A formality criterion for differential graded Lie algebras

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Deligne's principle (letter to J. Millson, 1986).
In characteristic 0, a deformation problem is controlled by a differential graded Lie algebra, with quasi-isomorphic DG-Lie algebras giving the same deformation theory.

In 1986 this was considered just a principle,

In 1994 evolved to a metatheorem (Kontsevich's Berkeley lectures in deformation theory),

Nowadays it is considered a theorem.

From now on we consider only fields of characteristic 0.

Definition

A *DG-vector space* is the data of a graded vector space $V = \oplus V^i$, $i \in \mathbb{Z}$, and a (linear) differential $d \colon V^i \to V^{i+1}$, $d^2 = 0$.

Definition

A *DG-Lie algebra* is the data of a DG-vector space (L, d) and a (bilinear) bracket $[-,-] \colon L^i \times L^j \to L^{i+j}$ such that:

1. (graded skewsymmetry) $[a, b] = -(-1)^{\deg(a)\deg(b)}[b, a]$.
2. (graded Jacobi) $[a, [b, c]] = [[a, b], c] + (-1)^{\deg(a)\deg(b)}[b, [a, c]]$
3. (graded Leibniz) $d[a, b] = [da, b] + (-1)^{\deg(a)}[a, db]$.

Example: The *Kodaira-Spencer* DG-Lie algebra of a complex manifold X. Denote by:
1) Θ_X the holomorphic tangent sheaf of X.
2) $A_X^{p,q}$ the space of differentiable (p,q)-forms on X and $A_X^{p,q}(\Theta_X)$ the space of (p,q)-forms with values in Θ_X.
The Kodaira-Spencer DGLA KS_X is by definition the Dolbeault complex

$$0 \to A_X^{0,0}(\Theta_X) \xrightarrow{\overline{\partial}} A_X^{0,1}(\Theta_X) \xrightarrow{\overline{\partial}} \cdots$$

equipped with the natural bracket.
By Dolbeault theorem $H^i(KS_X) = H^i(X, \Theta_X)$.

Important because it is the DGLA controlling deformations of X

A morphism $f\colon L \to M$ of DGLA is a morphism of graded vector spaces commuting with brackets and differentials.

A morphism $f\colon L \to M$ of DGLA is called a *quasi-isomorphism* if the induced map $f\colon H^i(L) \to H^i(M)$ is an isomorphism for every $i \in \mathbb{Z}$.

Two DGLA are homotopy equivalent if they are connected by a zigzag of quasi-isomorphisms. It is easy to prove that L, M are homotopy equivalent if and only if there exists a diagram

$$L \xleftarrow{f} H \xrightarrow{g} M$$

with both f, g quasi-isomorphisms of DG-Lie algebras.

Definition

A DG-Lie algebra is called **formal** if is homotopy equivalent to a DGLA with trivial differential.

Definition

A DG-Lie algebra is called **homotopy abelian** if is homotopy equivalent to a DGLA with trivial bracket.

Remark. A DGLA L is homotopy abelian if and only if it is formal and the bracket is trivial in cohomology.

Let X be a complex manifold. By deformation theory, if KS_X is homotopy abelian then the semiuniversal deformation space of X is smooth; if KS_X is formal then the semiuniversal deformation space is at most a quadratic singularity.

Some examples. 0) $KS_{\mathbb{P}^n}$ is formal.

1) If X is projective with trivial canonical bundle, then KS_X is homotopy abelian (Bogomolov-Tian-Todorov theorem).

2) Let $M = T/\Gamma$ be the Iwasawa manifold: here T is the Lie group of upper triangular unipotent 3×3 complex matrices and Γ is the subgroup of matrices with Gaussian integers coefficients. Then KS_M is formal but not homotopy abelian.

3) Let M as above and $Y = M \times \mathbb{P}^1$. Then KS_Y is not formal.

4) Let S be a complex surface with ample canonical bundle whose universal deformation space is defined by a non trivial cubic equation (it exists by Murphy's law). Then KS_S is not formal, although the bracket is trivial in cohomology.

(at least) Three possible proofs of BTT theorem:

1) By Tian-Todorov lemma, i.e., BV-algebra structure on polyvector fields;

2) By derived Griffiths period map (see talks by Di Natale and Fiorenza);

3) By Cartan homotopy formulas and derived brackets.

Here we give a sketch of 3).

