Mirror symmetry for minuscule flag varieties

with Thomas Lam (U. of Michigan)

“Everything should be made as simple as possible, but not simpler.”
\[ \text{Gr}(k, n) = \{ k\text{-planes in } \mathbb{C}^n \} = \{ \text{full rank } n \times k \text{ matrices} \} / \text{GL}(k) \]

Pliicker embedding in \( \mathbb{P}^{(\binom{n}{k}-1)} \), given by Pliicker coordinates

\( I: k \)-element subset. \( p_I = \prod \text{k minor} \).

cohomology given by Schubert calculus.

Consider the union \( J \) of the divisors \( (p_{\{1\ldots k-3\}} = 0), (p_{\{1\ldots k-1\}} = 0) \ldots (p_{\{n-1\ldots k-1\}} = 0) \)

Let \( \text{Gr}(k, n) := \text{Gr}(k, n) - J \)

\[ f := \frac{p_{\{1\ldots k-1\} k+1}}{p_{\{1\ldots k\}}} + \frac{p_{\{2\ldots k\} k+2}}{p_{\{2\ldots k+1\}}} + \ldots + \frac{p_{\{n-1\ldots k-2\} k+3}}{p_{\{n-1\ldots k-1\}}} \]

\[ \text{THM (Lam–T, ’17)} \]

\[ H_{dR}^i(\text{Gr}(k, n), f) = \begin{cases} \mathbb{C}^{\binom{n}{k}} & \text{if } i = k(n-k) \\ 0 & \text{otherwise} \end{cases} \]
Projective homogeneous spaces: $G/P$ with $P$ parabolic subgroup.

**Grassmannian:** $\text{Gr}(k,n) = \{ k\text{-subspaces in } \mathbb{C}^n \}$

There is a transitive action by $G=\text{GL}(n,\mathbb{C})$. Restricting the action to the torus $T$ of diagonal matrices inside $\text{GL}(n,\mathbb{C})$, the fixed points are the coordinate subspaces spanned by the choice of $k$ coordinate vectors. So there are $\binom{n}{k}$ fixed points. This is the dimension of cohomology, and also the number of Pl"ucker coordinates $p_i$.

The stabilizer of a standard coordinate subspace is a parabolic subgroup $P$, i.e. the subgroup $(k,n-k)$ block triangular matrices.

**Example:** $\text{Gr}(1,n) = \{ \text{lines in } \mathbb{C}^n \} = \text{projective space } \mathbb{P}^{n-1}$

**Example:** $\text{Gr}(2,4) = \{ \text{planes in } \mathbb{C}^4 \} = \text{Klein 4-dimensional quadric} = \text{GL}(4, \mathbb{C})/\begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$

Pl"ucker embedding inside $\mathbb{P}(\Lambda^2 \mathbb{C}^4)$

Pl"ucker relation:

$$p_{12}p_{34} + p_{14}p_{23} = p_{13}p_{24}$$
Motivation to study flag manifolds appear in many different fields.

**Lie theory:** $G/B$ parametrizes the Borel subgroups of $G$.

**Algebraic geometry:** Tautological vector bundle. Characteristic classes.

**Geometric representation theory:** Borel-Weil-Bott theorem.

**Combinatorics:** generalizations of toric varieties. Replace torus action by group action.

**Enumerative geometry:** Schubert calculus.

**Number theory:** Geometry at the boundary of Shimura varieties. Harish-Chandra structure theory.
Notation. \( G \) complex reductive Lie group \( \Rightarrow P \) parabolic subgroup.
\( G^\vee \) dual group \( \Rightarrow P^\vee \) parabolic with same nodes as \( P \).
Partial flag variety \( G^\vee/P \) - homogeneous, smooth, projective, Fano.

Question: Mirror symmetry for flag varieties?

The open Richardson \( G/P \) - smooth, affine Calabi-Yau, cluster variety.

**Conjecture** (Rietsch '08) \( G^\vee/P \) is mirror to \( (G/P, f) \).

Kim-Givental '95: complete flag varieties, i.e. \( P, P^\vee = Borel \).

The conjecture emerged in relation with works by Lusztig, Zelevinsky, Fomin and others on crystals, Peterson and others on quantum Schubert calculus, Witten, Vafa and others on Landau-Ginzburg models.

