Monodromy groups of $F$-isocrystals

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Let $p$ be a prime, $q$ a power of $p$ and $X_0$ a smooth geometrically connected variety over $\mathbb{F}_q$. Moreover, let $\mathbb{Q}_q$ be $\text{Frac}(W(\mathbb{F}_q))$.
We denote by $\text{Isoc}^\dagger(X_0)$ the category of overconvergent isocrystals on $X_0$.

Lisse sheaf on $X_0 \rightsquigarrow$ continuous $\ell$-adic representation of $\pi^\text{ét}_1(X_0)$

Thanks to the Tannakian formalism:

Overconvergent isocrystal $\rightsquigarrow$ representation of an affine group scheme
Definition

Let $\mathbb{K}$ be a field, a $\mathbb{K}$-linear neutral Tannakian category is an abelian $\mathbb{K}$-linear category $\mathcal{C}$ with the following additional properties:

1. It is endowed with a symmetric monoidal structure $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that is $\mathbb{K}$-linear, bi-additive, associative, commutative, and it admits a unit object $1$;
2. $\text{End}(1) \cong \mathbb{K}$;
3. $\forall M \in \mathcal{C}$ there exist $M^\vee$, $\text{ev} : M \otimes M^\vee \to 1$ and $\delta : 1 \to M^\vee \otimes M$ such that the compositions
   
   $$
   M \xrightarrow{id_M \otimes \delta} M \otimes M^\vee \otimes M \xrightarrow{\text{ev} \otimes id_M} M
   $$
   
   $$
   M^\vee \xrightarrow{\delta \otimes id_{M^\vee}} M^\vee \otimes M \otimes M^\vee \xrightarrow{id_{M^\vee} \otimes \text{ev}} M
   $$
   
   are the identity maps;
4. There exists a faithful exact $\mathbb{K}$-linear functor $\omega : \mathcal{C} \to \text{Vec}_\mathbb{K}$ that preserves the monoidal structure. We call such an $\omega$ a fiber functor for $\mathcal{C}$. 
Reconstruction theorem

Theorem (Grothendieck, Saavedra-Rivano, Deligne)

Let $\mathcal{C}$ be a $\mathbb{K}$-linear neutral Tannakian category. Every fiber functor $\omega$ induces an equivalence of Tannakian categories

$$\mathcal{C} \cong \text{Rep}_\mathbb{K}(\text{Aut}^\otimes(\omega)).$$

The group $\text{Aut}^\otimes(\omega)$ is an affine group scheme over $\mathbb{K}$. It is called the Tannakian group of $\mathcal{C}$ with respect to $\omega$, denoted by $G(\mathcal{C}, \omega)$. 
Let \((\mathcal{C}, \omega_\mathcal{C})\) and \((\mathcal{D}, \omega_\mathcal{D})\) be two \(K\)-linear Tannakian categories endowed with fiber functors \(\omega_\mathcal{C}\) and \(\omega_\mathcal{D}\). Let \(\varphi : \mathcal{C} \to \mathcal{D}\) be a functor of Tannakian categories commuting with the fiber functors. Then \(\varphi\) induces a natural morphism

\[ \varphi^* : G(\mathcal{D}, \omega_\mathcal{D}) \to G(\mathcal{C}, \omega_\mathcal{C}). \]
Monodromy of isocrystals

Proposition (Ogus, Crew)

If $X_0(\mathbb{F}_q) \neq \emptyset$, the category $\text{Isoc}^\dagger(X_0)$ is a $\mathbb{Q}_q$-linear neutral Tannakian category.

We will assume from now on that $X_0(\mathbb{F}_q) \neq \emptyset$. We introduce the following notation:

- $\pi_1^{\text{Isoc}^\dagger}(X_0) :=$ the Tannakian group of $\text{Isoc}^\dagger(X_0)$, called the *isocrystal fundamental group*;
- $G(M) :=$ the Tannakian group of $\langle M \rangle \otimes \subseteq \text{Isoc}^\dagger(X_0)$, called the *monodromy group of $M$*.

The affine group scheme $G(M)$ is of finite type over $\mathbb{Q}_q$ and it is a quotient of the isocrystal fundamental group.

$$\pi_1^{\text{Isoc}^\dagger}(X_0) \twoheadrightarrow G(M).$$
Frobneius structure

\[ F_{X_0} : X_0 \to X_0 \text{ the } q\text{-th power Frobenius endomorphism.} \]

**Definition**

A *Frobenius structure* for \( M \) is an isomorphism \( \Phi : F_{X_0}^* M \rightarrow M \). Such a pair \((M, \Phi)\) is called an *overconvergent F-isocrystal*.

