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Special Kähler geometry on the Hitchin base

joint w/ U. Bruzzo

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- In progress

Goal: Understand: • the special Kähler (SK) geometry for the (generalised) Hitchin system
• Behaviour under Langlands duality for $\mathfrak{g} \neq B_n, C_n$

① Review of SK geometry

$(B, \omega) \equiv (\underbrace{M, I}_{B}, \omega)$ - Kähler mfld

def A special Kähler (SK) structure is a connection ∇ on $T_M \equiv T_{B,\mathbb{R}}$

which is

1) flat 2) symplectic 3) torsion-free 4) special

$(d^\nabla)^2 = 0$ $\nabla \omega = 0$ $d^\nabla(\mathbb{1}) = 0$ $d^\nabla I = 0$

Note:

$\underline{\mathrm{End}}\, T_M \simeq \mathcal{A}^1_{\mathbb{R}}(T_M)$ and

$$\begin{array}{ccc} \underline{\mathrm{End}}\, T_M & \xrightarrow{\tilde{\nabla} = \nabla^* \otimes 1 + 1 \otimes \nabla} & \mathcal{A}^1_{\mathbb{R}}(\mathrm{End}\, T_M) \simeq \mathcal{A}^{1\,\otimes 2}_{\mathbb{R}} \otimes T_M \\ \wr\!\!\mid & \not\circlearrowleft & \downarrow \mathrm{Alt} \\ \mathcal{A}^1_{\mathbb{R}}(T_M) & \xrightarrow{d^\nabla} & \mathcal{A}^2_{\mathbb{R}}(T_M) \end{array}$$

$$\mathrm{Alt} \circ \tilde{\nabla} = d^\nabla - \underbrace{d^\nabla(\mathbb{1})}_{T^\nabla}$$

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In terms of $\pi^{1,0} = \frac{1}{2}(1 - iI) : T_{B,\mathbb{C}} \simeq T_B^{1,0} \oplus T_B^{0,1} \to T_B^{1,0}$

conditions 3) & 4) say:

$$d^\nabla(\pi^{1,0}) = 0 \iff \left| \begin{array}{l} d^\nabla 1 = 0 \\ d^\nabla I = 0 \end{array} \right. \iff \left| \begin{array}{l} d^\nabla 1 = 0 \\ \nabla I \in \Gamma(\mathrm{Sym}^2 T_M^\vee \otimes T_M) \end{array} \right.$$

In particular:

3), 4) $\not\Longrightarrow$ $\nabla I = 0$ and $\nabla \neq$ Levi-Civita connection on (M, I, ω)

Observation 1 (Hertling)

$(\underbrace{M, I}_{B}, \omega, \nabla)$ – SK $\longleftrightarrow$ weight one, polarised RVHS on $T_{B,\mathbb{C}}$

$$(\mathcal{F}^1 \subset \mathcal{F}^0, \mathcal{F}_{\mathbb{R}} \subset \mathcal{F}^0, \nabla^{GM}, Q) = (T_B \subset \underbrace{(T_{B,\mathbb{C}}, \nabla^{1,0})}_{\mathcal{F}^0}, T_M, \nabla^{0,1}, \omega)$$

NB $\quad 0 \to T_B \to \mathcal{F}^0 \to \Omega^1_B \to 0$

$\left(T_{B,\mathbb{C}} \simeq T^{1,0} \oplus T^{0,1} \text{ - c.v.b.} \qquad T_B \subset T^{1,0} = T_B \otimes \mathcal{C}^\infty \right)$

If the RVHS can be refined to a $\mathbb{Z}$VHS $\quad \mathcal{F}_{\mathbb{Z}} \subset T_M$

$\Rightarrow$

$$\mathcal{H} = \mathcal{F}^{1\vee} / \hat{\mathcal{F}}_{\mathbb{Z}} = T_B^\vee / \hat{\mathcal{F}}_{\mathbb{Z}} \longrightarrow B \qquad \text{family of ab. varieties}$$

