The moduli spaces of symplectic vortices on an orbifold Riemann surface

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Plan of this talk

1) **What are Symplectic Vortex Equations (SVE)?**

   Hamiltonian G-space $\to$ SVE $\to$ moduli $\to$ invariants

2) **Motivation of my research**

3) **The reason for differentiable stacks**

4) **Moduli spaces for special cases**

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Notation

- $\diamond$ G compact and connected Lie group.
- $\diamond$ $\mathfrak{g} = \mathfrak{g}^\vee$ thru an $\langle\ ,\ \rangle$ on $\mathfrak{g}$. $\qquad \mathfrak{g} := \mathrm{Lie}(G)$
- $\diamond$ A **Hamiltonian G-space** is a triple

  (G-manifold M, symp form ω, moment map μ).

- $\diamond$ A **moment map** is $\mu \in \mathcal{C}^\infty_G(M, \mathfrak{g})$ satisfying

$$d\langle \mu, \xi \rangle = -\iota(\xi^M)\omega \quad (\forall \xi \in \mathfrak{g})$$

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Examples of Hamiltonian G-spaces

1) Ham U_d-space $\left(\mathrm{Mat}(d\times n,\mathbb{C}),\omega,\mu_0\right)$ (Grassmannian)

- $\mathrm{Mat}(d\times n,\mathbb{C}) \curvearrowleft U_d$; $A\cdot g := g^{-1}A$
- $\mu_0 : \mathrm{Mat}(d\times n,\mathbb{C}) \to \mathfrak{u}_d$; $\mu_0(A) = -\frac{\mathrm{i}}{2}(AA^\dagger - \tau\mathbb{1})$ $(\tau\in\mathbb{R})$

2) Ham U_1-space $\left(\mathbb{C},\omega,\mu_a\right)$ $(a\in\mathbb{Z}_{>0})$ (WP pt)

- $\mathbb{C} \curvearrowleft U_1$; $z\cdot t = t^{-a}z$
- $\mu_a : \mathbb{C} \to i\mathbb{R}$; $\mu_a(z) = \frac{\mathrm{i}}{2}(a|z|^2 - \tau)$ $(\tau\in\mathbb{R})$

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Symplectic Vortex Equations

Fixed Data
$$\begin{cases} (M,\omega,\mu) & \text{Hamiltonian } G\text{-space} \\ (\Sigma, j, \mathrm{dvol}_\Sigma) & \text{closed Riemann surface} \\ P \to \Sigma & \text{principal } G\text{-bundle} \end{cases}$$

SVE
$$\begin{cases} \bar\partial_A u = 0 \\ *F_A + \mu \circ u = 0 \end{cases} \quad \text{for} \quad \begin{cases} A \in \mathcal{A}(P) \subset \Omega^1(P,\mathfrak{g})^G \\ u \in C^\infty_G(P,M) \end{cases}$$

- $\diamond\ \bar\partial_A u = 0 \iff u : \Sigma \to P \times_G M$ is pseudo-hol

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"Hitchin–Kobayashi correspondence"

- ◇ Hamiltonian U_d-space $\big(\mathrm{Mat}(d \times n, \mathbb{C}), \omega, \mu_0\big)$
- ◇ SVE: $\bar{\partial}_A u = 0$ and $*F_A - \frac{\mathrm{i}}{2}(uu^\dagger - \tau \mathbb{1}) = 0$ $\quad (\tau \in \mathbb{R})$

[H–K corresp. (Bertram–Daskalopoulos–Wentworth)]

$$\{\mathrm{SV}\ (A, u)\}/\mathrm{gauge} \leftrightarrow \{\tau\text{-stable } n\text{-pairs } (\bar{\partial}_E, s)\}/\mathrm{isom}$$

- ◇ $E = P \times_{U_d} \mathrm{Mat}(d \times n, \mathbb{C})$, $\quad s \in H^0(\Sigma, E^{\oplus n})$.
- ◇ τ-stable $\overset{\text{def}}{\Longleftrightarrow}$ $\dfrac{\deg(E')}{\mathrm{rk}(E')} < \dfrac{\tau \mathrm{Vol}(\Sigma)}{4\pi}$ $(\forall E' \subset E)$ and something

Moduli space and invariants

Assumptions: $\mu^{-1}(0) \curvearrowleft G$ is free and more.

[Thm (Cieliebak–Gaio–Mundet–Salamon)]

$\mathcal{M}(P) = \{(A, u) \mid \text{SVE}\}/(\text{gauge})$ is an oriented closed mfd.

◇ SVI: $H_G^{\dim \mathcal{M}(P)}(M) \to \mathbb{R};\ \alpha \mapsto \int_{\mathcal{M}(P)} \mathrm{ev}^* \alpha$ (Intuitive def'n!)

[Thm (Gaio–Salamon)]

Under several topological conditions,

SVI for M = GWI of $\overline{M}$ with fixed marked points

Here $\overline{M} := \mu^{-1}(0)/G$.

Q. GW theory is enough, isn't it?

A. No!!

- ◇ **Applications!**
  - Periodic orbits, SW inv, GW inv, $QH^*(\overline{M})$
- ◇ **Exciting Topics!**
  - Geometry and topology of moduli spaces
  - H–K correspondence
  - Hamiltonian invariants
  - Differentiable stacks (today)!

