Abel, Jacobi and the double homotopy fiber

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March 5, 2014

Joint work with Marco Manetti, (hopefully) soon on arXiv

Everything will be over the field $\mathbb{C}$ of complex numbers.

Questions like “does this work over an arbitrary characteristic zero algebraically closed field $\mathbb{K}$?” *are not allowed!*

(in any case the answer is “I guess so, but I don’t know”)

Let X be a smooth complex manifold and let $Z \subseteq X$ be a complex codimension p smooth complex submanifold.

Denote by $Hilb_{X/Z}$ the functor of infinitesimal deformations of Z inside X.

$$T_{b_0} Hilb_{X/Z} = H^0(Z; N_{X/Z})$$

$$\mathrm{obs}(Hilb_{X/Z}) \subseteq H^1(Z; N_{X/Z})$$

Actually one can control the obstructions better:

$$\mathrm{obs}(Hilb_{X/Z}) \subseteq \ker \left\{ H^1(Z; N_{X/Z}) \xrightarrow{\mathbf{i}} \mathbb{H}^{2p}(\mathcal{O}_X \to \Omega^1_X \to \cdots \to \cdots \Omega^{p-1}_X) \right\}$$

This has been originally shown by Bloch under a few additional hypothesis and recently by Iacono-Manetti and Pridham in full generality.

The aim of this talk is to illustrate a bit of the (infinitesimal) geometry behind these proofs.

Idea: to exhibit a morphism of (derived) infinitesimal deformation functors

$$AJ : Hilb_{X/Z} \to Jac^{2p}_{X/Z}$$

where

- $Jac^{2p}_{X/Z}$ is some deformation functor with $\mathrm{obs}(Jac^{2p}_{X/Z}) = 0$
- $\mathrm{obs}(AJ)$ is the restriction to $\mathrm{obs}(Hilb_{X/Z})$ of

$$\mathbf{i} : H^1(Z; N_{X/Z}) \to \mathbb{H}^{2p}(\mathcal{O}_X \to \Omega^1_X \to \cdots \to \cdots \Omega^{p-1}_X)$$

Infinitesimal deformation functors are "the same thing" as L_∞-algebras.

So the idea becomes: to exhibit a morphism of L_∞-algebras

$$\varphi : \mathfrak{g} \to \mathfrak{h}$$

such that:

- $\mathfrak{g} \leftrightsquigarrow Hilb_{X/Z}$
- $\mathfrak{h}$ is quasi-abelian (i.e. $\mathfrak{h}$ is quasi-isomorphic to a cochain complex)
- the linear morphism $H^2(\mathfrak{g}) \xrightarrow{H^2(\varphi)} H^2(\mathfrak{h})$ is naturally identified with

$$\mathbf{i} : H^1(Z; N_{X/Z}) \to \mathbb{H}^{2p}(\mathcal{O}_X \to \Omega^1_X \to \cdots \to \cdots \Omega^{p-1}_X)$$

Let $\chi : L \to M$ a morphism of dglas.

$$\begin{array}{ccc} \mathrm{hofiber}(\chi) & \longrightarrow & L \\ \big\downarrow & \swarrow\!\!\!\!\swarrow & \big\downarrow \chi \\ 0 & \longrightarrow & M \end{array}$$

A convenient model for $\mathrm{hofiber}(\chi)$ is the Thom-Whitney model

$$TW(\chi) = \{(l, m(t, dt)) \in L \oplus \big(M \otimes \Omega^{\bullet}(\Delta^1)\big) \mid m(0) = 0, m(1) = \chi(l)\}$$

It is a sub-dgla of $L \oplus \big(M \otimes \Omega^{\bullet}(\Delta^1)\big)$.

