Riemann-Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules

(joint work with Masaki KASHIWARA)

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The classical RH problem

Hilbert's 21st problem (1900)

> *"A problem that Riemann himself may have in mind"*
>
> *"To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group"*

Fuchsian ODEs

$$P(z,\partial_z) = a_m(z)\partial_z^m + \cdots + a_1(z)\partial_z + a_0(z), \quad a_j \in \mathcal{O}_{\mathbb{C}}$$

Definition

$z_0 \in \mathbb{C}$ Fuchsian singularity:

- $a_m(z_0) = 0$,
- $ord_{z=z_0} a_m - m \leq ord_{z=z_0} a_j - j \quad \forall j$

Basis of m local solutions at z_0 of the form:

$$u(z) = (z - z_0)^\lambda v(z) + (\text{log terms}), \quad \lambda \in \mathbb{C},\ v \in \mathcal{O}_{\mathbb{C},z_0}$$

$\lambda \leftrightsquigarrow$ monodromy

Corollary

$\{u \in \mathcal{O}_{\mathbb{C}}\,;\ Pu = 0\}$ *is a local system outside of the singular points*

$\mathcal{D}$-modules

- X: complex manifold
- $\mathcal{D}_X$: sheaf of linear differential operators

Definition

$\mathcal{M}$ a $\mathcal{D}_X$-module $\rightsquigarrow \mathcal{S}ol(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$

Example

$P \in \mathcal{D}_X \rightsquigarrow \mathcal{M} = \mathcal{D}_X/\mathcal{D}_X P$

$$H^0\mathcal{S}ol(\mathcal{M}) = \{u \in \mathcal{O}_X;\ Pu = 0\}$$

- holonomic $\mathcal{D}_X$-module $\leftrightsquigarrow$ ODE
- regular holonomic $\mathcal{D}_X$-module $\leftrightsquigarrow$ Fuchsian ODE
- $\mathbb{C}$-constructible sheaf $\leftrightsquigarrow$ local system

Regular RH correspondence

- $\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)$: bounded derived category of $\mathcal{D}_X$-modules
- $\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X)$: bounded derived category of sheaves of $\mathbb{C}$-vector spaces

Theorem (Kashiwara 1984)

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow{\mathcal{S}ol} & \mathrm{D}^{\mathrm{b}}(\mathbb{C}_X) \\
\uparrow & & \uparrow \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \searrow & \\
\uparrow & & \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow[\sim]{} & \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)
\end{array}$$

analysis *topology*

There is also an explicit reconstruction functor:

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X) \ni \mathcal{M} \rightsquigarrow F = \mathcal{S}ol(\mathcal{M}) \in \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X)$$
$$\rightsquigarrow \mathrm{R}\mathcal{H}om(F, \mathcal{O}^{\mathrm{t}}_X) \simeq \mathcal{M}$$

Subanalytic sheaves [Kashiwara-Schapira 2001]

- X_{sa} the subanalytic site:
  - open subanalytic subsets of X
  - locally finite covers
- $\mathrm{Mod}(\mathbb{C}_{X_{\mathrm{sa}}})$ subanalytic sheaves ($\rightsquigarrow$ ind-sheaves)
- Tempered distributions:

$$\mathcal{D}b^{\mathrm{t}}(U) = \mathrm{image}\big(\mathcal{D}b_X(X) \to \mathcal{D}b_X(U)\big)$$

- $\mathcal{O}_X^t$ = Dolbeault complex with coefficients in $\mathcal{D}b_X^{\mathrm{t}}$
- $\mathcal{S}ol^{\mathrm{t}}(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^t)$

Example

$$\mathcal{E}_{\mathbb{C}}^{1/z} = \mathcal{D}_{\mathbb{C}} e^{1/z} = \mathcal{D}_{\mathbb{C}}/\mathcal{D}_{\mathbb{C}} P, \quad P(z,\partial_z) = z^2\partial_z - 1 \text{ not Fuchsian}$$

$$H^0 \mathcal{S}ol^{\mathrm{t}}(\mathcal{E}_{\mathbb{C}}^{1/z}) = \text{“}\varinjlim_{c \to +\infty}\text{”}\ \mathbb{C}_{\{\mathrm{Re}(1/z) < c\}}$$

Caveat: $\mathcal{S}ol^{\mathrm{t}}(\mathcal{E}_{\mathbb{C}}^{1/z}) \simeq \mathcal{S}ol^{\mathrm{t}}(\mathcal{E}_{\mathbb{C}}^{2/z})$

Irregular ODEs

$P(z, \partial_z) = a_m(z)\partial_z^m + \cdots + a_0(z), \quad z_0 \in \mathbb{C}$ not Fuchsian

