Residues and Duality for Schemes and Stacks

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Outline

Some of the work discussed here was done with James Zhang several years ago.

1. Rigid Dualizing Complexes over Rings

All rings in this talk are commutative.

We fix a base ring $\mathbb{K}$, which is regular noetherian and finite dimensional (e.g. a field or $\mathbb{Z}$).

Let A be an *essentially finite type* $\mathbb{K}$-ring. Recall that this means A is a localization of a finite type $\mathbb{K}$-ring. In particular A is noetherian and finite dimensional.

We denote by $\mathsf{C}(\mathsf{Mod}\, A)$ the category of complexes of A-modules, and $\mathsf{D}(\mathsf{Mod}\, A)$ is the derived category.

There is a functor

$$\mathrm{Q} : \mathsf{C}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

which is the identity on objects. The morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ are all of the form $\mathrm{Q}(\phi) \circ \mathrm{Q}(\psi)^{-1}$, where ψ is a quasi-isomorphism.

Inside $\mathsf{D}(\mathsf{Mod}\, A)$ there is the full subcategory $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ of complexes with bounded finitely generated cohomology.

In [YZ3] we constructed a functor

$$\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

called the *squaring*.

It is a *quadratic functor*: if $\phi : M \to N$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)$, and $a \in A$, then

$$\mathrm{Sq}_{A/\mathbb{K}}(a\phi) = a^2 \,\mathrm{Sq}_{A/\mathbb{K}}(\phi).$$

If A is flat over $\mathbb{K}$ then there is an easy formula for the squaring:

$$\mathrm{Sq}_{A/\mathbb{K}}(M) = \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M).$$

But in general we have to use DG rings to define $\mathrm{Sq}_{A/\mathbb{K}}(M)$.

A *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}(M)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$.

A *rigid complex* over A relative to $\mathbb{K}$ is a pair (M, ρ), consisting of a complex $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and a rigidifying isomorphism ρ.

Suppose (N, σ) is another rigid complex. A *rigid morphism*

$$\phi : (M, \rho) \to (N, \sigma)$$

is a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A)$, such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathrm{Sq}_{A/\mathbb{K}}(M) \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \mathrm{Sq}_{A/\mathbb{K}}(\phi)} \\ N & \xrightarrow{\sigma} & \mathrm{Sq}_{A/\mathbb{K}}(N) \end{array}$$

is commutative.

We denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ the category of rigid complexes, and rigid morphisms between them.

Here is the important property of rigidity: if (M, ρ) is a rigid complex such that canonical morphism $A \to \mathrm{RHom}_A(M, M)$ is an isomorphism, then *the only automorphism of* (M, ρ) *in* $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ *is the identity*.

Rigid dualizing complexes were introduced by M. Van den Bergh [VdB] in 1997. Note that Van den Bergh considered dualizing complexes over a noncommutative ring A, and the base ring $\mathbb{K}$ was a field.

More progress (especially the passage from base field to base ring) was done in the papers "YZ" in the references.

Warning: the paper [YZ3] has several serious errors in the proofs, some of which were discovered (and fixed) by the authors of [AILN]. Fortunately all results in [YZ3] are correct, and an erratum is being prepared.

Further work on rigidity for commutative rings was done by Avramov, Iyengar, Lipman and Nayak. See [AILN, AIL] and the references therein.

2. Rigid Residue Complexes over Rings

Again A is an essentially finite type $\mathbb{K}$-ring.

The next definition is from [RD].

A complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called *dualizing* if it has finite injective dimension, and the canonical morphism $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism.

Grothendieck proved that for a dualizing complex R, the functor

$$\mathrm{RHom}_A(-, R)$$

is a duality (i.e. contravariant equivalence) of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R, ρ) such that R is dualizing.

We know that A has a rigid dualizing complex (R, ρ).

Moreover, any two rigid dualizing complexes are uniquely isomorphic in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.

If $A = K$ is a field, then its rigid dualizing complex R must be isomorphic to $K[d]$ for an integer d. We define the *rigid dimension* to be

$$\mathrm{rig.dim}_{\mathbb{K}}(K) := d.$$

Example 2.1. If the base ring $\mathbb{K}$ is also a field, then

$$\mathrm{rig.dim}_{\mathbb{K}}(K) = \mathrm{tr.deg}_{\mathbb{K}}(K).$$

On the other hand,

$$\mathrm{rig.dim}_{\mathbb{Z}}(\mathbb{F}_q) = -1$$

for any finite field $\mathbb{F}_q$.