Let $A = (A^{*,*}, d = \partial + \overline{\partial})$ be the de Rham complex of a projective manifold X. By $\partial\overline{\partial}$-lemma the subcomplex $Im\partial$ is exact.
The complexes $\mathrm{Hom}^*(A, A)[-1]$, $M := \mathrm{Hom}^*(ker\partial, coker\partial)[-1]$ have DGLA structures induced by the derived (in the sense of Koszul-Voronov) bracket $\{-, -\}_\partial$, defined as the graded commutator of the associative product

$$(f, g) \mapsto f\partial g\ .$$

The DGLA M is homotopy abelian since the natural map

$$\mathrm{Hom}^*(ker\partial, ker\partial)[-1] \to \mathrm{Hom}^*(ker\partial, coker\partial)[-1] = M$$

is a quasi-isomorphism of DGLA.

Let $i\colon A^{0,i}(\Theta_X) \to \mathrm{Hom}(A^{p,q}, A^{p-1,q+i})$ be the contraction. By Cartan homotopy formulas the induced map

$$i\colon KS_X \to M$$

is a morphism of DGLA. If X has a holomorphic volume form, then $i\colon KS_X \to M$ is injective in cohomology.
$(A^{0,i}(\Theta_X) = \mathrm{Hom}(A^{n,0}, A^{n-1,i})$, $n = \dim X$.)

Key Lemma. Let $f\colon L \to M$ be a morphism of DGLA. If:
1) $f\colon H^*(L) \to H^*(M)$ is injective;
2) M is homotopy abelian.
Then also L is homotopy abelian.

Proof. Homotopy classification of L_∞-algebras.

The key lemma is powerful and widely used in deformation theory.

Question: does there exists an analogue of key lemma for formality?

Beware. If $f\colon L \to M$ is injective in cohomology and M is formal, then L may **not** be formal. (the naive extension does not hold).

Need to replace the injectivity of $f\colon H^*(L) \to H^*(M)$ with a stronger condition.

Chevalley-Eilenberg cohomology.
Consider first the case of graded Lie algebras (DGLA with trivial differential). For every DGLA L, its cohomology $H^*(L)$ is a graded Lie algebra.
Given a morphism of graded Lie algebras $f\colon L \to M$, for every fixed integer p there is a sequence of cohomology groups

$$H^i(L, M)_p, \qquad i = 0, 1, \dots$$

defined as the cohomology of the Chevalley-Eilenberg complex $\operatorname{Hom}^p(\bigwedge^* L, M)$.

$$0 \to M^p \xrightarrow{\delta} \operatorname{Hom}^p(L, M) \xrightarrow{\delta} \operatorname{Hom}^p(L \wedge L, M) \cdots$$

Description (up to some signs) of δ:

0) For $m \in M^p$ we have $\pm(\delta m)(x) = [m, f(x)]$;

1) For $\phi \in \mathrm{Hom}^p(L, M)$ we have

$$\pm(\delta\phi)(x, y) = [\phi(x), f(y)] - (-1)^{\bar{x}\,\bar{y}}[\phi(y), f(x)] - \phi([x, y]);$$

≥ 2) For $\phi \in \mathrm{Hom}^p(L^{\wedge n-1}, M)$

$$\begin{aligned} &\pm (\delta\phi)(x_1, \ldots, x_p) = \\ &= \sum_i \chi_i \, [\phi(x_1, \ldots, \widehat{x_i}, \ldots, x_p), f(x_i)] \\ &- \sum_{i<j} \chi_{i,j} \, \phi(x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_p, [x_i, x_j]) \end{aligned}$$

(here χ_i, $\chi_{i,j}$ are the antisymmetric Koszul signs).

$$H^0(L,M)_p = \{m \in M^p \mid [f(L), m] = 0\}$$

$$H^1(L,M)_p = \frac{\{\text{derivations } h\colon L \to M \text{ of degree } p\}}{\{\text{inner derivations}\}}$$

Key lemma for formality. Let $f\colon L \to M$ be a morphism of DGLA, with M formal. If

$$f\colon H^n(H^*(L), H^*(L))_{2-n} \to H^n(H^*(L), H^*(M))_{2-n}$$

is injective for every $n \geq 3$, then L is formal.
(e.g. when L a direct summand of M as an L-module.)

Definition

A graded Lie algebra K is called **intrinsically formal** if every DGLA L such that $H^*(L) \simeq K$ is formal.