It is "big" even when L and M are small. However there is also another model which is just "as big as L and M".

$$\mathrm{cone}(\chi) = L \oplus M[-1],$$

$$[(l,m)]_1 = (dl, \chi(l) - dm)$$

$$[(l_1,m_1),(l_2,m_2)]_2 = \left([l_1,l_2], \frac{1}{2}[m_1,\chi(l_2)] + \frac{(-1)^{\deg(l_1)}}{2}[\chi(l_1),m_2]\right)$$

$$[(l_1,m_1),\cdots,(l_n,m_n)]_n = \left(0, \frac{B_{n-1}}{(n-1)!}\sum_{\sigma\in S_n} \pm[m_{\sigma(1)},[\cdots,[m_{\sigma(n-1)},\chi(l_{\sigma(n)})]\cdots]]\right),$$

for $n \geq 3$ where the B_n's are the Bernoulli numbers

Why is this relevant for us?

Let X be a complex manifold and let $Z \subseteq X$ be a complex submanifold. Let $A_X^{0,*}(\Theta_X)$ be the $p=0$ Dolbeault dgla with coefficients in holomorphic vector fields on X and

$$A_X^{0,*}(\Theta_X)(-\log Z) = \ker\{A_X^{0,*}(\Theta_X) \to A_Z^{0,*}(N_{X/Z})\}$$

the sub-dgla of $A_X^{0,*}(\Theta_X)$ of differential forms with coefficients vector fields tangent to Z. The deformation functor associated with

$$\mathrm{hofiber}\left(A_X^{0,*}(\Theta_X)(-\log Z) \hookrightarrow A_X^{0,*}(\Theta_X)\right)$$

is $Hilb_{X/Z}$.

Let L and M be two dglas, $\mathbf{i}: L \to M[-1]$ a morphism of *graded vector spaces*. Let

$$\begin{aligned} \mathbf{l}\colon L &\to M \\ a &\mapsto \mathbf{l}_a = d\mathbf{i}_a + \mathbf{i}_{da} \end{aligned}$$

be the differential of $\mathbf{i}$ in the cochain complex $\mathrm{Hom}(L, M)$.The map

$\mathbf{i}$ is called a *Cartan homotopy* for $\mathbf{l}$ if, for every $a, b \in L$, we have:

$$\mathbf{i}_{[a,b]} = [\mathbf{i}_a, \mathbf{l}_b], \qquad [\mathbf{i}_a, \mathbf{i}_b] = 0.$$

Note that $\mathbf{i}_{[a,b]} = [\mathbf{i}_a, \mathbf{l}_b]$ implies that $\mathbf{l}$ is a morphism of differential

graded Lie algebras: the *Lie derivative* associated with $\mathbf{i}$.

Example

let X be a differential manifold, $A^0_X(T_X)$ be the Lie algebra of vector fields on X, and $\mathrm{End}(A^*_X)$ be the dgla of endomorphisms of the de Rham complex of X. Then the contraction

$$\mathbf{i}\colon A^0_X(T_X) \to \mathrm{End}(A^*_X)[-1]$$

is a Cartan homotopy and its differential is the Lie derivative

$$\mathbf{l} = [d, \mathbf{i}] =\colon A^0_X(T_X) \to \mathrm{End}(A^*_X).$$

Example

let X be a complex manifold, $A_X^{0,*}(\Theta_X)$ be the $p=0$ Dolbeault dgla with coefficients in holomorphic vector fields on X, and $\mathrm{End}(A_X^{*,*})$ be the dgla of endomorphisms of the de Dolbeault complex of X. Then the contraction

$$\mathbf{i}\colon A_X^{0,*}(\Theta_X) \to \mathrm{End}(A_X^{*,*})[-1]$$

is a Cartan homotopy and its differential is the holomorphic Lie derivative

$$\mathbf{l} = [\partial, \mathbf{i}]\colon A_X^{0,*}(\Theta_X) \to \mathrm{End}(A_X^{*,*}).$$

Example

let X be a complex manifold, $A_X^{0,*}(\Theta_X)$ be the $p=0$ Dolbeault dgla with coefficients in holomorphic vector fields on X, and $\mathrm{End}(D_X)$ be the dgla of endomorphisms of the complex of smooth currents on X. Then the contraction

$$\hat{\mathbf{i}}\colon A_X^{0,*}(\Theta_X) \to \mathrm{End}(D_X)[-1]$$

is a Cartan homotopy and its differential is the holomorphic Lie derivative

$$\hat{\mathbf{l}} = [\hat{\partial}, \hat{\mathbf{i}}]\colon A_X^{0,*}(\Theta_X) \to \mathrm{End}(D_X).$$

The composition of a Cartan homotopy with a morphism of DGLAs (on either sides) is a Cartan homotopy. The corresponding Lie derivative is the composition of the Lie derivative of $\mathbf{i}$ with the given dgla morphisms.