In work with T. Lam we approach the problem via automorphic forms.
\[ a \frac{d}{da} - \begin{pmatrix} 0 & a \\ -1 & 0 \end{pmatrix} \]

\[ \int_{-\infty}^{\infty} e^{\frac{z^2}{2}} \frac{dz}{\sqrt{\pi}} = \sum_{r=0}^{\infty} \frac{a^r}{(r!)^2} = I_0(2\sqrt{a}) \]

\[ \int_{0}^{\infty} e^{\frac{z^2}{2}} \frac{dz}{\sqrt{\pi}} = 2 K_0(2\sqrt{a}) \]

\( I_0, K_0 \) are in the kernel of the Bessel operator

\[ \left( a \frac{d}{da} \right)^2 \] - a

Friedrich Wilhelm Bessel (1784 - 1846)
$CIP' = \frac{GL(2)}{(0\,*\,\star)}$

quantum connection is

$$a \frac{d}{da} - \begin{pmatrix} 0 & a \\ 1 & 0 \end{pmatrix}$$

Gelfand–Tsetlin

$$(\mathbb{C}^x, f_a)$$

$$(*)\quad f_a(z) = \sum_{\text{errors}} \frac{\text{head}}{\text{tail}} = z + \frac{a}{z}$$

$$\int_{-\infty}^{\infty} e^{z + \frac{a}{z}} \frac{dz}{z^2} = \sum_{r=0}^{\infty} \frac{a^r}{(r!)^2} = I_0(2\sqrt{a})$$

$$\int_{-\infty}^{0} e^{z + \frac{a}{z}} \frac{dz}{z} = 2 K_0(2\sqrt{a})$$

MIRROR $\Rightarrow$ $I_0, K_0$ are in the kernel of the Bessel operator

$$\left(\alpha \frac{d}{da}\right)^2 - a$$

Friedrich Wilhelm Bessel (1784-1846)
In 1838, Bessel made the first reliable measurement of the distance of a star. “61 Cygni” is a binary star about 10 light-years from Earth.
$D$-modules on $G_m$: $\mathcal{O} = \mathbb{C}[a, a^{-1}]$ structure sheaf

free $\mathcal{O}$-module = alg. vector bundle $V$ on $G_m$

Throughout this talk, $a$ will always denote the coordinate on $G_m$

$D = \mathbb{C}[a, a^{-1}] \langle \frac{d}{da} \rangle$ Weyl algebra. $[a, \frac{d}{da}] = 1$

scalar linear ODE $\leftrightarrow$ cyclic vector $\leftrightarrow$ matrix ODE $a \frac{d}{da} \psi = A \psi$

$Du = 0$

$D$-module that is $\mathcal{O}$-free $\leftrightarrow$ alg. vector bundle with connection $(V, \nabla)$

$D/V = M$ Hom$(V^\text{an}, \mathcal{O}^\text{an})$ flat sections $(V^\text{an})^D$

local system

example: $L = (a \frac{d}{da})^3 - \alpha$ matrix form $A = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$

$sF_2$ - hypergeometric
Example: Hypergeometric equation

\[ L = \frac{d}{da} \sum_{j=1}^{j} \left( a \frac{d}{da} + \beta_{j} - 1 \right) - \sum_{i=1}^{i} \left( a \frac{d}{da} + \alpha_{i} \right) \]

Hypergeometric D-module = \( \frac{D}{D_{L}} \)

Power series solution at the regular singular point \( a = 0 \)

\[ _{p}F_{q} \left( \alpha_{1}, \ldots, \alpha_{p} \mid \beta_{1}, \ldots, \beta_{q} \right) = \sum_{k=0}^{\infty} \frac{\left( \alpha_{1} \right)_{k} \cdots \left( \alpha_{p} \right)_{k}}{\left( \beta_{1} \right)_{k} \cdots \left( \beta_{q} \right)_{k}} \frac{a^{k}}{k!} \]

Special cases:

\[ \frac{d}{da} - 1 \quad _{0}F_{0} \left( a \right) = e^{a} \]

\[ \frac{d}{da} - (a \frac{d}{da} + \alpha) \quad _{1}F_{0} \left( \alpha \mid a \right) = (1 - a)^{-\alpha} \]

\[ \frac{d}{da} \left( a \frac{d}{da} + \beta - 1 \right) - 1 \quad _{0}F_{1} \left( \beta \mid a \right) = I_{0} \left( 2 \sqrt{a} \right) \text{ Bessel function} \]

