N}; \sigma_* \mathscr{T}^{\boxtimes N})$$

$\Rightarrow \mathcal{C}_N(-)$ becomes an object of $D(\mathbb{K}[\mathbb{Z}/N\mathbb{Z}]\text{-}\mathtt{mod})$.

Hamiltonian Loop Space



Each configuration $\{\mathbf{q}_{j,k}\}$ gives rise to a Hamiltonian broken loop trajectory in E with NM nodes, as shown on the left $(M = N = 3)$.

We have some conditions regarding action integrals on those loops. This allows us to compute cohomology with compact support of such loop space.

Ext Algebra $Ext^{\bullet}_{\mathbb{K}[\mathbb{Z}/N\mathbb{Z}]}(\mathbb{K},\mathbb{K})$

$$\begin{array}{ccccc} \mathcal{C}_N(\Phi(B_R \times \mathbb{S}^1)) & \xleftarrow{j^*} & \mathcal{C}_N(B_r \times \mathbb{S}^1) & \xleftarrow{{j_r}^*} & \mathcal{C}_N(\mathbb{R}^{2n} \times \mathbb{S}^1) \\ \downarrow \cong & & & \swarrow {j_R}^* & \\ \mathcal{C}_N(B_R \times \mathbb{S}^1) & & & & \end{array}$$

- N =prime allows us to compute each $\mathcal{C}_N(-)$ in a $\mathbb{Z}/N\mathbb{Z}$-equivariant way.
- We can show that both ${j_R}^*$ and ${j_r}^*$ are nonzero in $Ext^{\bullet}_{\mathbb{K}[\mathbb{Z}/N\mathbb{Z}]}(\mathbb{K},\mathbb{K})$.
- However, when N is large enough $\Rightarrow$ $deg({j_R}^*) < deg({j_r}^*)$, which leads to a contradiction.

THANK YOU