A period map for Global derived stacks
(Pedova 2014) 1

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X complex smooth projective variety

Consider the cohomology algebra

$$H^*(X,\mathbb{C}) := \bigoplus_{0\le k\le d} H^k(X,\mathbb{C})$$

and recall that each cohomology group $H^k(X,\mathbb{C})$ is endowed wit a Hodge structure of weight k, given by the decomposition

$$(*)\quad H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X) \qquad \text{where} \quad H^{p,q}(X) \cong H^q(X,\Omega$$

or equivalently by the Hodge filtration $(H^k(X,\mathbb{C}), F^\bullet)$

$$0 = F^{k+1} H^k(X,\mathbb{C}) \hookrightarrow F^k H^k(X,\mathbb{C}) \hookrightarrow F^{k-1} H^k(X,\mathbb{C}) \hookrightarrow \cdots \hookrightarrow F^1 H^k(X,\mathbb{C}) \hookrightarrow F^0 H^k(X,\mathbb{C}) =$$

where

$$F^m H^k(X,\mathbb{C}) := \bigoplus_{p\ge m} H^{p,q}(X)$$

Notation:

$$h^k := \dim H^k(X,\mathbb{C}) \qquad h^{p,q} := \dim H^{p,q}(X) \text{ (= Hodge numbers}$$

$$b^{p,k} := \dim F^p H^k(X,\mathbb{C})$$

Now consider a family of deformations of X

$$\begin{array}{c} \mathcal{X} \\ \varphi \downarrow \\ S \end{array}$$

φ proper holomorphic submersion over S contractib with a distinguished fibre

$$\varphi^{-1}(0) =: X_0 \cong X \quad (X_t := \varphi^{-1}(t))$$

?? Question 1: How the standard Hodge structures $(*)$ over X vary with respect to the family φ?

* Answer was originally given by Griffiths.

Classical Facts in Hodge Theory

(1) $\begin{array}{c}\mathcal{X}\\ \varphi\downarrow\\ S\end{array}$ family of deformations $\Rightarrow$ all fibres X_t are diffeomorphic to one another, and so they are diffeomorphic to X

$\Rightarrow$ we can view φ as a family of complex structures on X paramitrized by points in S.

(2) $\begin{array}{c}\mathcal{X}\\ \varphi\downarrow\\ S\end{array} \rightsquigarrow \varphi^*: Sh(\mathcal{X}) \longrightarrow Sh(S)$ push-forward functor

We can see that $\mathbb{R}^k\varphi^*\underline{\mathbb{C}}$ is a local system over S isomorphic to $\underline{H^k(X,\mathbb{C})}$

$\Rightarrow$ Chain of canonical isomorphisms (of vector spaces)

$$H^k(\mathcal{X},\mathbb{C}) \cong H^k(X_t,\mathbb{C}) \cong H^k(X,\mathbb{C}) \quad \forall t\in S$$

(3) E_1-degeneration of Hodge-to-De Rham (morally "$\partial\bar\partial$-lemma") $\Rightarrow$ $\exists$ a neighborhood of $0\in S$ s.t. $\dim H^{p,q}(X_t) = h^{p,q}$

$\Rightarrow$ Hodge numbers are constant under infinitesimal deformation.

Definition (Griffiths): Define the (p,k)-th local period map (associated to the family $\begin{array}{c}\mathcal{X}\\ \varphi\downarrow\\ S\end{array}$) to be

$$\mathcal{P}^{p,k}: S \longrightarrow \mathrm{Grass}(b^{p,k}, H^k(X,\mathbb{C}))$$
$$t \longmapsto F^pH^k(X_t,\mathbb{C})$$

RmK: by Fact (2) and Fact (3) $\mathcal{P}^{p,k}$ is well-defined.