\[ \frac{d}{da} \left( a \frac{d}{da} + \beta - 1 \right) - (a \frac{d}{da} + \alpha) \quad _{1}F_{1} \left( \alpha \mid a \right) \text{ is Kummer confluent hypergeometric} \]

\[ \frac{d}{da} \left( a \frac{d}{da} + \gamma - 1 \right) - (a \frac{d}{da} + \alpha) \left( a \frac{d}{da} + \beta \right) \quad _{2}F_{1} \left( \alpha, \beta \mid a \right) \text{ is Gauss hypergeometric function} \]
Crystal $D$-module is $\pi_! f^* D / D(\mathfrak{g}(\mathfrak{g}(-1)))$.
$W_p$: Weyl group of the Levi subgroup $L_p$ of $P$

$W_p^-$: minimal representative for $W/W_p$ in Bruhat order. $W_p^{-1}$: longest element of $W_p$.

$B_-$: opposite Borel. $\psi: U \rightarrow A'$ non-degenerate additive character.

**Berenstein-Kazhdan geometric crystal**: $U \cong (L_p) \otimes \rho \otimes U \cap B_-$

$a \in Z(L_p)$

$f_a(u, u_1, u_2) = \psi(u_1) + \psi(u_2)$ potential.

$w^{-1} \in G$

$X_a \xleftarrow{\sim} Fomin-Zelevinsky twist map $\Psi$

$B_- \otimes \rho \cap U \xrightarrow{\sim} \mathbb{R}^{w_o} \subset G/B$

Projected Richardson variety: $G/P$
Theorem (Lam-T'16)

If \( p \) is a minuscule parabolic, then there is an isomorphism

\[
G_C^\vee / p^\vee \cong \text{crystal } D\text{-module}
\]

for \( G^\vee / p^\vee \) and for \((G^\vee / p, \mathcal{F}_a)\)

\[
a \frac{d}{da} \rightarrow \sigma^* a
\]

connection 1-form = quantum multiplication by \( \sigma \), where \( \text{Pic}(G^\vee / p^\vee) = \mathbb{Z}_0 \)

pushforward \( D\)-module

\[
D = C [a, a^{-1}] \langle a \frac{d}{da} \rangle
\]

List of minuscule flag varieties (= compact Hermitian symmetric spaces)

- \( IP^n \) and Grassmannian \( \text{Gr}(k, n) \)
- Even-dimensional quadric
- Spinor variety = orthogonal Grassmannian \( \text{OG}(n, 2n) \)
- Cayley plane = projective (dim = 16)
- Octonions
- Freudenthal variety (dim = 27)
Example \( G/\rho = \mathbb{P}^2 \) \; anticanonical = \( O(3) \) \; \text{section} \; \text{vol}^{-1} = x_0 x_1 x_2 \rightarrow \text{divisor lines} \; \text{traditional lines}

complement of \( G/\rho =: G^\circ/\rho = \{ x_0 \neq 0, x_1 \neq 0, x_2 \neq 0 \} = G_m \times G_m \)

affine coordinates \( x_1 x_2 + x_1^2 + x_2^2 = \frac{a}{x_1 x_2} \)

intersection of two cubics: there are 9 indeterminacy points for the potential \( f_a \).

Fibers of \( f_a \) are elliptic curves.
Geometric summary

- $3$ anticanonical: $d_{\rho}$ multiplicity free union of Schubert divisors.
- Homogeneous projective $G/\rho$.
- $\Rightarrow$ $3$ vol$_{G/\rho}$ volume form with simple pole $d_{\rho}$ (Knutson-Lam-Speyer 09).
- $\Rightarrow G/\rho := \text{complement of } d_{\rho}$ is log CY.
  - Also $G/\rho \subset \mathbb{R}^{\nu}$ open Richardson.
  - $\mathbb{R}^{\nu} := B \cup B \cap B_{+\mu} B$ with $\mu \leq \nu$.
- $f_t: G/\rho \rightarrow \mathbb{A}^1$ regular function $H \in \mathbb{Z}(L_\rho)$ Bruhat opposite Bruhat (Berenstein-Kazhdan, Reidel 00 02).
- Upper cluster algebra (Berenstein-Fomin-Zelevinsky 03).