For a prime ideal $\mathfrak{p} \in \mathrm{Spec}\, A$ we define

$$\mathrm{rig.dim}_{\mathbb{K}}(\mathfrak{p}) := \mathrm{rig.dim}_{\mathbb{K}}(\boldsymbol{k}(\mathfrak{p})),$$

where $\boldsymbol{k}(\mathfrak{p})$ is the residue field.

The resulting function

$$\mathrm{rig.dim}_{\mathbb{K}} : \mathrm{Spec}\, A \to \mathbb{Z}$$

has the expected property: it drops by 1 if $\mathfrak{p} \subset \mathfrak{q}$ is an immediate specialization of primes.

For any $\mathfrak{p} \in \mathrm{Spec}\, A$ we denote by $J(\mathfrak{p})$ the injective hull of the A-module $\boldsymbol{k}(\mathfrak{p})$. This is an indecomposable injective module.

A *rigid residue complex* over A relative to $\mathbb{K}$ is a rigid dualizing complex $(\mathcal{K}_A, \rho_A)$, such that for every i there is an isomorphism of A-modules

$$\mathcal{K}_A^{-i} \cong \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}\, A \\ \mathrm{rig.dim}_{\mathbb{K}}(\mathfrak{p}) = i}} J(\mathfrak{p}) \, .$$

A morphism $\phi : (\mathcal{K}_A, \rho_A) \to (\mathcal{K}'_A, \rho'_A)$ between rigid residue complexes is a homomorphism of complexes $\phi : \mathcal{K}_A \to \mathcal{K}'_A$ in $\mathsf{C}(\mathsf{Mod}\, A)$, such that

$$\mathrm{Q}(\phi) : (\mathcal{K}_A, \rho_A) \to (\mathcal{K}'_A, \rho'_A)$$

is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.

We denote by $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$ the category of rigid residue complexes.

The algebra A has a rigid residue complex $(\mathcal{K}_A, \rho_A)$.

It is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$. So we call it *the rigid residue complex* of A.

Let me mention several important functorial properties of rigid residue complexes.

Suppose $A \to B$ is an *essentially étale homomorphism* of $\mathbb{K}$-algebras.

There is a unique homomorphism of complexes

$$\mathrm{q}_{B/A} : \mathcal{K}_A \to \mathcal{K}_B,$$

satisfying suitable conditions, called the *rigid localization homomorphism*.

The homomorphism $\mathrm{q}_{B/A}$ induces an isomorphism of complexes $B \otimes_A \mathcal{K}_A \cong \mathcal{K}_B$.

If $B \to C$ is another essentially étale homomorphism, then

$$\mathrm{q}_{C/A} = \mathrm{q}_{C/B} \circ \mathrm{q}_{B/A} .$$

In this way rigid residue complexes form a quasi-coherent sheaf on the étale topology of $\operatorname{Spec} A$. This will be important for us.

Now let $A \to B$ any homomorphism between essentially finite type $\mathbb{K}$-algebras.

There is a unique *homomorphism of graded A-modules*

$$\mathrm{Tr}_{B/A} : \mathcal{K}_B \to \mathcal{K}_A,$$

satisfying suitable conditions, called the *ind-rigid trace homomorphism*.

It is functorial: if $B \to C$ is another algebra homomorphism, then

$$\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B} .$$

When $A \to B$ is a *finite* homomorphism, then $\mathrm{Tr}_{B/A}$ is a homomorphism of complexes.

The ind-rigid traces and the rigid localizations commute with each other.

Example 2.2. Take an algebraically closed field $\mathbb{K}$ (e.g. $\mathbb{K} = \mathbb{C}$), and let $A := \mathbb{K}[t]$, polynomials in a variable t.

The rigid residue complex of A is concentrated in degrees $-1, 0$:

$$\mathcal{K}_A^{-1} = \Omega^1_{\mathbb{K}(t)/\mathbb{K}} \xrightarrow{\partial_A = \sum \partial_{\mathfrak{m}}} \mathcal{K}_A^0 = \bigoplus_{\mathfrak{m} \subset A \text{ max}} \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_{\mathfrak{m}}, \mathbb{K})$$

Note that for a maximal ideal $\mathfrak{m} = (t - \lambda)$, $\lambda \in \mathbb{K}$, the complete local ring is $\widehat{A}_{\mathfrak{m}} = \mathbb{K}[[t - \lambda]]$.

