Positivity of Lagrangians and vanishing of cohomology for microfunctions at the boundary

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Abstract

We compare the microlocal Levi forms of a pair of real submanifolds of class $C^2$ of a complex manifold, and discuss their geometric meaning under contact transformation.

This enables us to recover, and even improve, our former results in [2], [6] on vanishing of cohomology for microfunctions which are the “boundary version” of those in [1] and [9].

1 General Notations

Let $X$ be a complex analytic manifold, denote by $\pi : T^*X \longrightarrow X$ its conormal bundle, and by $\dot{\pi} : \dot{T}^*X \longrightarrow X$ the conormal bundle with the zero-section removed. Denote by $\overline{X}$ the complex conjugate to $X$, and by $X^R$ the underlying real analytic manifold. The identification of $X^R$ to the diagonal of $X \times \overline{X}$ induces an identification $T^*(X^R) \simeq (T^*X)^R$ that we will often use in the following. In particular, if $M$ is a $C^2$-submanifold of $X^R$, we identify its conormal bundle to a $C^1$-submanifold $T^*_M X$ of $(T^*X)^R$.

Let $M \subset X^R$ be a $C^2$-submanifold, let $p \in T^*_M X$, and set $x = \pi(p)$. Let $\phi$ be a $C^2$-function at $x$ such that $\phi|_M = 0$, and $d\phi(x) = p$. Choose a system of local coordinates $(z)$ on $X$. The Hermitian form $L_M(p)$ on the complex tangent space $T^C_x M = T_x M \cap iT_x M$ with matrix $(\partial^2_{z\bar{z}} \phi(x))_{ij}$, neither depends on the choice of the equation $\phi = 0$ of $M$, nor on the system of coordinates. This is called the Levi form of $M$ at $p$, and we denote by $s^+_M, s^-_M, s^0_M(p)$ the numbers of positive, negative, and null eigenvalues of $L_M(p)$.

We denote by $D^b(X)$ the derived category of the category of complexes of sheaves of $\mathbb{C}$-vector spaces on $X$ with bounded cohomology. For $F \in D^b(X)$,
we denote by $SS(F)$ the micro-support of $F$ in the sense of [9], [10]. This is a closed conic involutive set of $T^*X^\mathbb{R}$ which describes the directions of "non-propagation" for the cohomology of $F$. We denote by $D^b(X;p)$ the localization of $D^b(X)$ by the null-system $\{F \in D^b(X); p \notin SS(F)\}$.

Let $O_X$ be the sheaf of holomorphic functions on $X$. If $A$ is a locally closed set of $X$, we denote by $C_A$ the sheaf which is 0 on $X \setminus A$ and the constant sheaf with stalk $C$ on $A$. We will make use of the complex $\mu_A(O_X) = \mu_{hom}(C_A, O_X)$, where $\mu_{hom}(\cdot, \cdot)$ denotes the bifunctor of microlocalization defined by [9] (cf also [15]). Recall that if $M$ is a real analytic manifold, and $X$ is a complexification of $M$, then (up to the orientation and a shift) $\mu_M(O_X)$ is the sheaf of Sato’s microfunctions on $T^*_M X$. Moreover, according to [15], if $\Omega \subset M$ is an open subset, the complex $\mu_\Omega(O_X)$ is the natural framework for the study of microlocal boundary value problems.

2 Positivity

Let $X$ be a complex manifold of dimension $n$, and let $M$ be a $C^2$-submanifold of $X^\mathbb{R}$ of codimension $l$. Let $p \in T^*_M X$, let $x = \pi(p)$, and set:

\[
\begin{align*}
d^-_M(p) &= l + s_M(p) - \gamma_M(x), \\
d^+_M(p) &= n - s^+_M(p) + \gamma_M(x),
\end{align*}
\]

where $\gamma_M(x) = \dim ((T^*_M X)_x \cap i(T^*_M X)_x)$ is the “lack of genericity” of $M$ (recall that $M \subset X$ is generic if and only if $\gamma_M(x) \equiv 0$).