Theorem (Griffiths):

(1) The period map is holomorphic $\forall p \leq k$

(2) The differential $d\mathcal{P}^{p,k}$ is the same as the map

$$H^1(X, T_X) \longrightarrow \mathrm{Hom}\left(F^p H^k(X,\mathbb{C}), \frac{H^k(X,\mathbb{C})}{F^p H^k(X,\mathbb{C})}\right)$$

(3) The differential $d\mathcal{P}^{p,k}$ actually takes values in

$$\mathrm{Hom}\left(F^p H^k(X,\mathbb{C}), \frac{F^{p-1} H^k(X,\mathbb{C})}{F^p H^k(X,\mathbb{C})}\right)$$

(this property is called Griffiths transversality)

Upshot: (1) Hodge structures of the fibres X_t vary holomorphically in t

(2) The differential $d\mathcal{P}^{p,k}$ does not depend on the special Kuranishi family $\varphi \overset{\mathcal{X}}{\underset{S}{\downarrow}}$ but only on cohomological invariants of X, which actually have a deformation-theoretic meaning.

~~(Fiorenza-Manetti): Describe the period map as a morphisme of ... deformation functors ... deformation theoretic framework.~~

?? Question 2: Is it possible to describe the period map as a morphism of deformation functors?

* Answer was originally given by Fiorenza-Manetti.
Need suitable candidates for domain and codomain of such a morphism of deformation functors; in particular we need

(a) functor parametrizing deformations of X

(b) functor parametrizing variations of Hodge structures of X

(a) Deformations of X (Easy part)

Defn: A deformation of X over $A \in \mathrm{Art}_{\mathbb{C}}$ is a cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\Box"] & \mathcal{X} \arrow[d, "p"] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

with i closed immersion and p flat and proper.

Consider the functor

$$\mathrm{Def}_X : \mathrm{Art}_{\mathbb{C}} \longrightarrow \mathrm{Set}$$
$$A \longmapsto \frac{\{\text{defs of } X \text{ over } A\}}{\text{isomorphism}}$$

Fact (Kodaira, Kuranishi, Spencer, [...])

(1) Def_X is a (Schlessinger) deformation functor.

(2) $\mathrm{Def}_X \cong \mathrm{Def}_{KS_X}$ where

$$KS_X := \left(\Gamma(A^{0,*}(T_X), \bar{\partial}, [\ ]\right)$$

is the Kodaira-Spencer dgla.

(b) Variations of Hodge structures of X

Consider

$$\mathbb{R}\Gamma(X, \Omega^*_X) := \left(\Gamma(X, A^{**}_X), d\right) \quad \text{(derived global sections of the De Rham complex)}$$

$$F^p \mathbb{R}\Gamma(X, \Omega^*_X) \hookrightarrow \mathbb{R}\Gamma(X, \Omega^*_X)$$

Defn: A deformation of $F^p\mathbb{R}\Gamma(X, \Omega^*_X)$ inside $\mathbb{R}\Gamma(X, \Omega^*_X)$ over $A \in \mathrm{Art}_{\mathbb{C}}$ is

(a) a complex (V_A, d_A) of free A-modules whose residue modulo m_A is $\mathbb{R}\Gamma(X, \Omega^*_X)$

(b) $d_A(F^p\mathbb{R}\Gamma(X, \Omega^*_X)\otimes A) \subseteq F^p\mathbb{R}\Gamma(X, \Omega^*_X)\otimes A$

Consider the functor

$$\mathrm{Grass}_{F^p\mathbb{R}\Gamma,\,\mathbb{R}\Gamma} : \mathrm{Art}_{\mathbb{C}} \longrightarrow \mathrm{Set}$$
$$A \longmapsto \frac{\{\text{defs of } F^p\mathbb{R}\Gamma \text{ inside } \mathbb{R}\Gamma \text{ over } A\}}{\text{isomorphism}}$$

Now consider the dgla's

$\mathrm{End}^*(\mathbb{R}\Gamma(X,\Omega^*_X)) :=$ endomorphism algebra

$\mathrm{End}^{F^p}(\mathbb{R}\Gamma(X,\Omega^*_X)) := \{ f \in \mathrm{End}^*(\mathbb{R}\Gamma) \,/\, f(F^p\mathbb{R}\Gamma) \subseteq F^p\mathbb{R}\Gamma \}$

$$\chi : \mathrm{End}^{F^p} \hookrightarrow \mathrm{End}^* \qquad \text{(inclusion of dgla's)}$$

<u>Defn</u>: The mapping cone of χ is the homotopy cokernel

$$C_\chi := \underleftarrow{\mathrm{holim}} \left(\mathrm{End}^{F^p} \overset{\chi}{\underset{0}{\rightrightarrows}} \mathrm{End}^* \right)$$