The local component $\partial_{\mathfrak{m}}$ sends a meromorphic differential form α to the $\mathfrak{m}$-adically continuous functional $\partial_{\mathfrak{m}}(\alpha)$ on $\widehat{A}_{\mathfrak{m}}$ coming from the *residue pairing*:

$$\partial_{\mathfrak{m}}(\alpha)(a) := \mathrm{Res}_{\mathfrak{m}}(a\alpha) \in \mathbb{K}.$$

The rigid residue complex of $\mathbb{K}$ is just $\mathcal{K}_{\mathbb{K}}^0 = \mathbb{K}$.

Now consider the ring homomorphism $\mathbb{K} \to A$.

(cont.) The ind-rigid trace $\mathrm{Tr}_{A/\mathbb{K}}$ is the vertical arrows here:

$$\begin{array}{ccc}
\mathcal{K}_A^{-1} = \Omega^1_{\mathbb{K}(t)/\mathbb{K}} & \xrightarrow{\partial_A = \sum \partial_{\mathfrak{m}}} & \mathcal{K}_A^0 = \bigoplus_{\mathfrak{m} \subset A \text{ max}} \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{K}}(\widehat{A}_{\mathfrak{m}}, \mathbb{K}) \\
\Big\downarrow {\scriptstyle \mathrm{Tr}^{-1}_{A/\mathbb{K}} = 0} & & \Big\downarrow {\scriptstyle \mathrm{Tr}^{0}_{A/\mathbb{K}}} \\
\mathcal{K}_{\mathbb{K}}^{-1} = 0 & \xrightarrow{\partial_{\mathbb{K}} = 0} & \mathcal{K}_{\mathbb{K}}^0 = \mathbb{K}
\end{array}$$

The homomorphism $\mathrm{Tr}^0_{A/\mathbb{K}}$ is

$$\mathrm{Tr}^0_{A/\mathbb{K}}\big(\textstyle\sum_{\mathfrak{m}} \phi_{\mathfrak{m}}\big) := \textstyle\sum_{\mathfrak{m}} \phi_{\mathfrak{m}}(1) \in \mathbb{K}.$$

Taking $\alpha := \frac{dt}{t} \in \Omega^1_{\mathbb{K}(t)/\mathbb{K}}$, whose only pole is a simple pole at the origin, we have

$$(\mathrm{Tr}^0_{A/\mathbb{K}} \circ \partial_A)(\alpha) = 1.$$

We see that the diagram is not commutative; i.e. $\mathrm{Tr}_{A/\mathbb{K}}$ is *not a homomorphism of complexes*.

The last property I want to mention is *étale codescent*.

Suppose $u : A \to B$ is a faithfully étale ring homomorphism. This means that the map of schemes $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale and surjective.

Let $v_1, v_2 : B \to B \otimes_A B$ the two inclusions.

Then for every i the sequence of A-module homomorphisms

$$\mathcal{K}^i_{B \otimes_A B} \xrightarrow{\operatorname{Tr}_{v_1} - \operatorname{Tr}_{v_2}} \mathcal{K}^i_B \xrightarrow{\operatorname{Tr}_u} \mathcal{K}^i_A \to 0$$

is exact.

3. Rigid Residue Complexes over Schemes

Now we look at a finite type $\mathbb{K}$-scheme X. If $U \subset X$ is an affine open set, then $A := \Gamma(U, \mathcal{O}_X)$ is a finite type $\mathbb{K}$-ring.

Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. For any affine open set U, $\Gamma(U, \mathcal{M})$ is a $\Gamma(U, \mathcal{O}_X)$-module.

If $V \subset U$ is another affine open set, then

$$\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$$

is an *étale ring homomorphism*.

And there is a homomorphism

$$\Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

of $\Gamma(U, \mathcal{O}_X)$-modules.

A *rigid residue complex on* X is a complex $\mathcal{K}_X$ of quasi-coherent $\mathcal{O}_X$-modules, together with a rigidifying isomorphism ρ_U for the complex $\Gamma(U, \mathcal{K}_X)$, for every affine open set U.

There are two conditions:

(i) The pair $(\Gamma(U, \mathcal{K}_X), \rho_U)$ is a rigid residue complex over the ring $\Gamma(U, \mathcal{O}_X)$ relative to $\mathbb{K}$.