Example

$$\hat{\mathbf{i}}[2p] \colon A_X^{0,*}(\Theta_X)(-\log Z) \to \mathrm{End}(D_X[2p])[-1]$$

is a Cartan homotopy.

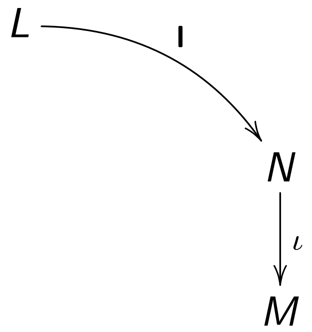
Cartan homotopies are compatible with base change/extension of scalars: if $\mathbf{i} \colon L \to M[-1]$ is a Cartan homotopy and Ω is a differential graded-commutative algebra, then its natural extension

$$\mathbf{i} \otimes \mathrm{Id} \colon L \otimes \Omega \to (M \otimes \Omega)[-1], \qquad a \otimes \omega \mapsto \mathbf{i}_a \otimes \omega,$$

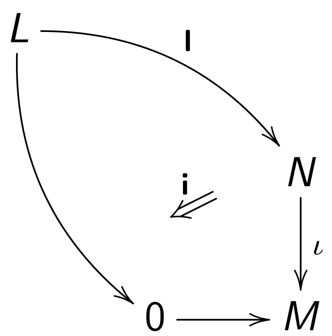
is a Cartan homotopy.

Cartan homotopies and homotopy fibers

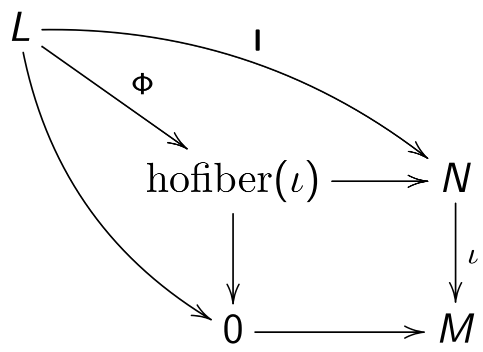
Let now $\mathbf{i} : L \to M[-1]$ be a Cartan homotopy with Lie derivative $\mathbf{l}$, and assume the image of $\mathbf{l}$ is contained in the subdgla N of M



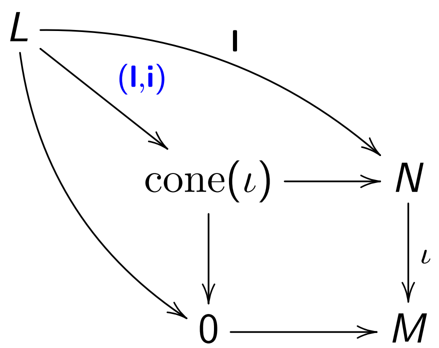
Then we have a homotopy commutative diagram of dglas



And so, by the universal property of the homotopy fiber we get



When we choose $\mathrm{cone}(\iota)$ as a model for the homotopy fiber we get a particularly simple expression for the L_∞ morpgism
$\Phi : L \to \mathrm{hofiber}(\iota)$:



A *Cartan square* is the following set of data:

- two morphisms of dglas $\varphi_L : L_1 \to L_2$ and $\varphi_M : M_1 \to M_2$;
- two Cartan homotopies $\mathbf{i}_1 : L_1 \to M_1[-1]$ and $\mathbf{i}_2 : L_2 \to M_2[-1]$

such that

$$\begin{array}{ccc} L_1 & \xrightarrow{\mathbf{i}_1} & M_1[-1] \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_M[-1]} \\ L_2 & \xrightarrow{\mathbf{i}_2} & M_2[-1] \end{array}$$

is a commutative diagram of graded vector spaces.