We can reformulate Rietsch conjecture 08 in a way compatible with recent work of Gross-Hacking-Keel (log CY), Katzarkov-Kontsevich-Pantev (compactified Fano) and Seidel (Lefschetz pencils):

$$G/\rho, d_{G/\rho}, \text{vol}_{G/\rho}, f_t \in \mathbb{C}(0),$$

mirror to

$$G^{\nu}/\rho, d_{G^{\nu}/\rho}, \text{vol}_{G^{\nu}/\rho}, f_t \in \mathbb{C}(\nu).$$
smooth projective Fano \quad \Downarrow \quad \text{Landau-Ginzburg model}

\quad (\text{quasi-projective Calabi-Yau, potential})

\quad \Downarrow \quad \text{pushforward D-module}

\text{small quantum differential equation} \quad \overset{\text{MIRROR}}{\sim}
Mirror theorem

Something you do not understand is equal to something you cannot compute.

B-side (complex geometry)  A-side (symplectic geometry)
Mirror theorem

Something you do not understand is equal to something you cannot compute.

B-side (complex geometry)
pushforward D-module $\mathcal{F}$
- Cohomology? Mixed Hodge structure?
- Weights (Deligne purity theorem)
- Compactification? Singularities?

A-side (symplectic geometry)
quantum differential equation $a \frac{d}{da} - 5a$
- How to solve for solutions?
- How to count number of rational curves?
  ( enumerative geometry, GW invariants)
Theorem (Lam - T '16)

If $\rho$ is a minuscule parabolic, then there is an isomorphism

$\text{quantum connection for } G/\rho \sim \text{crystal } D\text{-module for } (G/\rho, f\sigma)$

$$\frac{a}{d\sigma} \rightarrow$$

connection 1-form = quantum multiplication by $\sigma$, where $\text{Pic}(G/\rho) = \mathbb{Z}\sigma$.

$$\int_{G/\rho} e^{f\sigma}$$

pushforward $D$-module

$$D = \mathbb{C}[a, a^{-1}] \langle a, \frac{d}{da} \rangle$$
\[
\frac{d}{da} - \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}
\frac{f_a(z)}{z^2} := z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{a}{z_4}
\]

The mirror theorem implies that \( I(a) := \oint e^{f_a(z)} \frac{dz}{z^2} \) is the last entry of a solution.

The last entry of any solution of the quantum connection is also annihilated by the same operator (a binomial identity).

The mirror theorem also implies that we obtain an integral representation of all six independent solutions.
Example A_3

\[ \text{Gr}(2,4) = \frac{\text{GL}(4)}{\text{**} \times \text{**}} \]

= 4-dim quadric

quantum connection is given by

\[ \frac{d}{da} \quad a \to d \quad \left( \begin{array}{cccc}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right) \]

Gelfand-Teitlin coordinates

\[ \text{Landau-Ginzburg model} \]

given by the regular function

\[ f_a(z) := \sum \frac{\text{head}}{\text{tail}} \]

\[ = z_1 + \frac{z_2}{z_1} + \frac{z_3}{z_2} + \frac{z_4}{z_3} + \frac{z_5}{z_4} \]

The mirror theorem implies that \( I(a) := \oint e^{f_a(z)} \frac{dz}{z} \) is the last entry of a solution.

\underline{Elementary proof:} \( I(a) = \sum_{r \geq 0} \frac{(2r)!}{(r!)^6} a^r \) by Cauchy's residue theorem.

The series is annihilated by \( 15 - 2a(2d+1) \).

The last entry of any solution (a binomial identity)

of the quantum connection is also annihilated by the same operator. (Please excuse!)

1st order vector ODE \( \leftrightarrow \) high order scalar ODE \( \square \)

The mirror theorem also implies that we obtain an integral representation

of all six independent solutions.
Theorem (Lam-T '16)

If \( p^v \) is a minuscule parabolic, then there is an isomorphism

\[ 
\text{quantum connection} \quad \cong \quad \text{crystal D-module} 
\]

for \( G^v / p^v \) for \( (G/p, f_a) \)

\[ a \frac{d}{da} \rightarrow \sigma^* a \]

connection 1-form = quantum multiplication

by \( \sigma \), where \( \text{Pic} (G^v / p^v) = \mathbb{Z} \sigma \)

\[ \pi \quad \text{pushforward D-module} \]

\[ D = C \left[ a, a^{-1} \right] \langle a \frac{d}{da} \rangle \]

List of minuscule flag varieties (\( \subset \) compact Hermitian symmetric spaces)

- \( \mathbb{P}^n \) and Grassmannian \( \text{Gr}(k, n) \)
- Even-dimensional quadric
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- Cayley plane = projective (dim = 16) Octonions
- Freudenthal variety (dim = 27)
Corollary (Deligne-Yu filtration)
In particular, \( F^d H_{dR}^i(G/\rho, \Omega^d) = C \cdot \text{vol} \)

Deligne '84, '07: irregular Hodge filtration, that is not a Hodge structure.