(ii) For an inclusion $V \subset U$ of affine open sets, the canonical homomorphism

$$\Gamma(U, \mathcal{K}_X) \to \Gamma(V, \mathcal{K}_X)$$

is the unique rigid localization homomorphism between these rigid residue complexes.

We denote by $\boldsymbol{\rho}_X := \{\rho_U\}$ the collection of rigidifying isomorphisms, and call it a *rigid structure*.

Suppose $(\mathcal{K}_X, \boldsymbol{\rho}_X)$ and $(\mathcal{K}'_X, \boldsymbol{\rho}'_X)$ are two rigid residue complexes on X.

A morphism of rigid residue complexes

$$\phi : (\mathcal{K}_X, \boldsymbol{\rho}_X) \to \mathcal{K}'_X, \boldsymbol{\rho}'_X)$$

is a homomorphism $\phi : \mathcal{K}_X \to \mathcal{K}'_X$ of complexes of $\mathcal{O}_X$-modules, such that for every affine open set U, with $A := \Gamma(U, \mathcal{O}_X)$, the induced homomorphism $\Gamma(U, \phi)$ is a morphism in $\mathsf{C}(\mathsf{Mod}\, A)_{\mathrm{res}/\mathbb{K}}$.

We denote the category of rigid residue complexes by $\mathsf{C}(\mathsf{QCoh}\, X)_{\mathrm{res}/\mathbb{K}}$.

Every finite type $\mathbb{K}$-scheme X has a rigid residue complex $(\mathcal{K}_X, \boldsymbol{\rho}_X)$; and it is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{QCoh}\, X)_{\mathrm{res}/\mathbb{K}}$.

Suppose $f : X \to Y$ is any map between finite type $\mathbb{K}$-schemes.

The complex $f_*(\mathcal{K}_X)$ is a bounded complex of quasi-coherent $\mathcal{O}_Y$-modules.

The ind-rigid traces for rings that we talked about before induce a *homomorphism of graded quasi-coherent* $\mathcal{O}_Y$*-modules*

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y, \tag{3.1}$$

which we also call the *ind-rigid trace homomorphism*.

It is functorial: if $g : Y \to Z$ is another map, then

$$\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f .$$

It is not hard to see that if f is a finite map of schemes, then Tr_f is a homomorphism of complexes.

4. Residues and Duality for Proper Maps of Schemes

Theorem 4.1. (Residue Theorem, [Ye2])

Let $f : X \to Y$ *be a proper map between finite type* $\mathbb{K}$*-schemes.*

Then the ind-rigid trace

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y$$

is a homomorphism of complexes.

The idea of the proof (imitating [RD]) is to reduce to the case when $Y = \operatorname{Spec} A$, A is a local artinian ring, and $X = \mathbf{P}^1_A$ (the projective line).

For this special case we have a proof that relies on the following fact: the diagonal map $X \to X \times_A X$ endows the A-module $\mathrm{H}^1(X, \Omega^1_{X/A})$ with a canonical rigidifying isomorphism relative to A.

Theorem 4.2. (Duality Theorem, [Ye2])

Let $f : X \to Y$ *be a proper map between finite type* $\mathbb{K}$*-schemes.*

Then for any $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, X)$ *the morphism*

$$\mathrm{R}f_*\big(\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X)\big) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_*(\mathcal{M}), \mathcal{K}_Y\big)$$

in $\mathsf{D}(\mathsf{Mod}\, Y)$*, that is induced by the ind-rigid trace*

$$\mathrm{Tr}_f : f_*(\mathcal{K}_X) \to \mathcal{K}_Y,$$

is an isomorphism.

The proof of Theorem 4.2 imitates the proof of the corresponding theorem in [RD], once we have the Residue Theorem 4.1 at hand.

The proofs of Theorems 4.1 and 4.2 are sketched in the incomplete preprint [YZ1]. Complete proofs will be available in [Ye2].

One advantage of our approach – using rigidity – is that it is much cleaner and shorter than the original approach in [RD]. This is because we can avoid complicated diagram chasing (that was not actually done in [RD], but rather in follow-up work by Lipman, Conrad and others). See Lipman's book [LH] for a full account.

Another advantage, as we shall see next, is that the rigidity approach gives rise to a useful duality theory for stacks.