Caution $\quad \hat{\mathcal{F}}_{\mathbb{Z}} \subset T_B^\vee$ need not be Lagrangian

so $\mathcal{H}$ need not be holo symplectic

$\mathcal{H} \to B$ need not be Lagr. fibration

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def

An <u>integral</u> SK structure is SK str. for which $\hat{\mathcal{F}}_{\mathbb{Z}} \subset T_B^\vee$ is Lagrangian
$(M, I, \mathcal{F}_{\mathbb{Z}}, \nabla, \omega)$

<u>Observation 2</u> (Donagi-Witten)

$(M, I, \mathcal{F}_{\mathbb{Z}}, \nabla, \omega)$ - integral SK $\leftrightsquigarrow$ ACIHS $\mathcal{H} \to B$ w/ section

(further: ~~ACIHS~~ ~~B~~ ~~covers~~ ~~of~~ ~~B~~)

$$\begin{array}{c} \mathcal{H} \\ \downarrow \\ B \end{array} \leadsto (Alb_{\mathcal{H}/B} \to B)$$

<u>Observation 3</u> (Simpson)

To $(\mathcal{F}^\circ, \mathcal{F}_{\mathbb{R}}, \nabla^{GM}, Q)$ we can associate a Higgs bundle

$$(E, \vartheta) \quad \left| \begin{array}{l} E = \mathcal{F}^1 \oplus \mathcal{F}^{1\vee} \\ \vartheta = gr\, \nabla^{GM} : \mathcal{F}^1 \to \mathcal{F}^\circ/\mathcal{F}^1 \simeq \overline{\mathcal{F}^1} \underset{Q}{\simeq} \mathcal{F}^{1\vee} \end{array} \right.$$

In our case $E = T_B \oplus \Omega^1_B$

$$\vartheta = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, \quad c \in H^0(B, \Omega_B^{1 \otimes 3})$$

<u>Moreover:</u> (Integral case)

1) $c = d\underset{\uparrow}{\Phi} \in H^0(B, \Omega^1_B \otimes \underset{\uparrow}{Sym^2} \Omega^1_B)$

derivative of period map of $T_B^\vee/\hat{\mathcal{F}}_{\mathbb{Z}}$ Riemann's conditions

2) Donagi-Markman ('96) $\Rightarrow$ $c \in H^0(Sym^3 \Omega^1_B)$

3) (BCOV, Simpson) $\quad \nabla = \nabla^{LC} + \vartheta + \vartheta^*$

$$c = -\omega(\pi^{1,0}, \nabla \pi^{1,0})$$

4) "Donagi-Markman cubic", "Yukawa cubic"

<u>Locally on ~~the~~ B:</u> $c = d(\mathrm{Hess} F)$, $F : \underset{\overset{\cap}{B}}{U} \to \mathbb{C}$ "holo pre-potential"

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(2) The cubic for the generalised Hitchin system

Fix : X - smooth curve / $\mathbb{C}$

$D \geq 0$ divisor , $K(D)^2$ - very ample $\qquad L := K(D)$

$$\underset{\text{Cartan}}{T} \subset \underset{\text{Borel}}{B} \subset G \leftarrow \text{cx. semi-simple Lie grp}$$

Fact 1

$\exists$ coarse moduli space $\mathrm{Higgs}_{G,D} = \coprod_{c \in \pi_1(G)} \mathrm{Higgs}_{G,D,c}$

(semi-stable L-valued G-Higgs bundles (E,θ))

Fact 2 (Bottacin, Markman; Hurtubise; Hitchin, Beauville, Faltings, Donagi)

$\mathrm{Higgs}_{G,D,c}$ - holo Poisson and is an AC IHS via the Hitchin map

$$h : \mathrm{Higgs}_{G,D,c} \longrightarrow \mathcal{B} = H^0(X, \mathfrak{t} \otimes \mathbb{L} / W) \simeq H^0(\bigoplus_i L^{d_i})$$

$$(E, \theta) \longmapsto \text{invariants of } \theta \qquad d_i = \text{exponents of } G$$