Esnault-Sabbe-Yu '15
Kontsevich complex '12: \( t \)-adapted log forms \( \Omega^*_t \)

Example \( G/\rho = \mathbb{P}^2 \), \( G^0/\rho = \mathbb{G}_m^2 \), \( t = x_1 x_2 + \frac{1}{x_1 x_2} \)

\[
\begin{array}{cccc}
\circ & 0 & 0 & 0 \\
0 & \circ & 0 & 0 \\
0 & 0 & \circ & 0 \\
1 & 0 & 0 & h^0
\end{array}
\]

Hodge diamond of \( \mathbb{P}^2 \)

\[
\dim H^p(G_m^2, \Omega^q) = 0 \\
H^0(G_m^2, \Omega^2) = C \cdot \text{id} = C \frac{dx_1 dx_2}{x_1 x_2} \\
p + q = 2 \text{ (purity)}
\]
Example: $6$-dimensional quadric $\cong \frac{SO(8)}{P}$

Hasse diagram

middle cohomology is 2-dim.

quantum connection is

$$a \frac{d}{da} - \begin{pmatrix} 0 & a^0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

$7$-dim stable subspace generated by $\varepsilon$

$= \mathcal{D} / \mathcal{D} (7 - 2a (2a + i))$ where $\mathcal{D} = \mathbb{C}[a, a^i] \langle \varepsilon \rangle$

Thm (Katz, Frenkel-Grave) The monodromy group is $G_2$.

because of $S_3$-symmetry of $D_4$

because it is the $(1, 6)$-hypergeometric $F_6(\begin{pmatrix} \frac{1}{2} \\ 1 1 1 1 \end{pmatrix}; a)$

thm 4.1.5 in "Exponential sums and diff equations", Annals of Math Studies.
Mirror theorem

Something you do not understand is equal to something you cannot compute.

B-side (complex geometry)

A-side (symplectic geometry)
Mirror theorem

Something you do not understand is equal to something you cannot compute.

B-side (complex geometry)

A-side (symplectic geometry)

Our proof is via a complicated thing that you can half understand / compute.

 automorphic form (number theory)
Kloosterman sums, brief timeline.

- First appears in Poincare 1912: Fourier expansion of Poincare series.
  \[
  Kl(a) := \sum_{x \in \mathbb{F}_p^2} e\left(x + \frac{a}{x}\right)
  \]

- First application by Kloosterman 1926: quadratic forms in four variables.

- Weil’s bound, 1948, consequence of RH for curves:\[|Kl(a)| \leq 2\sqrt{p}\]

- Deligne SGA41/2, hyper-Kloosterman sums:
  \[
  Kl_n(a) := \sum_{x_1x_2\cdots x_n=a} e\left(x_1 + x_2 + \cdots + x_n\right) \quad |Kl_n(a)| \leq np^{n-1/2}
  \]


- Bump-Friedberg-Goldfeld: Fourier expansion of Poincare series and Peterson trace formula for GL(n).

- Jacquet-Ye fundamental lemma, Ngo Ph.D. 1997


idea of proof: via automorphic forms

quantum $G/V_{p^v}$

Langlands reciprocity

rigid automorphic form $A_G$ on $I_P$

Kloosterman sheaves as Hecke eigenvalues

Heinloth-Ngô-Yun '13: construct $A_G$ tame unipotent at $I_P$

Zhu '16: quantization Hitchin system in the ramified case.

we identify the crystal $D$-module as the automorphic side. (technically our main result)
Gross automorphic form $A_G$

Let G be a complex reductive group. Gross constructed an automorphic form $A_G$ which one can think as the simplest automorphic form.

**Theorem** (Gross) There exists a unique automorphic form $A_G$ over $P^1$ which is Steinberg at zero, *simple supercuspidal* at infinity and unramified otherwise.

It is rigid, similarly as Riemann’s theory of Gauss hypergeometric function. The proof is via the *simple trace formula*.

“Everything should be made as simple as possible, but not simpler”

On the Galois side it coincides with some of the local systems found by Katz.
Heinloth-Ngo-Yun constructed $A_G$ by writing down a newvector inside as the trace function of an $l$-adic sheaf.