A Cartan square induces a commutative diagram of dglas

$$\begin{array}{ccc} L_1 & \xrightarrow{\mathbf{l}_1} & M_1 \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow{\scriptstyle \varphi_M} \\ L_2 & \xrightarrow{\mathbf{l}_2} & M_2 \end{array},$$

where $\mathbf{l}_1$ and $\mathbf{l}_2$ are the Lie derivatives associated with $\mathbf{i}_1$ and $\mathbf{i}_2$, respectively.

It also induces a Cartan homotopy

$$(\mathbf{i}_1, \mathbf{i}_2) : TW(L_1 \to L_2) \to TW(M_1 \to M_2)[-1]$$

whose Lie derivative is

$$(\mathbf{l}_1, \mathbf{l}_2) : TW(L_1 \to L_2) \to TW(M_1 \to M_2).$$

Now assume the commutative diagram of dglas associated with a Cartan square factors as

$$\begin{array}{ccccc} L_1 & \xrightarrow{\mathbf{l}_1} & N_1 & \xrightarrow{\iota_1} & M_1 \\ \downarrow{\scriptstyle \varphi_L} & & \downarrow & & \downarrow{\scriptstyle \varphi_M} \\ L_2 & \xrightarrow{\mathbf{l}_2} & N_2 & \xrightarrow{\iota_2} & M_2 \end{array}$$

where ι_1 and ι_2 are inclusions of sub-dglas.

Then we have a linear L_∞ morphism

$$(\mathbf{l}_1, \mathbf{l}_2, \mathbf{i}_1, \mathbf{i}_2)\colon TW(L_1 \to L_2) \to \mathrm{cone}\,(TW(N_1 \to N_2) \to TW(M_1 \to M_2))\,.$$

If moreover also φ_L and φ_M are inclusions, then in the (homotopy) category of cochain complexes the linear L_∞-morphism $(\mathbf{l}_1, \mathbf{l}_2, \mathbf{i}_1, \mathbf{i}_2)$ is equivalent to the span

$$(L_2/L_1)[-1] \xleftarrow{\sim} \mathrm{cone}(L_1 \to L_2) \to (M_2/(M_1 + N_2))[-2],$$

where the quasi isomorphism on the left is induced by the projection on the second factor, and the morphism on the right is $(a_1, a_2) \mapsto \mathbf{i}_{2,a_2} \mod M_1 + N_2$.

Hence, at the cohomology level, the morphism $H^n(\mathbf{l}_1, \mathbf{l}_2, \mathbf{i}_1, \mathbf{i}_2)$ is naturally identified with the morphism

$$\begin{aligned} H^{n-1}(L_2/L_1) &\to H^{n-2}(M_2/(M_1 + N_2)) \\ [a] &\mapsto [\mathbf{i}_{2,\tilde{a}} \mod M_1 + N_2], \end{aligned}$$

where $\tilde{a} \in L_2$ is an arbitrary representative of $[a]$.

Where do we find Cartan squares?

Let V be a chain complex, and let $\mathrm{End}(V)$ and $\mathfrak{aff}(V)$ be the dgla of its linear endomorphisms and infinitesimal affine transformations, respectively.

$$\mathfrak{aff}(V) = \mathrm{End}(V) \oplus V = \{f \in \mathrm{End}(V \oplus \mathbb{C}, V \oplus \mathbb{C}) \,|\, \mathrm{Im}(f) \subseteq V\}.$$

$$[(f,v),(g,w)] = ([f,g], f(w) - (-1)^{\overline{f}\overline{g}} g(v))$$
$$d_{\mathfrak{aff}}(f,v) = (d_{\mathrm{End}} f, dv)$$

Every degree zero closed element v in V defines an embedding of dglas

$$j_v : \mathrm{End}(V) \to \mathfrak{aff}(V)$$
$$f \mapsto (f, -f(v))$$

This is the identification of $\mathrm{End}(V)$ with the stabilizer of v under the action of $\mathfrak{aff}(V)$ on V.

In particular j_0 is the canonical embedding of $\mathrm{End}(V)$ into $\mathfrak{aff}(V)$ given by $f \mapsto (f, 0)$.