Corollary (Hinich, Tamarkin) A graded Lie algebra K such that $H^n(K,K)_{2-n} = 0$ for every $n \geq 3$ is intrinsically formal.
Proof. take $M = 0$ in the Key lemma.

The Euler class $e \in H^1(K,K)_0$ of a graded Lie algebra K is the class of the Euler derivation

$$e\colon K \to K, \qquad e(x) = \deg(x)x\ .$$

Theorem

A graded Lie algebra K such that $e = 0 \in H^1(K,K)_0$ is intrinsically formal.

Examples of intrinsically formal graded Lie algebras (with trivial Euler class).

1) $\mathrm{Hom}^*(V, V)$ with V graded vector space.
Since $e = [u, -]$, $u \colon V \to V$, $u(v) = \deg(v)v$.

2) $\mathrm{Hom}^{\geq 0}(V, V)$, $\mathrm{Hom}^{\leq 0}(V, V)$, V as above.

3) $\mathrm{Der}^*(A, A)$, with A graded commutative algebra.

4) Differential operators of a graded commutative algebra.

Chevalley-Eilenberg cohomology (general case).

Given a DGLA $L = (L, d, [-,-])$, as above we have the Chevalley-Eilenberg differential

$$\delta\colon \operatorname{Hom}^p(\bigwedge^i L, L) \to \operatorname{Hom}^p(\bigwedge^{i+1} L, L)$$

together the natural differential induced by d:

$$d\colon \operatorname{Hom}^p(\bigwedge^i L, L) \to \operatorname{Hom}^{p+1}(\bigwedge^i L, L).$$

Since $d^2 = \delta^2 = d\delta + \delta d = 0$, this gives a double complex

$$(\operatorname{Hom}^*(\bigwedge^* L, L), d, \delta).$$

The natural decresasing complete filtration $F^p = \mathrm{Hom}^*(\bigwedge^{\geq p} L, L)$ gives the Chevalley-Eilenberg spectral sequence $(E(L,L)_r^{p,q}, d_r)$.
It is easy to see that:

1) $E(L,L)_2^{p,q} = H^p(H^*(L), H^*(L))_q$; in particular it is defined the Euler class $e \in E(L,L)_2^{1,0}$.

2) the pages of $(E(L,L)_r^{p,q}, d_r)$, for $r > 0$, and the Euler class are homotopy invariant of L. In particular, if L is formal, then the spectral sequence degenerate at E_2.

It is worth to mention a recent result by R. Bandiera.

Theorem (Bandiera, 2013)

A DGLA is homotopy abelian if and only if its Chevalley-Eilenberg spectral sequence degenerates at E_1.

We are now ready to state the main theorem:

Theorem

For a differential graded Lie algebra L the following conditions are equivalent:

1. *L is formal;*
2. *The Chevalley-Eilenberg spectral sequence degenerates at E_2;*
3. *$d_r(\text{Euler class}) = 0$ for every $r \geq 2$.*

The only non trivial implication is $3 \Rightarrow 1$. Bandiera's theorem follows immediately but his proof is more easy and elegant.

Main steps of the proof:

1) extend the definition of C.E. spectral sequence and Euler class to any L_∞-morphism $f\colon L \dashrightarrow M$ of L_∞-algebras.

2) using homotopy invariance and homotopy classification of L_∞-algebras, reduce to consider only minimal L_∞-algebras.

3) (inductive step) Let $e \in E(V,V)_2^{1,0}$ be the Euler class of a minimal L_∞-algebra

$$V = (V, 0, q_2, 0, \dots, 0, q_{k+1}, \dots), \quad k \geq 2, \quad q_n\colon V[1]^{\odot n} \to V[1]\,.$$

Then $d_r(e) = 0$ for every $2 \leq r < k$ and $d_k(e) = 0$ if and only if V is isomorphic to some $(V, 0, q_2, 0, \dots, 0, \hat{q}_{k+2}, \dots)$

4) push step 3 to the limit $k \to \infty$.

Some similar results (in the framework of DG associative algebras) were proved by Kaledin (arxiv:math/0509699) and Lunts (arXiv:0712.0996), by using the same strategy. In these papers the Euler class is not considered and the inductive step 3) is based on the properties of a well defined cohomology class in the Chevalley-Eilenberg cohomology of the deformation to normal cone.
The introduction of Euler class, not only gives simpler statement, but is also essential in the proof of the key lemma for formality.