Fact 3 (Markman)

Let $o \in \mathcal{B}$ - generic . The closure of the symplectic leaf through it is

$$\overline{S} = h^{-1}\left(\{o\} + H^0\left(\bigoplus_i L^{d_i}(-D)\right)\right) \xrightarrow{h} \left(\{o\} + H^0(\bigoplus L^{d_i}(-D))\right) \subset \mathcal{B}.$$

Restrict to generic locus

$$h^{-1}(\mathbb{B}) = S|_{\mathbb{B}} \xrightarrow{h} \mathbb{B} = \left(\{o\} + H^0(L^{d_i}(-D))\right) \cap (\mathcal{B} \setminus \mathfrak{D})$$

$\uparrow$ discr.

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Thm. (U. Bruzzo, R.D.)

Let $\pi_o : \tilde{X}_o \to X$ be the cameral cover, corresponding to $o \in B$.
Then the DM cubic at o is

$$c_o : H^0(\tilde{X}_o, \mathfrak{t} \otimes K_{\tilde{X}_o})^W \longrightarrow \mathrm{Sym}^2\left(H^0(\tilde{X}_o, \mathfrak{t} \otimes K_{\tilde{X}_o})^{W\,\vee}\right)$$

$$c_o(\xi)(\eta, \zeta) = \frac{1}{2} \sum_{p \in \mathrm{Ram}(\pi_o)} \mathrm{Res}_p^2 \left(\pi_o^* \left(\frac{\mathcal{L}_{v_\xi} \mathcal{D}}{\mathcal{D}} \Big|_{\{o\}\times X} \right) \eta \cdot \xi \right)$$

where
- $\mathcal{D} : B \times X \to \mathrm{tot}\, L^{|R|}$ discriminant
- v_ξ is the preimage of ξ under $T_{B,o} \simeq H^0(\tilde{X}_o, \mathfrak{t} \otimes K_{\tilde{X}_o})^W$

Explanation / Idea of pf:

$\chi : \mathfrak{t} \to \mathfrak{t}/W$: W-cover, ramified over $Z(\mathcal{D} = \prod_{\alpha \in R} \alpha)$

$\mathbb{C}[\mathfrak{t}]^W \subset \mathbb{C}[\mathfrak{t}]$

non-can. $\cong$

$\mathbb{C}[I_1, \dots, I_\ell]$, I_k-homog., $\deg I_k = d_k$

$\ell = \mathrm{rk}\, \mathfrak{g} = \dim \mathfrak{t}$

Twist χ by L :

$$\begin{tikzcd}
 & & \mathrm{tot}(L^{\oplus \ell}) & \\
\tilde{X}_o \arrow[r, hook] \arrow[d, "\pi_o"] & \mathcal{X} \arrow[r] \arrow[d, "\pi"'] & \mathrm{tot}\, \mathfrak{t} \otimes_{\mathbb{C}} L \arrow[u, phantom, "\cong"] \arrow[d, "W\text{-cov.}"', "p"] \arrow[dr, "\tilde{\mathcal{D}}"] & \\
\{o\} \times X \arrow[r, hook] & B \times X \arrow[r, "ev"] \arrow[dr] & \mathrm{tot}\, \mathfrak{t} \otimes_{\mathbb{C}} L/W \arrow[r, "\mathcal{D}"'] \arrow[d, "q"'] \arrow[dr, phantom, "\cong"] & q^* L^{|R|} \\
 & & X & \mathrm{tot}(\oplus_i L^{d_i})
\end{tikzcd}$$

- $\mathrm{tot}(\mathfrak{t} \otimes L)$ carries mero $\mathfrak{t}$-valued 2-form $\omega_{\mathfrak{t}} \in H^0(\Omega^2(r^* D) \otimes \mathfrak{t})^W$