Let $\mathbf{i} : L \to \mathrm{End}(V)[-1]$ be a Cartan homotopy and let v be a degree zero closed element in V. Then

$$\begin{aligned} \mathbf{i}^v : L &\to \mathfrak{aff}(V)[-1] \\ a &\mapsto (\mathbf{i}_a, -\mathbf{i}_a(v)) \end{aligned}$$

is a Cartan homotopy. The corresponding Lie derivative is

$$\begin{aligned} \mathbf{l}^v : L &\to \mathfrak{aff}(V) \\ a &\mapsto (\mathbf{l}_a, -\mathbf{l}_a(v)) \end{aligned}$$

Indeed, the linear map $\mathbf{i}^v$ is the composition of the Cartan homotopy $\mathbf{i}$ with the dgla morphism j_v, hence it is a Cartan homotopy. The corresponding Lie derivative is the composition of $\mathbf{l}$ with j_v.

So we have built a Cartan homotopy $\mathbf{i}^v$ out of a Cartan homotopy $\mathbf{i} : L \to \mathrm{End}(V)[-1]$ and of a closed element v in V.

Let us now use the same ingredients to cook up a sub-dgla of L.

$$L_v = \{a \in L \text{ such that } \mathbf{i}_a(v) = 0 \text{ and } \mathbf{l}_a(v) = 0\}$$

For any sub-dgla $\tilde{L} \subseteq L_v$, the diagram

$$\begin{array}{ccc} \tilde{L} & \xrightarrow{\mathbf{i}|_{\tilde{L}}} & \mathrm{End}(V)[-1] \\ \downarrow & & \downarrow{\scriptstyle j_0[-1]} \\ L & \xrightarrow{\mathbf{i}^v} & \mathfrak{aff}(V)[-1]\,, \end{array}$$

where the left vertical arrow is the inclusion $\tilde{L} \hookrightarrow L$, is a Cartan square.

Let now F be a subcomplex of V such that the dgla morphism $\mathbf{l}^v : L \to \mathfrak{aff}(V)$ takes its values in

$$\mathfrak{aff}(V)(-F) = \{(f, v) \in \mathfrak{aff}(V) \,|\, f(F) \subseteq F,\, v \in F\}.$$

Then we have a linear L_∞-morphism

$$TW\begin{pmatrix} \tilde{L} \\ \downarrow \\ L \end{pmatrix} \xrightarrow{(\mathbf{l}|_{\tilde{L}}, \mathbf{l}^v, \mathbf{i}|_{\tilde{L}}, \mathbf{i}^v)} \mathrm{cone}\left(TW\begin{pmatrix} \mathrm{End}(V)(-F) \\ \downarrow \\ \mathfrak{aff}(V)(-F) \end{pmatrix} \to TW\begin{pmatrix} \mathrm{End}(V) \\ \downarrow \\ \mathfrak{aff}(V) \end{pmatrix}\right).$$

At the n-th cohomology level, this L_∞-morphism gives the map

$$\begin{aligned} H^{n-1}(L/\tilde{L}) &\to H^{n-2}(V/F) \\ [a] &\mapsto -[\mathbf{i}_{\tilde{a}}(v) \mod F]. \end{aligned}$$

where $\tilde{a}$ is any representative of $[a]$ in L.

The L_∞-algebra

$$\mathrm{cone}\left(TW\begin{pmatrix} \mathrm{End}(V)(-F) \\ \downarrow \\ \mathfrak{aff}(V)(-F) \end{pmatrix} \to TW\begin{pmatrix} \mathrm{End}(V) \\ \downarrow \\ \mathfrak{aff}(V) \end{pmatrix}\right)$$

is a model for the double homotopy fiber of the commutative diagram

$$\begin{array}{ccc} \mathrm{End}(V)(-F) & \longrightarrow & \mathrm{End}(V) \\ \downarrow & & \downarrow \\ \mathfrak{aff}(V)(-F) & \longrightarrow & \mathfrak{aff}(V) \end{array}$$

$$
\begin{array}{ccccc}
\mathit{cone} & \longrightarrow & TW & \longrightarrow & TW \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{End}(V)(-F) & \longrightarrow & \mathrm{End}(V) \\
 & & \downarrow & & \downarrow \\
 & & \mathfrak{aff}(V)(-F) & \longrightarrow & \mathfrak{aff}(V)
\end{array}
$$