**Lemma 2.1.** (cf [3]) Let $S \subset M \subset X$ be $C^2$-submanifolds, and let $p \in S \times_M \dot{T}^*_M X$. Then

\[
d^\pm(p) \leq d^\pm_S(p) \leq d^\pm_M(p) + \text{codim}_M S.
\]

In [4] (see [7]) a notion of positivity for Lagrangian manifolds is introduced, after the works [11], [12], [14]. Restricting the attention to conormal bundles, the aim of this section is to relate this notion to the following conditions (2.4)–(2.6) (cf Remark 2.4).

Let $(M_1, M_2)$ be a pair of $C^2$-submanifolds of $X$, and let $p \in T^*_M X \cap \dot{T}^*_M X$. We shall consider the conditions:

\[
R = M_1 \cap M_2 \text{ is a submanifold of class } C^2; \tag{2.4}
\]

\[
i p \notin T^*_R X, \tag{2.5}
\]

\[
d_M(p) = \text{codim}_{M_2} R + d_{M_2}(p). \tag{2.6}
\]

Notice that if $S \subset M$, $i p \notin T^*_S X$ and $\text{codim}_M S = 1$, then Lemma 2.1 implies that either $(M, S)$ or $(S, M)$ satisfy (2.6).
Theorem 2.2. Let \((M_1,M_2)\) satisfy (2.4)–(2.6). Then there exists a germ of complex homogeneous contact transformation \(\chi\) at \(p \in \tilde{T}^*X\) such that, for \(q = \chi(p), y = \pi(q)\):

\[
\begin{cases}
\chi(T_{M_1}^*X) = T_{N_1}^*X \text{ for } N_1 \subset X \text{ with codim}_X N_1 = 1, \ s_{N_1}^-(q) = 0, \\
\chi(T_{M_2}^*X) = T_{N_2}^*X \text{ for } N_2 \subset X \text{ with codim}_X N_2 = 1, \ s_{N_2}^-(q) = 0, \\
N_1^+ \supset N_2^+ \text{ at } y,
\end{cases}
\tag{2.7}
\]

where \(N_1^+, N_2^+\) denote the closed half-spaces with boundary \(N_1, N_2\) and interior conormal \(q\).

Proof. (a) By Lemma 2.1 one has \(d_{M_1} \leq d_R \leq d_{M_2} + \text{codim}_{M_2} R\), and hence (2.6) is satisfied if and only if \(d_{M_1} = d_R = d_{M_2} + \text{codim}_{M_2} R\).

(b) Conditions (2.4)–(2.5) ensure that there exists a germ of contact transformation \(\chi\) at \(p\) such that:

\[
\begin{cases}
\chi(T_{M_1}^*X) = T_{N_1}^*X, \ \text{codim}_X N_1 = 1, \ s_{N_1}^- = 0, \\
\chi(T_{M_2}^*X) = T_{N_2}^*X, \ \text{codim}_X N_2 = 1, \\
\chi(T_R^*X) = T_{S}^*X, \ \text{codim}_X S = 1,
\end{cases}
\tag{2.8}
\]

(c) Notice that since \(N_i\) and \(S\) are hypersurfaces, one has the isomorphisms \(\mathcal{C}_{N_i} \simeq \mathcal{C}_{N_i}^+, \mathcal{C}_S \simeq \mathcal{C}_S^+\) in \(D^b(X;q)\). According to [10, Chapter 7], let

\[
\Phi_K : D^b(X;p) \xrightarrow{\sim} D^b(X;q)
\]

be a quantization of \(\chi\). Here \(K \in D^b(X \times X; (p, -q))\) is a simple sheaf of shift \(-n\) at \((p, -q)\) along the Lagrangian manifold associated to \(\chi\). The restriction morphism \(\mathcal{C}_{M_1} \rightarrow \mathcal{C}_R\) is transformed by \(\Phi_K\) to a non null morphism

\[
\mathcal{C}_{N_1}^+ \rightarrow \mathcal{C}_S^+ [d_R(p) - d_{M_1}(p) - s_S^-] = \mathcal{C}_S^+ [-s_S^-].
\]

Since

\[
\text{Hom}_{D^b(X;q)}(\mathcal{C}_{N_1}^+, \mathcal{C}_S^+)[-s_S^-] \simeq H^{-s_S^-}(\text{Rf}_{N_1}^+(\mathcal{C}_S^+))_y,
\]

and the first term is non zero, it follows that \(s_S^- = 0\) and \(S^+ \subset N_1^+\) at \(y\).