<u>Fact</u>: C_χ is a L_∞-algebra (homotopy Lie algebra)

<u>Proposition</u> (Fiorenza - Manetti)

- $\mathrm{Grass}_{F^p\mathbb{R}\Gamma,\mathbb{R}\Gamma} \simeq \mathrm{Def}_{C_\chi}$

(In particular $\mathrm{Grass}_{F^p\mathbb{R}\Gamma,\mathbb{R}\Gamma}$ is a deformation functor)

In the same fashion we can define the functor

$$\mathrm{Grass}_{F^pH^*,H^*} : \mathrm{Art}_{\mathbb{C}} \to \mathrm{Set}$$

and

$$\mathrm{Grass}_{F^pH^*,H^*} \cong \underline{\mathrm{Def}}_{C_{\hat\chi}} \qquad \text{where } \hat\chi : \mathrm{End}^{F^p}(H^*(X)) \hookrightarrow \mathrm{End}^*(H^*(X))$$

<u>Fact</u>: The E_1-degeneracy of Hodge-to De Rham of X implies that

(a) There exists a homotopy equivalence of L_∞-algebras
$$h : C_{\chi_p} \longrightarrow C_{\hat\chi_p}$$

(b) $H^* : \mathrm{Grass}_{F^p\mathbb{R}\Gamma,\mathbb{R}\Gamma} \longrightarrow \mathrm{Grass}_{F^pH^*,H^*}$ is an isomorphism of def functors

Now what is the map?

We have a pair of morphisms (i, l)

$$i : \mathcal{T}_X \longrightarrow \mathcal{E}nd^*(\Omega^*_X)[-1]$$
$$\xi \longmapsto \xi \lrcorner - \quad \text{contraction map}$$

$$l : \mathcal{T}_X \longrightarrow \mathcal{E}nd^*(\Omega^*_X) \quad \text{Lie derivative}$$
$$\xi \longmapsto d i(\xi)$$

<u>Proposition</u> (Fiorenza–Manetti)

The linear map

$$fm^p : KS_X \longrightarrow C_{\chi_p}$$
$$\xi \longmapsto (l_\xi, i^p_\xi)$$

is a L_∞-morphism.

<u>Defn</u> (Fiorenza–Manetti)

The p^{th} algebraic Fiorenza–Manetti local period map is the composite

$$FM^p : \mathrm{Def}_{KS_X} \xrightarrow{fm^p} \mathrm{Def}_{C_{\chi_p}} \xrightarrow{h^{-1}} \mathrm{Def}_{C_{\hat{\chi}_p}}$$

<u>Defn</u> (Fiorenza–Manetti)

The p^{th} geometric Fiorenza–Manetti local period map is the composite

$$\mathcal{P}^p : \mathrm{Def}_X \longrightarrow \mathrm{Grass}_{F^pH^*, H^*}$$
$$(\mathcal{O}_A \xrightarrow{\varphi} \mathcal{O}_X) \longmapsto F^pH^*(X, \mathcal{O}_A)$$

<u>Thm</u>: (1) The diagram of Schlessinger def. functors

$$\begin{array}{ccc} \mathrm{Def}_{KS_X} & \xrightarrow{FM^p} & \mathrm{Def}_{C_{\hat{\chi}_p}} \\ \downarrow & & \downarrow \\ \mathrm{Def}_X & \xrightarrow{\mathcal{P}^p} & \mathrm{Grass}_{F^pH^*, H^*} \end{array}$$

commutes

(2) The tangent morphism to $\mathcal{P}^p$ is the same as the differential of Griffiths period map

?? <u>Question 3</u>: (a) Can we get a universal version (i.e. independent on the filtration parameter p) of the Fiorenza - Manetti - Martinengo period map?

(b) Can we lift such a map to the level of derived deformation functors?