- Relate the inf. period map for $h : S|_B \to B$ w/ Kodaira-Spencer map for $\tilde{X}_o \to \{o\} \times X$, which is computable

$$\begin{matrix} \tilde{X}_o & \longrightarrow & \{o\} \times X \\ \cap & & \cap \\ \mathcal{X}|_B & \longrightarrow & B \times X \end{matrix}$$

Remark

For $D=0$ this gives the formula of Balduzzi & Pantev

③ Further question (in progress)

$D=0$, $G=$ simple, $\mathfrak{g} \neq B_n, C_n$

Thm A (0604617 / Donagi–Pantev) $\Rightarrow$ $\exists$ automorphisms ℓ^{base}, ℓ^{cam}

$$\begin{array}{ccc} \mathcal{X}_G & \xrightarrow{\ell^{cam}} & \mathcal{X}_{{}^LG} \\ \downarrow & & \downarrow \\ B & \xrightarrow{\ell^{base}} & B \end{array}$$

preserving $\mathcal{D}$

such that for $b \in B \setminus \mathcal{D}$ ℓ^{cam} induces isom of polarised ab. var.

$$P_b \simeq \left({}^L P_{\ell^{base}(b)}\right)^D$$

where $P_b =$ Hitchin fibre

Ex. $G = G_2$ $\qquad B \simeq H^0(K^2) \oplus H^0(K^6)$

$\ell^{base}: (a,b) \mapsto (a, b - \frac{1}{54} a^3)$

Q: Is ℓ^{base} an isometry of the SK structure?

(Expect "yes")

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Appendix

Coordinate description of the SK condition

$\exists$ Flat Darboux coordinates $\{x_i, y_i\}$ $\quad \nabla(dx_i)=0$

$\nabla(dy_i)=0$

$$\omega = \sum dx_i \wedge dy_i$$

$$d^\nabla \pi^{1,0} = 0 \Rightarrow \pi^{1,0} = \nabla \zeta \text{ locally} \quad \zeta \in \Gamma(U, T_{B,\mathbb{C}})$$

$$\zeta = \frac{1}{2} \sum_i z_i \otimes \frac{\partial}{\partial x_i} - w_i \otimes \frac{\partial}{\partial y_i}, \quad z_i, w_i - \text{holomorphic}$$

$$\mathrm{Re}(dz_i) = dx_i, \quad \mathrm{Re}(dw_i) = -dy_i$$

$$\frac{\partial}{\partial z_i} = \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sum_j \tau_{ij} \frac{\partial}{\partial y_j}\right), \text{ where } dw_i = \sum \tau_{ji}\, dz_j$$

$$\left(\tau_{ji} = \frac{\partial w_i}{\partial z_j}\right)$$

$$\omega \text{ of type } (1,1) \Rightarrow \tau_{ij} = \tau_{ji} = \frac{\partial^2 \mathcal{F}}{\partial z_i \partial z_j}$$

$$\mathcal{F}: U \to \mathbb{C}, \text{ holo}$$

$$U \subset \mathcal{B}$$

$$\omega - \text{positive} \Rightarrow \mathrm{Im}\,\tau > 0$$

Kähler potential: $\omega = i \partial\bar\partial K = \frac{i}{2} \sum \mathrm{Im}\,\tau_{jk}\, dz_j \wedge d\bar z_k$

$$K = \frac{1}{2} \mathrm{Im}\left(\sum w_j \bar z_j\right)$$

Cx. Str:

$$\pi^{1,0} = \sum_i dz_i \otimes \frac{\partial}{\partial z_i} = \frac{1}{2}\left(\sum_i dz_i \otimes \frac{\partial}{\partial x_i} - dw_i \otimes \frac{\partial}{\partial y_i}\right)$$

$$I = i(2\pi^{1,0} - \mathbb{1}) = -\sum_j \mathrm{Im}(dz_j) \otimes \frac{\partial}{\partial x_j} + \sum_j \mathrm{Im}\, dw_j \otimes \frac{\partial}{\partial y_j}$$

NB $w = iz$, $\mathcal{F} = iz^2/2$ gives $\mathbb{C}$