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4. D-critical loci

Theorem (Joyce arXiv:1304.4508)

Let $X$ be a classical $\mathbb{K}$-scheme. Then there exists a canonical sheaf $S_X$ of $\mathbb{K}$-vector spaces on $X$, such that if $R \subseteq X$ is Zariski open and $i : R \hookrightarrow U$ is a closed embedding of $R$ into a smooth $\mathbb{K}$-scheme $U$, and $I_{R,U} \subseteq O_U$ is the ideal vanishing on $i(R)$, then

$$S_X|_R \cong \text{Ker}\left(\frac{O_U}{I_{R,U}^2} \xrightarrow{d} \frac{T^*U}{I_{R,U} \cdot T^*U}\right).$$

Also $S_X$ splits naturally as $S_X = S_X^0 \oplus \mathbb{K}_X$, where $\mathbb{K}_X$ is the sheaf of locally constant functions $X \to \mathbb{K}$.

The meaning of the sheaves $S_X, S_X^0$

If $X = \text{Crit}(f : U \to \mathbb{A}^1)$ then taking $R = X$, $i =$inclusion, we see that $f + I_{X,U}^2$ is a section of $S_X$. Also $f|_{X_{\text{red}}} : X_{\text{red}} \to \mathbb{K}$ is locally constant, and if $f|_{X_{\text{red}}} = 0$ then $f + I_{X,U}^2$ is a section of $S_X^0$. Note that $f + I_{X,U} = f|_X$ in $O_X = O_U/I_{X,U}$. The theorem means that $f + I_{X,U}^2$ makes sense intrinsically on $X$, without reference to the embedding of $X$ into $U$.

That is, if $X = \text{Crit}(f : U \to \mathbb{A}^1)$ then we can remember $f$ up to second order in the ideal $I_{X,U}$ as a piece of data on $X$, not on $U$. Suppose $X = \text{Crit}(f : U \to \mathbb{A}^1) = \text{Crit}(g : V \to \mathbb{A}^1)$ is written as a critical locus in two different ways. Then $f + I_{X,U}^2$, $g + I_{X,V}^2$ are sections of $S_X$, so we can ask whether $f + I_{X,U}^2 = g + I_{X,V}^2$. This gives a way to compare isomorphic critical loci in different smooth classical schemes.
The definition of d-critical loci

**Definition (Joyce arXiv:1304.4508)**

An (algebraic) d-critical locus \((X, s)\) is a classical \(\mathbb{K}\)-scheme \(X\) and a global section \(s \in H^0(S^0_X)\) such that \(X\) may be covered by Zariski open \(R \subseteq X\) with an isomorphism \(i : R \to \text{Crit}(f : U \to \mathbb{A}^1)\) identifying \(s|_R\) with \(f + I^2_{R, U}\), for \(f\) a regular function on a smooth \(\mathbb{K}\)-scheme \(U\).

That is, a d-critical locus \((X, s)\) is a \(\mathbb{K}\)-scheme \(X\) which may Zariski locally be written as a critical locus \(\text{Crit}(f : U \to \mathbb{A}^1)\), and the section \(s\) remembers \(f\) up to second order in the ideal \(I_{X, U}\).

We also define complex analytic d-critical loci, with \(X\) a complex analytic space locally modelled on \(\text{Crit}(f : U \to \mathbb{C})\) for \(U\) a complex manifold and \(f\) holomorphic.

**Theorem (Joyce arXiv:1304.4508)**

Let \((X, s)\) be an algebraic d-critical locus and \(X^{\text{red}}\) the reduced \(\mathbb{K}\)-subscheme of \(X\). Then there is a natural line bundle \(K_{X, s}^{\text{red}}\) on \(X^{\text{red}}\) called the canonical bundle, such that if \((X, s)\) is locally modelled on \(\text{Crit}(f : U \to \mathbb{A}^1)\) then \(K_{X, s}^{\text{red}}\) is locally modelled on \(K_U^{\otimes 2}|_{\text{Crit}(f)^{\text{red}}}\), for \(K_U\) the usual canonical bundle of \(U\).

**Definition**

Let \((X, s)\) be a d-critical locus. An orientation on \((X, s)\) is a choice of square root line bundle \(K^{1/2}_{X, s}\) for \(K_{X, s}\) on \(X^{\text{red}}\).