* <u>Answer</u> was originally given by Fiorenza - Manetti - Martinengo in a <u>purely algebraic (i.e. Lie-theoretic)</u> context

The pair of linear maps of sheaves of dgla's

$$i : \mathcal{T}_X \longrightarrow \mathcal{E}nd^*(\Omega^*_X)[-1] \qquad \text{contraction map}$$

$$l : \mathcal{T}_X \longrightarrow \mathcal{E}nd^*(\Omega^*_X) \qquad \text{Lie derivative}$$

induces a pair of linear maps of dgla's

$$\tilde{i} : \mathbb{R}\Gamma(X, \mathcal{T}_X) \longrightarrow End^*(\mathbb{R}\Gamma(X, \Omega^*_X))[-1]$$

$$\tilde{l} : \mathbb{R}\Gamma(X, \mathcal{T}_X) \longrightarrow End^*(\mathbb{R}\Gamma(X, \Omega^*_X))$$

which is a <u>Cartan homotopy</u>, meaning that

$$\forall a, b \in \mathbb{R}\Gamma(X, \mathcal{T}_X) \approx KS_X \qquad \tilde{i}([a,b]) = [\tilde{i}(a), \tilde{l}(b)]$$

$$[\tilde{i}(a), \tilde{i}(b)] = 0$$

This follows from Cartan magic formulas.

<u>Rmk</u>: $(\tilde{i}, \tilde{l})$ Cartan homotopy $\Rightarrow e^{\tilde{i}}$ gives an homotopy (of morphisms of dgla's) between $\tilde{l}$ and 0.

Now consider

$$End^{\geq 0}(\mathbb{R}\Gamma(X, \Omega^*_{X/k})) \underset{\text{sub-dgla}}{\hookrightarrow} End^*(\mathbb{R}\Gamma(X, \Omega^*_{X/k}))$$

$End^{\geq 0}(\mathbb{R}\Gamma(X, \Omega^*_{X/k}))$ "=" dgla of filtration-preserving endomorphisms of $\mathbb{R}\Gamma(X, \Omega^*_{X/k})$

We get a diagram of dgla's

$$KS_X \cong \mathbb{R}\Gamma(X, T_X) \xrightarrow{\tilde{e}} End^{\geq 0}(\mathbb{R}\Gamma(X, \Omega^*_{X/k})) \underset{0}{\overset{incl}{\rightrightarrows}} End^*(\mathbb{R}\Gamma(X, \Omega^*_{X/k}))$$

inducing a morphism to the homotopy limit

$$KS_X \xrightarrow{(\tilde{e}, e^{\tilde{}})} \underleftarrow{holim}\left(End^{\geq 0}(\mathbb{R}\Gamma(X, \Omega^*_{X/k})) \underset{0}{\overset{incl}{\rightrightarrows}} End^*(\mathbb{R}\Gamma(X, \Omega^*_{X/k}))\right)$$

So we get a morphism of derived deformation functors just by taking the Hinich nerve of ~~each~~ such.

Recall that the Hinich nerve of $g \in DGLA_{\mathbb{C}}$ is the derived deformation functor

$$\mathbb{R}Def_g : dg\,Art_{\mathbb{C}}^{\leq 0} \longrightarrow sSet$$

$$A \longmapsto \mathbb{R}MC_g(A)$$

where $\mathbb{R}MC_g(A)_n = \{ x \in (g \otimes \Omega^*(\Delta^n) \otimes m_A)^1 \mid dx + \frac{1}{2}[x,x] = 0 \}$

<u>Defn</u>: The ~~algebraic~~ (universal) algebraic Fiorenza-Manetti-Martinengo (Fiorenza-Martinengo) ~~local~~ period map is the morphism of d.d.f.

$$FMM : \mathbb{R}Def_{KS_X} \longrightarrow \mathbb{R}Def_{\underleftarrow{holim}(\quad \rightrightarrows \quad)}$$

induced by the morphism $(\tilde{e}, e^{\tilde{}})$

Moreover one can see that ~~the dgla~~

$$\underleftarrow{holim}(\quad \rightrightarrows \quad)$$

is actually an abelian dgla.

?? <u>Question 4</u>: Can we get a more geometric interpretation of map FMM?

In order to answer that question we first need to define "geometric" versions of $\mathbb{R}Def_{KS_X}$ and $\mathbb{R}Def_{\underleftarrow{holim}(\rightrightarrows)}$.