But actually, due to the fact that we have sections

$$
0 \longrightarrow V \longrightarrow \mathfrak{aff}(V) \rightleftarrows \mathrm{End}(V) \longrightarrow 0
$$

$$
0 \longrightarrow F \longrightarrow \mathfrak{aff}(V)(-F) \rightleftarrows \mathrm{End}(V)(-F) \longrightarrow 0
$$

there is a simpler model:

$$
\begin{array}{ccccc}
(V/F)[-2] & \longrightarrow & F[-1] & \longrightarrow & V[-1] \\
 & & \downarrow & & \downarrow \\
 & & \mathrm{End}(V)(-F) & \longrightarrow & \mathrm{End}(V) \\
 & & \downarrow & & \downarrow \\
 & & \mathfrak{aff}(V)(-F) & \longrightarrow & \mathfrak{aff}(V)
\end{array}
$$

Let now X be a compact complex manifold and let $Z \subseteq X$ be a codimension p complex submanifold.
Then integration over Z defines a closed (p,p)-current, which we will denote by the same symbol Z. By shifting the degrees, we can look at Z as a closed degree zero element v in the chain complex $V = D(X)[2p]$.
Let $F = (F^p D(X))[2p]$ be the sub-complex of V obtained by shifting the p-th term in the Hodge filtration on currents,

$$F^p D(X) = \bigoplus_{i \geq p} D^{i,*}(X).$$

Finally, let $L = A_X^{0,*}(\Theta_X)$, let $\tilde{L} = A_X^{0,*}(\Theta_X)(-\log Z)$ and let $\mathbf{i}: L \to \mathrm{End}(V)[-1]$ be the (shifted) contraction operator on currents:

$$\hat{\mathbf{i}}[2p] : A_X^{0,*}(\Theta_X) \to \mathrm{End}(D(X)[2p], D(X)[2p])[-1].$$

The 6-ple $(L, \tilde{L}, V, F, v, \mathbf{i})$ defined this way satisfies the hypothesis of the slides above, so we get an L_∞-morphims

$$TW\left(\begin{array}{c} A_X^{0,*}(\Theta_X)(-\log Z) \\ \downarrow \\ A_X^{0,*}(\Theta_X) \end{array}\right) \to (D(X)/F^p D(X))[2p-2]$$

inducing in cohomology

$$\begin{aligned} H^0(Z; N_{X/Z}) &\to H^{2p-1}(D(X)/F^p D(X)) \\ H^1(Z; N_{X/Z}) &\to H^{2p}(D(X)/F^p D(X)) \\ [x] &\mapsto -[\hat{\mathbf{i}}_{\tilde{x}} Z \mod F^p D(X)] \end{aligned}$$

in degrees 1 and 2, where $\tilde{x}$ is any representative of $[x]$ in $A_X^{0,*}(\Theta_X)$.

Since

$$H^\bullet(D(X)/F^pD(X)) = \mathbb{H}^\bullet(X; \mathcal{O}_X \to \Omega^1_X \to \cdots \Omega^{p-1}_X),$$

if we define $Jac^{2p}_{X/Z}$ to be the deformation functor associated to the abelian dgla $(D(X)/F^pD(X))[2p-2]$ then we get from the L_∞-morphis exhibited above a morphism of deformation functors

$$AJ : Hilb_{X/Z} \to Jac^{2p}_{X/Z}$$

with

$$dAJ : H^0(Z; N_{X/Z}) \xrightarrow{\mathbf{i}} \mathbb{H}^{2p-1}(X; \mathcal{O}_X \to \Omega^1_X \to \cdots \Omega^{p-1}_X)$$

and

$$obs(AJ) : H^1(Z; N_{X/Z}) \xrightarrow{\mathbf{i}} \mathbb{H}^{2p}(X; \mathcal{O}_X \to \Omega^1_X \to \cdots \Omega^{p-1}_X)$$