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Motivation: SVI=GWI for orbifold

- ◇ **Unnatural** assumption: $\mu^{-1}(0) \curvearrowleft G$ is free.
- ◇ $\overline{M}$ $(= \mu^{-1}(0)/G)$ is usually an orbifold.
- ◇ Orbifold GWI
- ◇ (SVI of M) $\neq$ (GWI for orbifold $\overline{M}$)
- ◇ $\because$ SVE do not care about singularities.

[Conjecture]

"SVI=GWI" holds for orbifolds after modifying SVE.

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Idea: a "variation" of the eqn of J-hol curve

Don't Look at solutions!

[Idea for "SVI=GWI"]

$$\begin{cases} \overline{\partial}_A u = 0 \\ *F_A + \mu \circ u = 0 \end{cases} \xrightarrow{\mathrm{dvol}_\Sigma \to +\infty} \begin{cases} \overline{\partial}_A u = 0 \\ \mu \circ u = 0 \end{cases}$$

SVE — Eqn of J-hol $\Sigma \to \overline{M}$

- ⋄ Orbi GW: pseudo-hol maps from **orbifold Riemann surf**
- ⋄ Everything is an orbifold → Terrible!
- ⋄ Strategies: 1) "holonomy data" on smooth Σ (majority)
  2) differentiable stacks! (today)

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Q. Why do we need diff stacks?

A. SVE are PDEs on differentiable stacks!

◇ SVE $\begin{cases} \overline{\partial}_A u = 0 \\ *F_A + \mu \circ u = \tau \end{cases}$ for $\begin{cases} A \in \mathcal{A}(P) \\ u \in C^\infty_G(M, P) \end{cases}$

◇ $\begin{array}{ccc} P & \xrightarrow{u} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \Sigma & \overset{\varphi}{\dashrightarrow} & [M/G] \end{array}$ $(\pi, u) \longleftrightarrow$ map of stacks $\varphi : \Sigma \to [M/G]$

(smooth Σ)

◇ Idea: An orbifold Riemann surf Σ as a stacks.

Q. What should we do for P?

A. Use the cat $\mathfrak{P}^G(\Sigma)$. (orbifold Σ)

- $\mathfrak{P}^G(\Sigma) =$ cat of prin G-bdl over Σ with smooth total space.
- $(P \to \Sigma) \in \mathfrak{P}^G(\Sigma) \implies \mathcal{A}(P) \neq \varnothing$

[Theorem]

Take $P \to \Sigma$ in $\mathfrak{P}^G(\Sigma)$. Then

$$\begin{cases} \bar{\partial}_A u = 0 \\ *F_A + \mu(u) = 0 \end{cases} \xrightarrow{\mathrm{dvol}_\Sigma \to +\infty} \begin{cases} \bar{\partial}_A u = 0 \\ \mu(u) = 0 \end{cases}$$

SVE (nothing to change!) — Eqn of J-hol orbicurve $\Sigma \to \overline{M}$

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Moduli spaces (special cases)

◇ Recall: Ham U_1-space $(\mathbb{C}, \omega_{\mathrm{std}}, \mu_a)$ $\quad (a \in \mathbb{Z}_{>0})$

$$z \cdot t = t^{-a} z \quad (z \in \mathbb{C}, t \in \mathrm{U}_1), \quad \mu_a(z) = \frac{\mathrm{i}}{2}\left(a|z|^2 - \tau\right) \quad (\tau \in \mathbb{R})$$

[Theorem]

◇ $\pi_1(\underline{\Sigma}) = 1$ for $\Sigma = (\underline{\Sigma}; \overbrace{z_1, \ldots, z_k}^{\text{sing pts}}; \overbrace{m_1, \ldots, m_k}^{\text{order}})$

◇ $a \in \mathrm{lcm}(m_1, \ldots, m_k)\mathbb{Z} \quad (\Leftarrow \exists$ of J-hol orbicurve$)$

$$\Longrightarrow \quad \mathcal{M}(P) \cong \mathbb{C}\mathrm{P}^{ad} \text{ if } d < \frac{\tau \mathrm{Vol}(\Sigma)}{4\pi} \qquad \left(d := \frac{\mathrm{i}}{2\pi}\int_{\Sigma} F_A\right)$$

What will come next?

1) Moduli spaces

- ◇ $\pi_1(\underline{\Sigma}) \neq 1$ $\left(\mathcal{M}(P) \overset{?}{\cong} \text{covering sp of } \mathrm{Sym}^{ad}(\underline{\Sigma})?\right)$
- ◇ Linear Hamiltonian $\mathbb{T}^r$-space $(\mathbb{C}^n, \omega_{\mathrm{std}}, \mu)$
  [WANTED] Specialist of geometric analysis of G-mfd!

2) Construction of SVI

3) SVI=GWI for orbifold $\overline{M}$

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Summary

- ◇ Hamiltonian G-space → SVE → moduli → invariants
- ◇ Exciting topics and appl: H–K corresp, GW theory, etc.
- ◇ SVE as PDEs on differentiable stacks work!

Thank you for your attention!

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