This is related to orientation data in Kontsevich–Soibelman 2008.
Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Let \((X, \omega)\) be a \(-1\)-shifted symplectic derived \(\mathbb{K}\)-scheme. Then the classical \(\mathbb{K}\)-scheme \(X = t_0(X)\) extends naturally to an algebraic d-critical locus \((X, s)\). The canonical bundle of \((X, s)\) satisfies \(K_{X,s} \cong \det \mathbb{L}X|_{X\text{red}}\).

That is, we define a truncation functor from \(-1\)-shifted symplectic derived \(\mathbb{K}\)-schemes to algebraic d-critical loci. Examples show this functor is not full. Think of d-critical loci as classical truncations of \(-1\)-shifted symplectic derived \(\mathbb{K}\)-schemes.

An alternative semi-classical truncation, used in D–T theory, is schemes with symmetric obstruction theory. D-critical loci appear to be better, for both categorified and motivic D–T theory.

The corollaries in lecture 1, §2 imply:

Corollary

Let \(Y\) be a Calabi–Yau 3-fold over \(\mathbb{K}\) and \(\mathcal{M}\) a classical moduli \(\mathbb{K}\)-scheme of coherent sheaves, or complexes of coherent sheaves, on \(Y\). Then \(\mathcal{M}\) extends naturally to a d-critical locus \((\mathcal{M}, s)\). The canonical bundle satisfies \(K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}\text{red}}\), where \(\phi : \mathcal{E}^\bullet \to \mathbb{L}\mathcal{M}\) is the (symmetric) obstruction theory on \(\mathcal{M}\).

Corollary

Let \((S, \omega)\) be a classical smooth symplectic \(\mathbb{K}\)-scheme, and \(L, M \subseteq S\) be smooth algebraic Lagrangians. Then \(X = L \cap M\) extends naturally to a d-critical locus \((X, s)\). The canonical bundle satisfies \(K_{X,s} \cong K_L|_{X\text{red}} \otimes K_M|_{X\text{red}}\). Hence, choices of square roots \(K_L^{1/2}, K_M^{1/2}\) give an orientation for \((X, s)\).

Bussi (in progress) extends the second corollary to complex Lagrangians in complex symplectic manifolds.
5. D-critical stacks

To generalize the d-critical loci in §4 to Artin stacks, we need a good notion of sheaves on Artin stacks. This is already well understood. Roughly, a sheaf $S$ on an Artin stack $X$ assigns a sheaf $S(U, \varphi)$ on $U$ (in the usual sense for schemes) for each smooth morphism $\varphi : U \to X$ with $U$ a scheme, and a morphism $S(\alpha, \eta) : \alpha^*(S(V, \psi)) \to S(U, \varphi)$ (often an isomorphism) for each 2-commutative diagram

$$
\begin{array}{ccc}
U & \xrightarrow{\alpha} & V \\
\eta \downarrow & & \psi \downarrow \\
\varphi & \xrightarrow{} & X
\end{array}
$$

with $U, V$ schemes and $\varphi, \psi$ smooth, such that $S(\alpha, \eta)$ have the obvious associativity properties. So, we pass from stacks $X$ to schemes $U$ by working with smooth atlases $\varphi : U \to X$.

The definition of d-critical stacks

Generalizing d-critical loci to stacks is now straightforward. As in §6, on each scheme $U$ we have a canonical sheaf $S_0^U$. If $\alpha : U \to V$ is a morphism of schemes we have pullback morphisms $\alpha^* : \alpha^{-1}(S_0^V) \to S_0^U$ with associativity properties.

So, for any classical Artin stack $X$, we define a sheaf $S_0^X$ on $X$ by $S_X(U, \varphi) = S_0^U$ for all smooth $\varphi : U \to X$ with $U$ a scheme, and $S(\alpha, \eta) = \alpha^*$ for all diagrams (2).

A global section $s \in H^0(S_0^X)$ assigns $s(U, \varphi) \in H^0(S_0^U)$ for all smooth $\varphi : U \to X$ with $\alpha^*[\alpha^{-1}(s(V, \psi))] = s(U, \varphi)$ for all diagrams (2). We call $(X, s)$ a d-critical stack if $(U, s(U, \varphi))$ is a d-critical locus for all smooth $\varphi : U \to X$.

That is, if $X$ is a d-critical stack then any smooth atlas $\varphi : U \to X$ for $X$ is a d-critical locus.
A truncation functor from $-1$-symplectic derived stacks

As for the scheme case in §4, we prove:

**Theorem (Ben-Bassat, Brav, Bussi, Joyce)**

Let $(X, \omega)$ be a $-1$-shifted symplectic derived Artin stack. Then the classical Artin stack $X = t_0(X)$ extends naturally to a $d$-critical stack $(X, s)$, with canonical bundle $K_{X,s} \cong \det \mathbb{L}_X|_{X^{\text{red}}}$.