(d) The restriction morphism \(\mathcal{C}_{M_2} \rightarrow \mathcal{C}_R\) induces by duality a non zero morphism \(\mathcal{C}_R \rightarrow \mathcal{C}_{M_2} [\text{codim}_{M_2} R]\). Applying \(\Phi_K\), we get a non zero morphism

\[
\mathcal{C}_S^+ \rightarrow \mathcal{C}_{N_2}^+ [\text{codim}_{M_2} R + d_{M_2}(p) - d_R(p) - s_{N_2}^-] = \mathcal{C}_{N_2}^+ [-s_{N_2}^-].
\]

Thus again, \(s_{N_2}^- = 0\) and \(N_2^+ \supset S^+\) at \(y\). This proves the statement. \(\square\)
Recall that an object \( F \in D^b(X) \) which verifies \( SS(F) \subset T^*_M X \) at \( p \), is microlocally isomorphic (i.e. isomorphic in the category \( D^b(X; p) \)) to \( C_M \) for a complex of \( \mathbb{C} \)-vector spaces \( C \). This criterion, stated in [9, Proposition 6.2.1] for \( C^2 \)-submanifolds \( M \), easily extends to \( C^1 \)-submanifolds (cf [2]).

Recall from [4] that a complex \( L_M \in D^b(X; p) \) is called a Levi simple sheaf along \( T^*_M X \) at \( p \), when it is simple along \( T^*_M X \) with shift \(-d_M(p) + \frac{1}{2} \text{codim}_X M \) at \( p \). Levi simple sheaves are unique up to isomorphism, and we recall from [9, Chapter 7] that if \( \chi \) is a germ of a contact transformation which interchanges \( T^*_M X \) with \( T^*_N X \), and if \( \Phi_K \) is a quantization of \( \chi \) as above, then \( \Phi_K(L_M) = L_N \) for a Levi simple sheaf \( L_N \) along \( T^*_N X \) at \( q \). In particular, if \( \chi \) is as in (2.7), then

\[
\chi_* \mu \text{hom}(L_{M_1}, L_{M_2})_q \simeq \mu \text{hom}(L_{N_1}, L_{N_2})_q \simeq \mu \text{hom}(C_{N_1}, C_{N_2})_q \simeq \mu \text{hom}(C_{N_1^+}, C_{N_2^+})_q \simeq R\Gamma_{N_1^+}(C_{N_2^+})_y \simeq \mathbb{C},
\]

i.e.

\[
\mu \text{hom}(L_{M_1}, L_{M_2})_p \simeq \mathbb{C}. \tag{2.10}
\]

On the other hand, if (2.10) holds for a pair \((M_1, M_2)\) satisfying (2.4)–(2.5), then a transformation \( \chi \) with the above properties may be constructed as in the proof of Theorem 2.2.

**Proposition 2.3.** Let \( S \subset M \subset X^R \) be \( C^2 \)-submanifolds with \( p \in S \times_M T^*_M X \) and \( ip / T^*_S X \). Then \((M, S)\) (resp. \((S, M)\)) satisfies (2.6) if and only if it satisfies (2.10).

**Proof.** One has \( \mu \text{hom}(C_M, C_S) \simeq C_{S \times M T^*_M X} \), and hence:

\[
\mu \text{hom}(L_M, L_S)_p \simeq \mu \text{hom}(C_M, C_S)_p[-d_S(p) + d_M(p)] = \mathbb{C}[-d_S(p) + d_M(p)].
\]

Moreover

\[
\mu \text{hom}(L_S, L_M)_p \simeq \mu \text{hom}(C_M, C_S)_{-p}[-d_S(p) + d_M(p) + \text{codim}_M S] = \mathbb{C}[-d_S(p) + d_M(p) + \text{codim}_M S].
\]

\[\square\]

**Remark 2.4.** In view of the above discussion, we conclude that for a pair \((M_1, M_2)\) satisfying (2.4)–(2.5), condition (2.6) is equivalent to \( T^*_M X \succ T^*_S X \succ T^*_M X \), where \( \succ \) means that the conormal bundles are relatively positive in the sense of [5].
Let $S \subset M$ with $ip \notin T^*_s X$, assume codim$_M S = 1$, and let $\Omega$ be an open component of $M \setminus S$. We denote by $\rho$ the projection $M \times_X T^*_X \longrightarrow T^*_M$, and by $N^*(\Omega)$ the conormal cone to $\Omega$ in $M$. Let $\chi$ satisfy (2.7), and let $\Phi_K$ be a quantization of $\chi$ as above.