An overconvergent *F*-isocrystal is said to be *unit-root* if \( \forall x_0 \in |X_0| \) the roots of the Frobenius characteristic polynomial at \( x_0 \) are \( p \)-adic units.
Main theorem on unit-root $F$-isocrystals

**Theorem (Katz, Crew, Tsuzuki, Kedlaya, Shiho)**

There exists a canonical equivalence of $\mathbb{Q}_q$-linear neutral Tannakian categories

\[
\begin{pmatrix}
\text{unit-root} \\
\text{overconvergent} \\
F\text{-isocrystals}
\end{pmatrix}
\sim
\begin{pmatrix}
\text{continuous } \mathbb{Q}_q\text{-linear representations of } \pi_1^{\text{ét}}(X_0) \\
satisfying a certain condition at infinity
\end{pmatrix}
\]

We denote by $\rho(M, \Phi)$ the representation associated to $(M, \Phi)$. 
Main theorem on unit-root $F$-isocrystals

$M$ is controlled by the restriction of $\rho_{(M,\Phi)}$ to $\pi^\text{ét}_1(X_0 \otimes \overline{F}_q)$.

Fact

*There exists a dominant embedding for the $p$-adic topology*

$$\rho_{(M,\Phi)}(\pi^\text{ét}_1(X_0 \otimes \overline{F}_q)) \hookrightarrow G(M)(\mathbb{Q}_q).$$
**Theorem (MD’A)**

Let $A_0$ be an abelian variety over $\overline{\mathbb{F}}_q$ and $M$ be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on $Y_0$.

**Corollary (Tsuzuki’s theorem)**

For every $F$-isocrystal on $A_0$, the Newton polygon of the Frobenius characteristic polynomials at closed points is independent of the point.
The global monodromy theorem

**Proposition (Crew, Abe)**

Let $M$ be an overconvergent isocrystal of rank 1 such that $F_{X_0}^* M \simeq M$, then $G(M)$ is finite.

**Sketch of the proof.**

1. $M$ is a rank 1 overconvergent isocrystal $M$ that admits a Frobenius structure, thus it also admits a Frobenius structure $\Phi$ such that $(M, \Phi)$ is a unit-root overconvergent $F$-isocrystal.

2. As the representation $\rho_{(M, \Phi)}$ is of rank 1, its image is commutative.

3. (2) and the condition at infinity on $\rho_{(M, \Phi)}$ imply, by class field theory, that $\rho(\pi_1^{\text{ét}}(X_0 \otimes \overline{F}_q))$ is finite.

4. As $\rho(\pi_1^{\text{ét}}(X_0 \otimes \overline{F}_q))$ is dense in $G(M)(\mathbb{Q}_q)$ we conclude.
Theorem (The global monodromy theorem; Crew)

Let $M$ be an overconvergent isocrystal such that $F_{X_0}^* M \simeq M$. The radical subgroup of $G(M)$ (i.e. the greatest connected normal solvable subgroup) is unipotent.
The case of abelian varieties

A lemma

Let $A_0$ be an abelian variety over $\mathbb{F}_q$.

Lemma

For every overconvergent isocrystal $M$ on $A_0$, the algebraic group $G(M)$ is commutative.

Proof of the lemma.

Let $m_0 : A_0 \times A_0 \to A_0$ be the multiplication map of $A_0$, we take

$$\tilde{m}_* : \pi_1^{\text{Isoc}^\dagger}(A_0) \times \pi_1^{\text{Isoc}^\dagger}(A_0) \simeq \pi_1^{\text{Isoc}^\dagger}(A_0 \times A_0) \xrightarrow{m_*} \pi_1^{\text{Isoc}^\dagger}(A_0).$$

It endows $\pi_1^{\text{Isoc}^\dagger}(A_0)$ with a second group structure compatible with the structural one. By an Eckmann–Hilton argument, $\pi_1^{\text{Isoc}^\dagger}(A_0)$ is commutative. Hence the same is true for its quotient $G(M)$. 

□
Theorem (MD’A)

Let $A_0$ be an abelian variety over $\mathbb{F}_q$ and $M$ be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on $Y_0$.

Proof of the theorem

1. $M$ semi-simple $\Rightarrow$ $G(M)$ is a reductive group.
2. Previous lemma + (1) $\Rightarrow$ $G(M) \simeq \text{torus} \times \text{commutative finite group}$. In particular, the radical of $G(M)$ is $G(M)^\circ$.
3. Global monodromy theorem $\Rightarrow$ $G(M)^\circ$ is unipotent, hence trivial. Thus $G(M)$ is finite.
Theorem (MD’A)

Let $A_0$ be an abelian variety over $\overline{\mathbb{F}}_q$ and $M$ be a semi-simple overconvergent isocrystal such that $F_{A_0}^* M \simeq M$. Then there exists a finite étale cover $f_0 : Y_0 \to A_0$ such that $f_0^* M$ is trivial on $Y_0$.

Proof of the theorem.

4 An overconvergent isocrystal with finite monodromy admits a unit-root Frobenius structure. We denote by $\Phi$ one of these Frobenius structures of $M$.

5 $\rho_{(M,\Phi)}(\pi_1^\text{ét}(A_0 \otimes \overline{\mathbb{F}}_q))$ is finite, thus there exits $f_0 : Y_0 \to A_0$ finite étale such that $\rho_{(M,\Phi)}(\pi_1^\text{ét}(Y_0 \otimes \overline{\mathbb{F}}_q)) = 1$. Hence $f_0^* M$ is trivial.
References