From now on discussion will be less precise (rigorous defns would require more sophisticated tools)

Derived deformations of X

Defn: A derived deformation of X over $A \in dgArt_{\mathbb{C}}^{\leq 0}$ is a homotopy pull-back of derived schemes

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[d] \arrow[dr, phantom, "\square^h"] & \mathcal{X} \arrow[d, "p"] \\ \operatorname{Spec} k \arrow[r] & \mathbb{R}\operatorname{Spec} A \end{tikzcd}$$

where p is homotopy flat.

Equivalently: it is a (quasi-smooth surjective) morphism $\mathcal{O}_{A,X} \longrightarrow \mathcal{O}_X$ of presheaves of dgca's over A s.t. (a) $\mathcal{O}_{A,X}$ homotopy flat

(b) $\mathcal{O}_{A,X} \otimes_A^{\mathbb{L}} k \longrightarrow \mathcal{O}_A$ weak equivalence

Defn: The functor of derived defs of X "is" the d.d.f.

$$\mathbb{R}\mathrm{Def}_X : dgArt_{\mathbb{C}}^{\leq 0} \longrightarrow sSet$$

$$A \longmapsto \{\text{derived defs of } X \text{ over } A\}$$

1-morph $\varphi \longmapsto \{1\text{-morphisms}\}$

2-morph $\psi \longmapsto \{2\text{-morphisms}\}$

$\vdots \qquad \vdots$

Theorem (–): Functors $\mathbb{R}\mathrm{Def}_X$ and $\mathbb{R}\mathrm{Def}_{KS_X}$ are weakly equiv.

<u>Derived defs of $(\mathbb{R}\Gamma(X, \Omega^{\bullet}_X), F^{\cdot})$</u>

<u>Defn</u>: A derived def of $(\mathbb{R}\Gamma(X, \Omega^{\bullet}_X), F^{\cdot})$ over A is a pair $((V_A, F^{\cdot}_A), \varphi)$ where

(a) $(V_A, F^{\cdot}_A)$ is a filt (perfect) complex of A-modules

(b) $\varphi : (V_A, F^{\cdot}_A) \longrightarrow (\mathbb{R}\Gamma, F^{\cdot})$ is (surjective) morph. s.t. $F^p V_A \otimes^{\mathbb{L}}_A k \longrightarrow F^p \mathbb{R}\Gamma$ is a quasi-iso $\forall p$.

<u>Defn</u>: Define the functor of derived defs of ~~$\mathbb{R}\Gamma(X, \Omega^{\bullet}_X)$~~ $(\mathbb{R}\Gamma, F^{\cdot})$ to be

$$\mathrm{hoFlag}^{F}_{\Omega^{\bullet}_{X/k}} : dg\,\mathrm{Art}^{\leq 0}_{\varphi} \longrightarrow s\mathrm{Set}$$

$$A \longmapsto \{\text{der defs of } (\mathbb{R}\Gamma, F)\}$$

$$1\text{-morp} \quad \varphi \longmapsto \{1\text{-morph}\}$$

$$\vdots \qquad\qquad \vdots$$

<u>Theorem</u> (Fiorenza–Martinengo, Pridham, $\circledast$): $\mathrm{hoFlag}^{F}_{\Omega^{\bullet}_{X/k}}$ is weakly equivalent to $\mathbb{R}\mathrm{Def}_{\mathrm{holim}(\Rightarrow)}$

<u>Rmk</u>: Think of hoFlag as some sort of <u>homotopy flag variety</u>

<u>Defn</u> (–): The ~~geometric~~ (universal) geometric Fiorenza–Manetti–Martinengo local period map "is"

$$\mathbb{R}\mathcal{P} : \mathbb{R}\mathrm{Def}_X \longrightarrow \mathrm{hoFlag}^{F^{\cdot}}_{\mathbb{R}\Gamma}$$

$$(\mathcal{O}_{A,*} \xrightarrow{\varphi} \mathcal{O}_X) \longmapsto [(\mathbb{R}\Gamma(\pi^{0}\mathcal{X}, \Omega^{\bullet}_{\mathcal{X}/A}), F^{\cdot}), \tilde{\varphi}]$$

<u>Theorem</u> (–): $\mathbb{R}\mathcal{P}$ is equivalent to FMM. Moreover it lifts to the level of global derived stacks, as $\mathbb{R}\mathrm{Def}_X$ & hoFlag are formal neighborhoods of derived stacks.