**Corollary**

Let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli stack of coherent sheaves $F$ on $Y$, or complexes $F^\bullet$ in $D^b_{\text{coh}}(Y)$ with $\text{Ext}^0(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ extends naturally to a $d$-critical locus $(\mathcal{M}, s)$ with canonical bundle $K_{\mathcal{M},s} \cong \det(\mathcal{E}^\bullet)|_{\mathcal{M}^{\text{red}}}$, where $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$ is the natural obstruction theory on $\mathcal{M}$.

Canonical bundles and orientations

For schemes, a $d$-critical locus $(U, s)$ has a canonical bundle $K_{U,s} \to U^{\text{red}}$, and an orientation on $(U, s)$ is a square root $K_{U,s}^{1/2}$. Similarly, a $d$-critical stack $(X, s)$ has a canonical bundle $K_{X,s} \to X^{\text{red}}$. For any smooth $\varphi : U \to X$ with $U$ a scheme we have $K_{X,s}(U^{\text{red}}, \varphi^{\text{red}}) = K_{U,s}(U, \varphi) \otimes (\det \mathbb{L}_{U/X})^{\otimes -2}$. An orientation on $(X, s)$ is a choice of square root $K_{X,s}^{1/2}$ for $K_{X,s}$.

Note that as $(\det \mathbb{L}_{U/X})^{\otimes -2}$ has a natural square root, an orientation for $(X, s)$ gives an orientation for $(U, s(U, \varphi))$ for any smooth atlas $\varphi : U \to X$. 
6. Categorification using perverse sheaves

**Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)**

Let \((X, s)\) be an algebraic d-critical locus over \(\mathbb{K}\), with an orientation \(K_\frac{1}{2}^{1/2}\). Then we can construct a canonical perverse sheaf \(P_{X,s}^\bullet\) on \(X\), such that if \((X, s)\) is locally modelled on \(\text{Crit}(f : U \to \mathbb{A}^1)\), then \(P_{X,s}^\bullet\) is locally modelled on the perverse sheaf of vanishing cycles \(PV_{U,f}^\bullet\) of \((U, f)\).

Similarly, we can construct a natural \(\mathcal{D}\)-module \(D_{X,s}^\bullet\) on \(X\), and when \(\mathbb{K} = \mathbb{C}\) a natural mixed Hodge module \(M_{X,s}^\bullet\) on \(X\).

**Sketch of the proof of the theorem**

Roughly, we prove the theorem by taking a Zariski open cover \(\{R_i : i \in I\}\) of \(X\) with \(R_i \cong \text{Crit}(f_i : U_i \to \mathbb{A}^1)\), and showing that \(PV_{U_i,f_i}^\bullet\) and \(PV_{U_j,f_j}^\bullet\) are canonically isomorphic on \(R_i \cap R_j\), so we can glue the \(PV_{U_i,f_i}^\bullet\) to get a global perverse sheaf \(P_{X,s}^\bullet\) on \(X\).

In fact things are more complicated: the (local) isomorphisms \(PV_{U_i,f_i}^\bullet \cong PV_{U_j,f_j}^\bullet\) are only canonical up to sign. To make them canonical, we use the orientation \(K_\frac{1}{2}^{1/2}\) to define natural principal \(\mathbb{Z}_2\)-bundles \(Q_i\) on \(R_i\), such that \(PV_{U_i,f_i}^\bullet \otimes_{\mathbb{Z}_2} Q_i \cong PV_{U_j,f_j}^\bullet \otimes_{\mathbb{Z}_2} Q_j\) is canonical, and then we glue the \(PV_{U_i,f_i}^\bullet \otimes_{\mathbb{Z}_2} Q_i\) to get \(P_{X,s}^\bullet\).
The first corollary in lecture 1, §2 implies:

**Corollary**

Let $Y$ be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on $Y$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. orientation data, $K$–$S$). Then we have a natural perverse sheaf $P_{\mathcal{M}, s}$ on $\mathcal{M}$.

(Compare Kiem and Li arXiv:1212.6444).