**Proposition 2.5.** ([16]) In the above situation, the pair $(M, S)$ (resp. $(S, M)$) satisfies (2.6) if and only if $\Phi_K(C_{\Omega}) = C_Y[d_M(p) - 1]$ (resp. $\Phi_K(C_{\Omega}) = C_Y[d_M(p) - 1]$), where $Y \subset X$ is a $C^1$-hypersurface containing $R$.

**Proof.** We put $\Lambda^+_1 = \Omega \times_M T^*_M X$, $\Lambda^+_2 = \rho^{-1}(\pm N^*(\Omega)|_S)$, $\Sigma = S \times_M T^*_M X$, $\Lambda'^+_1 = \Lambda^+_1 \setminus \Sigma$, $\Lambda'^+_2 = \Lambda^+_2 \setminus \Sigma$. We have $S\Sigma(C_{\Omega}) = \Lambda'^+_1 \cup \Lambda'^+_2$, $S\Sigma(C_{\Omega}) = \Lambda'^+_1 \cup \Lambda'^+_2$.

Let $\chi$ satisfy (2.7). Then either $\chi(\Lambda'^+_1 \cup \Lambda'^+_2)$ or $\chi(\Lambda'^+_1 \cup \Lambda'^+_2)$ is the conormal bundle to a $C^1$-hypersurface $Y \subset X$. This implies that either $\Phi_K(C_{\Omega})$ or $\Phi_K(C_{\Omega})$ is isomorphic to $C_Y$ in $D^b(X; q)$. Note that $\Phi_K(C_{\Omega})$ (resp. $\Phi_K(C_{\Omega})$) has constant shift $d_M(p) - 1/2$ in $\Lambda'^+_1 \cup \Lambda'^+_2$ (resp. $\Lambda'^+_1 \cup \Lambda'^+_2$) if and only if $d_M(p) = d_S(p)$ (resp. $d_M(p) = d_S(p) - 1$). The conclusion follows.

### 3 Application to vanishing of cohomology for microfunctions at the boundary

Using the results of the previous section, we may recover and even improve our former results in [5]. Let $X$ be a complex manifold of dimension $n$, $M$ a real $C^2$-submanifold of codimension $l$, $\Omega$ an open set of $M$ with $C^2$-boundary $S = \partial \Omega$ of codim$_M S = r$ ($\Omega$ locally on one side of $S$ when $r = 1$).

**Theorem 3.1.** (i) Let $p \in S \times_M \tilde{T}^*_M X$ and assume $ip \notin T^*_s X$. Then:

$$H^j(\mu_\Omega(\mathcal{O}_X)) = 0 \quad \forall j \notin [d_M(p), d_M(p) + r - 1]. \quad (3.1)$$

(ii) Assume in addition that $d_M(p')$ and $d_M(p'')$ are constant for $p' \in \Omega \times_M T^*_M X$, $p'' \in S \times_X \rho^{-1}(-N^*(\Omega))$ near $p$, and that $d_S(p) = d_M(p) + 1$. Then

$$H^j(\mu_\Omega(\mathcal{O}_X)) = 0 \quad \forall j \notin d_M(p). \quad (3.2)$$