5. Finite Type DM Stacks

Unfortunately I do not have time to give background on stacks. For those who do not know about stacks, it is useful to think of a Deligne-Mumford stack $\mathfrak{X}$ as a scheme, with an extra structure: the points of $\mathfrak{X}$ are clumped into finite groupoids.

Here are some good references on algebraic stacks: [LMB], [SP] and [Ol].

Before going on, I should mention the paper [Ni] by Nironi, that also addresses Grothendieck duality on stacks. The approach is based on Lipman's work in [LH]. Not all details in that paper are clear to me.

Dualizing complexes on stacks are also discussed in [AB], but that paper does not touch Grothendieck duality for maps of stacks.

We will only consider noetherian finite type DM $\mathbb{K}$-stacks.

Let $\mathfrak{X}$ be such a stack. If $g : U \to \mathfrak{X}$ is an étale map from an affine scheme, then $\Gamma(U, \mathcal{O}_U)$ is a finite type $\mathbb{K}$-ring.

The definition of a rigid residue complex on $\mathfrak{X}$ is very similar to the scheme definition.

A *rigid residue complex on* $\mathfrak{X}$ is a complex of quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules $\mathcal{K}_{\mathfrak{X}}$, together with a rigid structure $\boldsymbol{\rho}_{\mathfrak{X}}$.

However here the indexing of the rigid structure $\boldsymbol{\rho}_{\mathfrak{X}} = \{\rho_{(U,g)}\}$ is by étale maps $g : U \to \mathfrak{X}$ from affine schemes.

For any such (U, g) there is a rigidifying isomorphism $\rho_{(U,g)}$ for the complex $\Gamma(U, g^*(\mathcal{K}_{\mathfrak{X}}))$, and the pair

$$\big(\Gamma(U, g^*(\mathcal{K}_{\mathfrak{X}})), \rho_{(U,g)}\big)$$

is a rigid residue complex over the ring $\Gamma(U, \mathcal{O}_U)$ relative to $\mathbb{K}$.

The compatibility condition is this: suppose we have a commutative diagram of étale maps

$$\begin{array}{ccc} U_2 & \xrightarrow{h} & U_1 \\ & \underset{g_2}{\searrow} & \downarrow{\scriptstyle g_1} \\ & & \mathfrak{X} \end{array}$$

where U_1 and U_2 are affine schemes.

Then the homomorphism of complexes

$$h^* : \Gamma(U_1, g_1^*(\mathcal{K}_{\mathfrak{X}})) \to \Gamma(U_2, g_2^*(\mathcal{K}_{\mathfrak{X}}))$$

is the unique rigid localization homomorphism, w.r.t. $\rho_{(U_1,g_1)}$ and $\rho_{(U_2,g_2)}$.

Theorem 5.1. ([Ye3]) *Let $\mathfrak{X}$ be a finite type DM stack over $\mathbb{K}$.*

The stack $\mathfrak{X}$ has a rigid residue complex $(\mathcal{K}_{\mathfrak{X}}, \boldsymbol{\rho}_{\mathfrak{X}})$. It is unique up to a unique rigid isomorphism.

The proof is by étale descent for quasi-coherent sheaves.

Theorem 5.2. ([Ye3]) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between finite type DM $\mathbb{K}$-stacks.*

There is a homomorphism of graded quasi-coherent $\mathcal{O}_{\mathfrak{Y}}$-modules

$$\mathrm{Tr}_f : f_*(\mathcal{K}_{\mathfrak{X}}) \to \mathcal{K}_{\mathfrak{Y}}$$

called the ind-rigid trace, *extending the ind-rigid trace on $\mathbb{K}$-algebras.*

The proof relies on the étale codescent property of the ind-rigid trace.

The obvious question now is: do the Residue Theorem and the Duality Theorem hold for a proper map $f : \mathfrak{X} \to \mathfrak{Y}$ between stacks?

I only know a partial answer.

By the Keel-Mori Theorem, a separated stack $\mathfrak{X}$ has a *coarse moduli space* $\pi : \mathfrak{X} \to X$. The map π is proper and quasi-finite, and X is, in general, an *algebraic space*.

Let us call $\mathfrak{X}$ a *coarsely schematic stack* if its coarse moduli space X is a scheme.

This appears to be a rather mild restriction: most DM stacks that come up in examples are of this kind.