The hypercohomology $\mathbb{H}^*(P_{\mathcal{M}, s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P_{\mathcal{M}, s})$ is the Behrend function $\nu_\mathcal{M}$ of $\mathcal{M}$. Thus

$$\sum_{i \in \mathbb{Z}} (-1)^i \dim \mathbb{H}^i(P_{\mathcal{M}, s}) = \chi(\mathcal{M}, \nu_\mathcal{M}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M}) = \chi(\mathcal{M}, \nu_\mathcal{M})$. So, $\mathbb{H}^*(P_{\mathcal{M}, s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a categorification of $DT(\mathcal{M})$.

The second corollary in lecture 1, §2 implies:

**Corollary**

Let $(S, \omega)$ be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L, M \subseteq S$ be smooth algebraic Lagrangians, with square roots $K_{L}^{1/2}, K_{M}^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P_{\mathcal{L}, \mathcal{M}}$ on $X = L \cap M$.

Bussi (in progress) extends this to complex Lagrangians in complex symplectic manifolds. This is related to Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P_{\mathcal{L}, \mathcal{M}})$ as being morally related to the Lagrangian Floer cohomology $HF^*(L, M)$ by

$$\mathbb{H}^i(P_{\mathcal{L}, \mathcal{M}}) \approx HF^{i+n}(L, M).$$

We are working on defining ‘Fukaya categories’ for algebraic/complex symplectic manifolds using these ideas (§8).
Let \((X, s)\) be a d-critical stack, with an orientation \(K_{X,s}^{1/2}\). Then for any smooth \(\varphi : U \to X\) with \(U\) a scheme, \((U, s(U, \varphi))\) is an oriented d-critical locus, so as above, BBDJS constructs a perverse sheaf \(P_{U,\varphi}\) on \(U\). Given a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\eta} & V \\
\downarrow{\varphi} & & \downarrow{\psi} \\
& X
\end{array}
\]

with \(U, V\) schemes and \(\varphi, \psi\) smooth, we can construct a natural isomorphism \(P_{\alpha,\eta} : \alpha^*(P_{V,\psi})[\dim \varphi - \dim \psi] \to P_{U,\varphi}\).

All this data \(P_{U,\varphi}, P_{\alpha,\eta}\) is equivalent to a perverse sheaf on \(X\).

Thus we prove:

**Theorem (Ben-Bassat, Brav, Bussi, Joyce)**

Let \((X, s)\) be a d-critical stack, with an orientation \(K_{X,s}^{1/2}\). Then we can construct a canonical perverse sheaf \(P_{X,s}\) on \(X\).

**Corollary**

Suppose \(Y\) is a Calabi–Yau 3-fold and \(\mathcal{M}\) a classical moduli stack of coherent sheaves \(F\) on \(Y\), or of complexes \(F^\bullet\) in \(D^b_{\text{coh}}(Y)\) with \(\text{Ext}^{<0}(F^\bullet, F^\bullet) = 0\), with (symmetric) obstruction theory \(\phi : \mathcal{E}^\bullet \to \mathcal{L}_\mathcal{M}\). Suppose we are given a square root \(\det(\mathcal{E}^\bullet)^{1/2}\) for \(\det(\mathcal{E}^\bullet)\). Then we construct a natural perverse sheaf \(P_{\mathcal{M},s}\) on \(\mathcal{M}\).

The hypercohomology \(\mathbb{H}^*(P_{\mathcal{M},s})\) is a categorification of the Donaldson–Thomas theory of \(Y\).
7. Algebraic structures on perverse sheaves

Let \((X, \omega_X)\) be a \(-1\)-shifted symplectic derived scheme, and \(i : L \to X\) a Lagrangian, in the sense of PTVV.

Choose an orientation \(K_{X,s}^{1/2}\) for \((X, \omega_X)\). There is then a notion of relative orientation for \(i : L \to X\), choose one of these.

We get a perverse sheaf \(P_{X, \omega_X}^\bullet\) on \(X\), by BBDJS in \(\S 6\).

**Conjecture**

There is a natural morphism in \(D_b^c(L)\)

\[
\mu_L : \mathbb{Q}_L[\text{vdim} L] \to i'_!(P_{X, \omega_X}^\bullet),
\]

with given local models in ‘Darboux form’ presentations for \(X, L\).

This Conjecture has important consequences (\(\S 8, \S 11\)).

I already know local models for \(i : L \to X\) and \(\mu_L\) in (2). What makes the Conjecture difficult is that local models are not enough: \(\mu_L\) is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine \(\mu_L\) to be globally nonzero, but zero on the sets of an open cover of \(L\).