**Proof.** For the completeness of our exposition we begin by repeating some arguments of [9, Theorems 11.3.1, 11.3.5].

Consider two germs of contact transformations $\chi, \tilde{\chi}$ at $p$, such that, setting $q = \chi(p)$, $\tilde{q} = \tilde{\chi}(p)$, $y = \pi(q)$, $\tilde{y} = \pi(q)$:

\[
\begin{align*}
(\chi(T^*_M X) &= T^*_N X \text{ for } N \subset X \text{ with codim}_X N = 1, \ s_N(q) = 0, \\
\tilde{\chi}(T^*_N X) &= T^*_\tilde{N} X \text{ for } \tilde{N} \subset X \text{ with codim}_X \tilde{N} = 1, \ s_\tilde{N}(\tilde{q}) = 0.
\end{align*}
\]
Quantizing $\chi$ and $\tilde{\chi}$ with kernels $K$ and $\tilde{K}$ as in the proof of Theorem 2.2, we get:

$$\Phi_K(C_M) = C_{M_1}[d^+_M(p) - n], \quad \Phi_{\tilde{K}} = (C_M) = C_{M_2}[d^-_M(p) - 1].$$

This implies (as in (2.9)):

$$\chi_*\mu_M(\mathcal{O}_X)_q \simeq R\Gamma_{N^+}(\mathcal{O}_X)_y[n - d^+_M(p)], \quad \tilde{\chi}_*\mu_M(\mathcal{O}_X)_{\tilde{q}} \simeq R\Gamma_{\tilde{N}^+}(\mathcal{O}_X)_{\tilde{y}}[-d^-_M(p) + 1].$$

Hence:

$$H^j(\mu_M(\mathcal{O}_X))_p = 0 \quad \forall j \notin [d^-_M(p), d^+_M(p)]. \quad (3.3)$$

The same arguments hold for $S$.

We set now $\Omega^- = M \setminus \Omega$ and consider the distinguished triangle

$$\mu_S(\mathcal{O}_X) \longrightarrow \mu_M(\mathcal{O}_X) \longrightarrow \mu_0(\mathcal{O}_X) \oplus \mu_{\Omega^-}(\mathcal{O}_X) \overset{+1}{\longrightarrow}. \quad (3.4)$$

Formulas (2.3), (3.3), and the analogous formula for $S$, imply the vanishing of (3.1) for $j > d^+_M(p) + r - 1$ and for $j < d^-_M(p)$, when $d^-_S(p) > d^-_M(p)$.

In order to treat the case $d^-_S(p) = d^-_M(p)$, it remains to prove the injectivity of the morphism

$$H^{d^-_M(p)}(\mu_S(\mathcal{O}_X)_p) \longrightarrow H^{d^-_M(p)}(\mu_M(\mathcal{O}_X)_p). \quad (3.5)$$

Applying Theorem 2.2 for $(M_1, M_2) = (M, S)$, we reduce (3.5) to the morphism

$$H^{1}_{N_2^+}(\mathcal{O}_X)_y \longrightarrow H^{1}_{N_1^+}(\mathcal{O}_X)_y, \quad (3.6)$$

which is clearly injective since $N_2^+ \subset N_1^+$.

(i) If $r > 1$, then $\Omega = M$, $N^+(\Omega) = T^* M$, and therefore $d^-_M(p')$, $d^-_S(p'')$ are constant in a full neighborhood of $p$ in $T^*_X$ and $T^*_S X$ respectively. We recall that $(s^-_M(p') - \gamma_M(x')) - s^-_{N_1^+}(q')$ is constant for $q' = \chi(p')$. Since $s^-_{N_1^+}(q) = 0$, then $X \setminus N_1^+$ is pseudoconvex (cf. [8]), whence the complex $\mu_M(\mathcal{O}_X)_p \simeq R\Gamma_{N_1^+}(\mathcal{O}_X)_y[-d^-_M(p) + 1]$ is concentrated in degree $d^-_M(p)$. The same is true for $S$ and thus we get the conclusion by the aid of (3.4).

(ii) Let now $r = 1$, and assume $d^-_S(p) = d^-_M(p) + 1$. According to Proposition 2.5 we have

$$\chi_*\mu_0(\mathcal{O}_X)_p \rightarrow R\Gamma_{Y^+}(\mathcal{O}_X)_y[-d^-_M(p) + 1]. \quad (3.7)$$

We observe that $Y$ is $C^1$ and even $C^2$ outside $R$. Moreover,

$$s^-_Y(q') \equiv 0 \quad \forall q' \in T^*_Y X, \pi(q') \notin R, \ q' \text{ near } q.$$ 

It follows that $X \setminus Y^+$ is pseudoconvex whence (3.7) is concentrated in degree $d^-_M(p)$. \qed
References