A map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *coarsely schematic map* if for some surjective étale map $V \to \mathfrak{Y}$ from an affine scheme V, the stack

$$\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} V$$

is coarsely schematic.

Theorem 5.3. (Residue Theorem, [Ye3])

Suppose $f : \mathfrak{X} \to \mathfrak{Y}$ is a proper coarsely schematic map between finite type DM $\mathbb{K}$-stacks.

Then the rigid trace

$$\mathrm{Tr}_f : f_*(\mathcal{K}_{\mathfrak{X}}) \to \mathcal{K}_{\mathfrak{Y}}$$

is a homomorphism of complexes of $\mathcal{O}_{\mathfrak{Y}}$-modules.

It is not expected that duality will hold in this generality. In fact, there are easy counter examples. The problem is *finite group theory in positive characteristics*!

Following [AOV], a separated stack $\mathfrak{X}$ is called *tame* if for every algebraically closed field K, the automorphism groups in the finite groupoid $\mathfrak{X}(K)$ have orders prime to the characteristic of K.

A separated map $f : \mathfrak{X} \to \mathfrak{Y}$ is called a *tame map* if for some surjective étale map $V \to \mathfrak{Y}$ from an affine scheme V, the stack $\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} V$ is tame.

Theorem 5.4. (Duality Theorem, [Ye3])

Suppose $f : \mathfrak{X} \to \mathfrak{Y}$ is a proper tame coarsely schematic map between finite type DM $\mathbb{K}$-stacks.

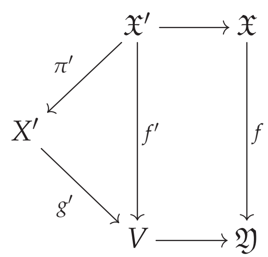
Then Tr_f induces duality (as in Theorem 4.2).

Remark 5.5. It is likely that the "coarsely schematic" condition could be removed from these theorems; but I don't know how.

Here is a sketch of the proofs of Theorems 5.3 and 5.4.

Take a surjective étale map $V \to \mathfrak{Y}$ from an affine scheme V such that the stack $\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} V$ is coarsely schematic.

Consider the commutative diagram of maps of stacks



where f' is gotten from f by base change, and X' is the coarse moduli space of $\mathfrak{X}'$.

It suffices to prove "residues" and "duality" for the map f'.

Because X' is a scheme, the proper map g' satisfies both "residues" and "duality" (by Theorems 4.1 and 4.2).

It remains to verify "residues" and "duality" for the map $\pi' : \mathfrak{X}' \to X'$.

These properties are étale local on X'.

Namely let $U'_1, \ldots, U'_n$ be affine schemes, and let

$$\coprod_i U'_i \to X' \tag{5.6}$$

be a surjective étale map.

For any i let

$$\mathfrak{X}'_i := \mathfrak{X}' \times_{X'} U'_i .$$

It is enough to check "residues" and "duality" for the maps $\pi'_i : \mathfrak{X}'_i \to U'_i$.

$$\begin{array}{ccc} \coprod_i \mathfrak{X}'_i & \longrightarrow & \mathfrak{X}' \\ {\scriptstyle \coprod_i \pi'_i} \downarrow & & \downarrow {\scriptstyle \pi'} \\ \coprod_i U'_i & \longrightarrow & X' \end{array}$$

Note that U'_i is the coarse moduli space of the stack $\mathfrak{X}'_i$.

It is possible to choose a covering (5.6) such that

$$\mathfrak{X}'_i \cong [W_i / G_i] \quad \text{and} \quad U'_i \cong W_i / G_i .$$

Here W_i is an affine scheme, G_i is a finite group acting on W_i, $[W_i / G_i]$ is the quotient stack, and W_i / G_i is the quotient scheme.

Moreover, in the tame case we can assume that the order of the group G_i is invertible in the ring $\Gamma(U'_i, \mathcal{O}_{U'_i})$.

We have now reduced the problem to proving "residues" and "duality" for the map of stacks

$$\pi : [W/G] \to W/G,$$

where $W = \operatorname{Spec} A$ for some ring A, and G is a finite group acting on A.

The proofs are by direct calculations, using the fact that

$$\mathsf{QCoh}\,[W/G] \approx \mathsf{Mod}^G A\,,$$

the category of G-equivariant A-modules, and under this equivalence the functor π_* becomes $\pi_*(M) = M^G$.

- END -

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