Detailed summary

- Definition of Lie algebra. Every (associative) algebra $A$ can be endowed with a structure of a Lie algebra by defining
  $$[a,b] := ab - ba$$
  for every $a, b \in A$.

- Let $G$ be an (affine) algebraic group over the algebraically closed field $K$ of characteristic 0. Three ways of seeing the tangent space to $G$ at 1.
  1. Let $D := K[t]/(t^2) \cong K \oplus K \varepsilon$, with $\varepsilon = t + (t^2)$. Let $G \subseteq \mathbb{A}^n$. Then $v \in \mathbb{A}^n$ is tangent to $G$ at $x \in G$ if the map
     $$\phi_v : K[G] \to D$$
     $$f \mapsto f(x + \varepsilon v)$$
     is an algebra map.
   2. $T_1(G) := \{ \delta \in \text{Hom}(K[G], K) : \delta(fg) = (\delta f)g(1) + f(1)\delta g \text{ for every } f, g \in K[G] \}$.
   3. Let $\text{Der}(K[G])$ be the set of derivations of $K[G]$. Let $g \in G$ and $f \in K[G]$ we define $\lambda_g : K[G] \to K[G]$ as
     $$\lambda_g(f)(h) = f(g^{-1}h)$$
     for every $h \in G$, and
     $$\text{Lie}(G) = \text{Der}_G(K[G]) := \{ \delta \in \text{Der}(K[G]) : \delta \lambda_g = \lambda_g \delta \text{ for every } g \in G \}.$$  
     $\text{Lie}(G)$ is a Lie algebra.

Sketch of the proof of the 1-to-1 correspondence between these three objects and examples of the tangent spaces to $\text{GL}_n(K)$ and $\text{SL}_n(K)$ in 1, namely $\mathfrak{gl}_n(K)$ and $\mathfrak{sl}_n(K)$. 

• There is a correspondence between some structure of the algebraic group $G$ and the structure of $T_1(G)$, which is a Lie algebra. Moreover, to every rational representation of an algebraic group $G$, we can associate a representation of the Lie algebra $T_1(G)$. These are some motivations that lead us to study in detail Lie algebras and its representations.

• Let $L$ be a Lie algebra. Definitions of subalgebras and ideals of $L$. Definition of the adjoint representation

$$\text{ad}_L : L \to \text{Der}(L)$$

$$x \mapsto [x, -].$$

Simple Lie algebras, and $\mathfrak{sl}_2(K)$ as example.

• **Nilpotent Lie algebras**: definitions, first properties. $L$ is nilpotent if and only if every element of $L$ is ad-nilpotent. If $L \subseteq \mathfrak{gl}(V)$ for some fin. dim. $K$-vector space $V$, then there exists a basis for $V$ such that $L$ is a subalgebra of the algebra of strictly upper triangular matrices.

• **Solvable Lie algebras**: definitions, first properties. Definition of $\text{Rad}(L)$ as the maximal solvable ideal of every Lie algebra $L$. If $L \subseteq \mathfrak{gl}(V)$ for some fin. dim. $K$-vector space $V$, then there exists a basis for $V$ such that $L$ is a subalgebra of the algebra of upper triangular matrices. $L$ solvable if and only if $[L, L]$ is nilpotent. Cartan solvability criterion.

• **Semisimple Lie algebras**: definitions, first properties. The Killing form $k$. $L$ is semisimple if and only if $k$ is nondegenerate on $L$. Decomposition of every semisimple Lie algebra into a sum of simple ideals. Every fin. dim. representation of a semisimple Lie algebra is completely reducible (Weyl’s theorem).

• Abstract and usual Jordan decomposition: if $L$ is semisimple, any element $x$ of $L$ can be written uniquely as a sum of a nilpotent and a semisimple element. If $L \subseteq \mathfrak{gl}(V)$ then usual=abstract.

• Toral subalgebras of semisimple Lie algebras: any toral subalgebra is abelian, any toral subalgebra is maximal if and only if it is self-centralizing.

• Let $L$ be a semisimple Lie algebra, $H \subseteq L$ a maximal toral subalgebra. Decomposition of $L$ with respect to the action of $\text{ad}_L(H)$ on $L$. Example: decomposition of $\mathfrak{sl}_2(K)$ and (maybe) characterization of the irreducible representations of $\mathfrak{sl}_2(K)$. 