- For $w = (z - z_0)^{1/r}$, basis of m formal solutions:

$$\widehat{u}(w) = e^{\varphi(w)} w^\lambda \widehat{v}(w) + (\text{log terms}), \quad \varphi \in \mathbb{C}[w^{-1}],\ \lambda \in \mathbb{C},\ v \in \widehat{\mathcal{O}}_{\mathbb{C},z_0}$$

- $\forall$ direction θ, $\exists$ analytic solution u with $u \sim \widehat{u}$ on a sector S

Caveat: $u + u_1 \sim \widehat{u}$ if $\operatorname{Re}\varphi_1 < \operatorname{Re}\varphi$ at $\theta \rightsquigarrow$ Stokes phenomenon

Theorem ([Deligne] and [Malgrange] in the 80s)

Irregular RH in dimension one, for fixed singular locus

Idea: order as above the exponents, so that u is well defined in the graded part

Caveat: difficult to extend in higher dimensions, cf [Sabbah 2013]

Enhanced sheaves

- $X \times \mathbb{R}_\infty$ = bordered subanalytic site
  - open subanalytic subsets of $X \times \mathbb{P}^1(\mathbb{R})$ included in $X \times \mathbb{R}$
  - locally finite covers

Definition (influenced by [Tamarkin 2008])

$\mathrm{E}^{\mathrm{b}}(\mathbb{C}_X) = \mathrm{D}^{\mathrm{b}}(\mathbb{C}_{X\times\mathbb{R}_\infty})/\{K\colon K \simeq \pi^{-1}\mathrm{R}\pi_* K\}$ $\qquad \pi\colon X \times \mathbb{R}_\infty \to X_{\mathrm{sa}}$

It is a commutative tensor category

- $K_1 \overset{+}{\otimes} K_2 = \mathrm{R}\mu_{!!}(q_1^{-1}K_1 \otimes q_2^{-1}K_2)$ $\qquad$ convolution
- unit: $\mathbb{C}_{\{t=0\}} = \mathbb{C}_{X\times\{0\}}$

Six operations: $\overset{+}{\otimes}$, $\mathcal{I}hom^+$, $\mathrm{R}f_*$, $\mathrm{R}f_{!!}$, f^{-1}, $f^!$

Lemma

$\mathrm{D}^{\mathrm{b}}(\mathbb{C}_X) \hookrightarrow \mathrm{E}^{\mathrm{b}}(\mathbb{C}_X)$ $\qquad F \mapsto \mathbb{C}_X^{\mathrm{E}} \otimes \pi^{-1}F,\quad \mathbb{C}_X^{\mathrm{E}} = \underset{c\to+\infty}{\text{“}\varinjlim\text{”}}\, \mathbb{C}_{\{t\geq c\}}$

Reconstruction of exponential modules

- $\mathcal{D}b^{\mathsf{E}}_X = \big(\mathcal{D}b^{\mathsf{t}}_{X\times\mathbb{R}_\infty} \xrightarrow{\partial_t - 1} \underset{0}{\mathcal{D}b^{\mathsf{t}}_{X\times\mathbb{R}_\infty}}\big) = \mathcal{H}om_{\mathcal{D}_{\mathbb{R}_\infty}}(\mathcal{E}^t_{\mathbb{R}_\infty}, \mathcal{D}b^{\mathsf{t}}_{X\times\mathbb{R}_\infty})[1]$
- $\mathcal{O}^{\mathsf{E}}_X =$ (Dolbeault complex with coefficients in $\mathcal{D}b^{\mathsf{E}}_X$)[1]
- $\mathcal{S}ol^{\mathsf{E}}(\mathcal{M}) = \mathrm{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}^{\mathsf{E}}_X) \sim \mathcal{S}ol^{\mathsf{t}}(\mathcal{M} \boxtimes \mathcal{E}^t_{\mathbb{R}_\infty})[1]$

Theorem

$$\mathcal{S}ol^{\mathsf{E}}(\mathcal{E}^{\varphi}_X) \simeq \underset{c\to+\infty}{\text{“}\varinjlim\text{”}}\ \mathbb{C}_{\{t+\mathrm{Re}\,\varphi(z)\geq c\}} \qquad \mathcal{E}^{\varphi}_X = \mathcal{D}_X e^{\varphi}(*D),\ \varphi \in \mathcal{O}_X(*D)$$

Generalizes the example of $\mathcal{E}^{1/z}_{\mathbb{C}}$

Theorem

$$\mathcal{E}^{\varphi}_X \simeq \mathrm{R}\mathcal{H}om\big(\underset{c\to+\infty}{\text{“}\varinjlim\text{”}}\ \mathbb{C}_{\{t+\mathrm{Re}\,\varphi(z)\geq c\}}, \mathcal{O}^{\mathsf{E}}_X\big), \qquad \mathrm{R}\mathcal{H}om \sim \mathrm{R}\pi_*\mathcal{I}hom^+$$

Related to [D’A 2013]