So to construct \(\mu_L\), we have to do a gluing problem in an \(\infty\)-category, probably using hypercovers. I have a sketch of one way to do this (over \(\mathbb{C}\)). It is not easy.

Maybe gluing local models naïvely is not the best approach for this problem, need some more advanced Lurie-esque technology? Any help would be appreciated.
8. ‘Fukaya categories’ of complex symplectic manifolds

Let \((S, \omega)\) be a complex symplectic manifold, with \(\dim_{\mathbb{C}} S = 2n\), and \(L, M \subset S\) be complex Lagrangians (not supposed compact or closed). The intersection \(L \cap M\), as a complex analytic space, has a d-critical structure \(s\) (Vittoria Bussi, work in progress). Given square roots of canonical bundles \(K_{L}^{1/2}, K_{M}^{1/2}\), we get an orientation on \((L \cap M, s)\), and so a perverse sheaf \(P_{L,M}^{\bullet}\) on \(L \cap M\).

I claim that we should think of the shifted hypercohomology \(\mathbb{H}^{* -n}(P_{L,M}^{\bullet})\) as a substitute for the Lagrangian Floer cohomology \(HF^*(L, M)\) in symplectic geometry. But \(HF^*(L, M)\) is the morphisms in the derived Fukaya category \(D^bFuk(S, \omega)\).

Problem

Given a complex symplectic manifold \((S, \omega)\), build a ‘Fukaya category’ with objects \((L, K_{L}^{1/2})\) for \(L\) a complex Lagrangian, and graded morphisms \(\mathbb{H}^{* -n}(P_{L,M}^{\bullet})\).

Extend to derived Lagrangians \(L\) in \((S, \omega)\).

Work out the ‘right’ way to form a ‘derived Fukaya category’ for \((S, \omega)\) out of this, as a (Calabi–Yau?) triangulated category.

Show that (derived) Lagrangian correspondences induce functors between these derived Fukaya categories.

Question

Can we include complex coisotropic submanifolds as objects?

Maybe using \(\mathcal{D}\)-modules?
The Conjecture in §7 is what we need to define composition of morphisms in this ‘Fukaya category’, as follows. If $L, M, N$ are Lagrangians in $(S, \omega)$, then $M \cap L, N \cap M, L \cap N$ are $-1$-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (lecture 1, §1). Applying the Conjecture to $L \cap M \cap N$ and rearranging gives a morphism of constructible complexes

$$
\mu_{L,M,N} : P^\bullet_{L,M} \otimes P^\bullet_{M,N}[n] \rightarrow P^\bullet_{L,N}.
$$

Taking hypercohomology gives the multiplication

$$\text{Hom}^\ast(L, M) \times \text{Hom}^\ast(M, N) \rightarrow \text{Hom}^\ast(L, N).$$

Relation to deformation quantization and DQ-modules

Kashiwara and Schapira (Astérisque 345) develop a theory of deformation quantization modules, or DQ-modules, on a complex symplectic manifold $(S, \omega)$, which are roughly symplectic versions of $\mathcal{D}$-modules. Holonomic DQ-modules are supported on (singular) Lagrangians. If $L$ is a closed, embedded complex Lagrangian in $(S, \omega)$ with $K^1_L$, D’Agnolo and Schapira construct a simple holonomic DQ-module $D^\bullet_L$ supported on $L$. For Lagrangians $L, M$, Kashiwara and Schapira show that $R\mathcal{H}om(D^\bullet_L, D^\bullet_M)$ is a perverse sheaf over $\mathbb{C}((\hbar))$ supported on $X = L \cap M$. Schapira (private communication) explained that this perverse sheaf should be isomorphic to the perverse sheaf $P^\bullet_{L,M}$ we construct, over base ring $A = \mathbb{C}((\hbar))$. 
All this looks very similar to our ‘complex Fukaya category’ picture, but there are some puzzling differences:

- Our perverse sheaf picture works over (nearly) any base ring $A$, e.g. $A = \mathbb{Z}, \mathbb{Q}$. DQ-modules work only over $A = \mathbb{C}((\hbar))$. Is our picture related to ‘microlocal perverse sheaves’?

- We have natural monodromy and Verdier duality operators on our perverse sheaves. Does $\hbar$ encode the monodromy?

- Our objects live on $i: L \to S$, where $i$ need not be an embedding, and $L$ can be derived, with classical singularities. Holonomic DQ-modules live on embedded Lagrangians $L \subset S$. They can be singular, but the singularities allowed look very different to those in our picture.

I would like to understand the relation between the theories better.