I like to think of their construction as a far-reaching generalization of Poincaré $q$-expansion of Poincaré series. Nowadays known as Petersson trace formula.

We are going to use the construction of Heinloth-Ngo-Yun which also works over the complex numbers in the sense of geometric Langlands.

Historical note: This was Poincaré’s last paper written in 1912 a few days before he died. Whereas Poincaré series was the first major work of Poincaré, during the years 1880-1882, when he discovered automorphic forms, the theory of Fuchsian and Kleinian groups, the uniformization theorem, monodromy, etc.
idea of proof: via automorphic forms

\[ G^{\omega}/\rho \quad \text{MIRROR THM} \quad (G^{\omega}_{/\rho}, \phi) \]

- Langlands reciprocity
- rigid automorphic form \( A_G \) on \( \Gamma' \)

Kleisterman sheaves as Hecke eigenvalues

Heinloth-Ngô-Yun '13: construct \( A_G \) tame ramified at \( 0 \)
Zhu '16: quantization Hitchin system in the ramified case

remark. Witten "gauge theory and wild ramification" '07 relates Langlands reciprocity and T-duality of the Hitchin systems for \( G \) and \( G' \)
See also Hausel-Thaddeus, Kapustin-Witten, Gukov-Witten, Baalch, Donagi-Pantev, ...
summary

Q. What number theory brings to mirror symmetry?

Purity, weight-monodromy, Ramanujan conj, are at the heart of number theory and automorphic forms. There are statements inside mirror symmetry that do involve purity. In this talk I focus on those statements.

Q. What mirror symmetry brings to number theory?

Hodge structures, which could be transported to congruences via p-adic Hodge theory. Asymptotics, Purity, which could be exploited directly.
Deligne’s purity theorem

Let \( p \) be a prime number, the finite field \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \), and \( n \geq 2 \). Define the hyper Kloosterman sums by

\[
Kl_n = \sum_{x_1, \ldots, x_{n-1} \in \mathbb{F}_p^*} e^{\frac{2\pi i}{p} \left( x_1 + \cdots + x_{n-1} + \frac{1}{x_1 \cdots x_{n-1}} \right)}
\]

Each term is a \( p^{\text{th}} \)-root of unity and there are \((p-1)^{n-1}\) terms. Very important in number theory is that there is “square-root cancellation”

\[
\text{Deligne’s bound: } |Kl_n| \leq np^{\frac{n-1}{2}}
\]

This can be thought as the Riemann Hypothesis over finite fields. For \( n = 2 \), this is Weil’s bound for Kloosterman sums.
\[ Kl_n(a) = \sum_{x_i \cdots x_n \in \mathbb{F}_p^\times} e_p\left(x_1 + \cdots + x_{n-1} + \frac{a}{x_1 \cdots x_{n-1}}\right) \]

\text{THM (Deligne SGA 4\frac{1}{2}, Sperber 77)} \quad (n=2, \text{ Weil 48, Drinfeld 74})

Wan 04

for every \( a \in \mathbb{F}_p^\times \)

(i) \( Kl_n(a) = \alpha_1 + \cdots + \alpha_n \) is a sum of \textbf{Weil numbers} of \textbf{wt} \( n-1 \)

(ii) \( \nu_p(\alpha_1) = 0, \nu_p(\alpha_2) = 1 \cdots \nu_p(\alpha_n) = n-1 \)

The proof is deep: \textbf{Weil's conjecture, Drinfeld p-adic cohomology}

(i) is Deligne \textbf{purity: cohomological calculation}.

(ii) are the \textbf{slopes} of the Newton polygon of the \textbf{L-function}. 

\[ \left( G^2_m, x_1 + x_2 + \frac{a}{x_1 x_2} \right) \]

\[ \textit{Mirror} \]

\[ p^2 \]

"Hodge numbers for $K3(a)"\]

Hodge diamond of $p^2$ generated by the hyperplane class $\sigma$ (purely algebraic)

\[ h^p_9 = \begin{cases} 1 & \text{if } p+q=2 \\ 0 & \text{otherwise} \end{cases} \]

\[ H^p_9(p^2) = \begin{cases} C \cdot \sigma^p & \text{if } p=q \\ 0 & \text{o/w} \end{cases} \]

\[ \text{purity} \xleftarrow{\text{Mirror}} \text{Hodge-Tate type} \]

\[ \text{slope} \xleftarrow{\text{Mirror}} \text{degrees} \]