Structure of holonomic modules

Key result from [Mochizuki 2011] and [Kedlaya 2011]

Lemma

A statement $Q_X(\mathcal{M})$ is true for any $\mathcal{M} \in \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)$ and any X if:

- $Q_X(\mathcal{M}) \iff Q_{U_i}(\mathcal{M}|_{U_i})\ \forall i \in I$, *for* $X = \bigcup_{i\in I} U_i$ *an open cover.*
- $Q_X(\mathcal{M}) \implies Q_X(\mathcal{M}[n])\ \forall n \in \mathbb{Z}$.
- $Q_X(\mathcal{M}')\&Q_X(\mathcal{M}'') \implies Q_X(\mathcal{M})$, *for* $\mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \xrightarrow{+1}$ *a d.t.*
- $Q_X(\mathcal{M} \oplus \mathcal{M}') \implies Q_X(\mathcal{M})$.
- $Q_X(\mathcal{M}) \implies Q_Y(\mathrm{D}f_*\mathcal{M})$, *for* $f\colon X \to Y$ *projective.*
- $Q_X(\mathcal{M})$ *holds for* $\mathcal{M}$ *with a normal form along a n.c. divisor.*

- $\mathcal{M}$ has normal form if it is a direct sum of exponential $\mathcal{D}$-modules on polysectors along the divisor.

RH correspondence

Theorem

- $\mathcal{S}ol^{\mathsf{E}} \colon \mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} \to \mathsf{E}^{\mathrm{b}}(\mathbb{C}_X)$ *is fully faithful*
- $\mathcal{S}ol^{\mathsf{E}}(\mathcal{M})$ *is* $\mathbb{R}$*-constructible*
- *Reconstruction holds:*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X) \ni \mathcal{M} \rightsquigarrow K = \mathcal{S}ol^{\mathsf{E}}(\mathcal{M}) \in \mathsf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X)$$
$$\rightsquigarrow R\mathcal{H}om(K, \mathcal{O}^{\mathsf{E}}_X) \simeq \mathcal{M}$$

- *Compatibility with the regular case:*

$$\begin{array}{ccccc}
\mathrm{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_X)^{\mathrm{op}} & \xhookrightarrow{\mathcal{S}ol^{\mathsf{E}}} & \mathsf{E}^{\mathrm{b}}_{\mathbb{R}\text{-c}}(\mathbb{C}_X) & \xrightarrow{R\mathcal{H}om(*, \mathcal{O}^{\mathsf{E}}_X)} & \mathrm{D}^{\mathrm{b}}(\mathcal{D}_X)^{\mathrm{op}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}} & \xrightarrow[\sim]{\mathcal{S}ol^{\mathrm{t}}} & \mathrm{D}^{\mathrm{b}}_{\mathbb{C}\text{-c}}(\mathbb{C}_X) & \xrightarrow[\sim]{R\mathcal{H}om(*, \mathcal{O}^{\mathrm{t}}_X)} & \mathrm{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)^{\mathrm{op}}
\end{array}$$

Stokes phenomenon

- $\varphi_1, \varphi_2 \in \mathcal{O}_{\mathbb{C}}(*0) \rightsquigarrow K_j = \mathcal{S}ol^{\mathsf{E}}(\mathcal{E}_{\mathbb{C}}^{\varphi_j}) \simeq \underset{c \to +\infty}{\text{“}\varinjlim\text{”}} \mathbb{C}_{\{t + \mathrm{Re}\,\varphi_j(z) \geq c\}}$
- $\mathcal{M}$ flat meromorphic connection
- $\forall \theta \colon \mathcal{M} \sim_\theta \mathcal{E}_{\mathbb{C}}^{\varphi_1} \oplus \mathcal{E}_{\mathbb{C}}^{\varphi_2} \rightsquigarrow K = \mathcal{S}ol^{\mathsf{E}}(\mathcal{M}) \sim_\theta K_1 \oplus K_2$
- $\{\mathrm{Re}(\varphi_1 - \varphi_2) = 0\} = \sqcup \ell_n \rightsquigarrow L_n$ Stokes lines

Lemma

S an open sector

$$\mathrm{End}_{\mathsf{E}^{\mathrm{b}}(\mathbb{C}_{\mathbb{C}})}(\pi^{-1}\mathbb{C}_S \otimes (K_1 \oplus K_2)) \simeq \begin{cases} \mathfrak{b}^{\pm}, & S \subset \{\pm \mathrm{Re}(\varphi_1 - \varphi_2) > 0\} \\ \mathfrak{t}, & S \supset L_{n_0},\ S \cap L_n = \varnothing,\ n \neq n_0 \end{cases}$$

($\mathfrak{b}^{\pm}$ upper/lower triangular in $M_2(\mathbb{C})$, $\mathfrak{t} = \mathfrak{b}^+ \cap \mathfrak{b}^-$)