GAGA functor & Kiehl's thm

$K = \text{complete nonarchimedean field (non-trivial)}$

Reference: "Lectures on formal and rigid geometry" Bosch

§ GAGA

Based on Serre's work over $\mathbb{C}$

"Géométrie algébrique et géométrie analytique", now adapted to nonarchimedean setting.

Qn: How to construct a rigid analytic $A^n$?

We know how to define rigid analytic closed balls e.g. $Sp K\langle \xi_1, \ldots, \xi_n \rangle = \text{unit ball} < A^n$
Idea: Take larger and larger balls, and pass to the limit.

Pick $c \in K$ s.t. $|c| > 1$. For $i \geq 0$, let $B_i = \text{closed ball of radius } |c|^i$

i.e. $B_i = \text{Sp} K(c^i \xi_1, \ldots, c^i \xi_n)$

Have a tower of inclusions

$K\langle \xi_1, \ldots, \xi_n \rangle \leftarrow K\langle c^i \xi_1, \ldots, c^i \xi_n \rangle \leftarrow K\langle c^{2i} \xi_1, \ldots, c^{2i} \xi_n \rangle \leftarrow \cdots$

corresponding to $B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots$

We can glue these spaces to get
$A^\infty_k := \lim_{i \to 0} B_i$

This is a rigid analytic $k$-space with admissible affinoid covering $(B_i)_{i \geq 0}$.

Moreover,

$\mathcal{O}_{A^\infty_k}(A^\infty_k) = \lim_{i \to 0} K\langle c^{-i} \xi_1, \ldots, c^{-i} \xi_n \rangle$

$$= \left\{ \sum_{x \in \mathbb{N}^n} a_x \xi^x \mid a_x \in K \text{ and } |c^{i}| |a_x| \to 0 \text{ as } |x| \to \infty, \text{ for all } i \geq 0 \right\}$$

$\left( |x| = d_1 + \ldots + d_n \right)$
Can generalise previous construction to any affine $K$-scheme of finite type $X = \text{Spec} \left( K[\xi_1, \ldots, \xi_n]/\mathfrak{m} \right)$ by replacing the $B_i$ with $X_i := \text{Sp} \left( K[c_1 \xi_1, \ldots, c_n \xi_n]/\mathfrak{m} \right)$.

Then gluing the $X_i$ one obtains $X = \lim X_i$.

Fact: Doesn't depend on choice of $c$. 
In general, given a $K$-scheme $X$ locally of finite type, can cover $X$ by open affines $X_i$ which are of finite type over $K$, then can apply previous construction to get $X_i$.$\mapsto$ glue to get $X$.$\mapsto$ an
This way we get a functor
\[
\begin{array}{c}
\{\text{$K$-schemes} \} \\
\{\text{locally of finite type} \} \\
\end{array}
\quad \longrightarrow 
\begin{array}{c}
\{\text{rigid analytic} \} \\
\{\text{$K$-spaces} \} \\
\end{array}
\]
\[
X \mapsto X^{\text{can}}
\]
called GAGA.
Properties: \* Universal property: there is a morphism of locally ringed spaces \((i, i^\ast): (X^{\text{an}}, \mathcal{O}_{X^{\text{an}}}) \rightarrow (X, \mathcal{O}_X)\) s.t. for any rigid \(K\)-space \((Y, \mathcal{O}_Y)\), any morphism of loc. ringed spaces \((Y, \mathcal{O}_Y) \rightarrow (X, \mathcal{O}_X)\) factors uniquely through \((i, i^\ast)\).

The underlying map of sets \(X^{\text{an}} \rightarrow X\) identifies \(X^{\text{an}}\) with the closed points of \(X\).

\* \(\mathbb{A}_K^n\) satisfies the usual universal property of affine \(n\)-space i.e. given a rigid \(K\)-space \(Y\), \(\text{Hom}(Y, \mathbb{A}_K^n) \cong \mathcal{O}_Y(Y)^n\).
\( \cong \text{ Hom}_{k-\text{alg}}(K[[t_1, \ldots, t_n]], \mathcal{O}_x(x)) \) via the morphism \( A^n_{k} \to A^n_k \).

\* \( X \to X^{\text{an}} \) is faithful but not full e.g. \( \exists A^1_{k} \to A^1_{k} \) corresponding to \( \xi \mapsto \mathbb{Z} c_{n^2} \xi^n \). This doesn't come from a morphism of schemes \( A^n_{k} \to A^n_{k} \).

\* Given an \( \mathcal{O}_x^* \)-module \( F \), can construct an \( \mathcal{O}_x^{\text{an}} \)-module \( F^{\text{an}} := i^{-1}(F) \otimes_{i^{-1}(\mathcal{O}_x^*)} \mathcal{O}_x^{\text{an}} \).

Moreover:

\( F \) is coherent \( \iff \) \( F^{\text{an}} \) is coherent

& \( F \to F^{\text{an}} \) is exact.
Example: \( \mathbb{P}^{n, an}_K \) is by defn covered by

\[
U_{i}^{an} = \lim_{\delta \to 0} \text{Sp} \ K \langle c^{-\delta} \frac{f_0}{s_i}, \ldots, c^{-\delta} \frac{f_n}{s_i} \rangle
\]

\[
= A_{i}^{an, an}_K \quad (i = 0, \ldots, n)
\]

In fact, there is an admissible affinoid covering of \( \mathbb{P}^{n, an}_K \) by closed unit balls

\[
V_i = \text{Sp} \ K \langle \frac{f_0}{s_i}, \ldots, \frac{f_n}{s_i} \rangle \subset U_{i}^{an}
\]

Idea: for \( x = (x_0, \ldots, x_n) \ (x_i \in K) \), let \( i \)

be s.t. \( |x_i| = \max \{ |x_1, \ldots, |x_n| \} \). Then \( x \in V_i \).
§ Proper Mapping Theorem

First need notion of a proper morphism.

Def\(^n\): A morphism \(\phi: X \rightarrow Y\) of rigid \(k\)-spaces is a **closed immersion** if there is an admissible affinoid covering \((V_j = \text{Sp } A_j)\) of \(Y\) s.t \(\phi^{-1}(V_j) = \text{Sp } B_j\) is affinoid & the corresponding map \(A_j \rightarrow B_j\) is surjective.

Remarks: * This def\(^n\) doesn't depend on choice of covering \((V_j)\)

* If \(F\) is a coherent \(\mathcal{O}_X\)-module then \(\phi_*(F)\) is coherent
Defn. Let \( f : X \to Y \) be a morphism of rigid \( \mathbb{K} \)-spaces.

1. Say \( f \) is quasi-compact if for each quasi-compact \( Y' \subset Y \) (i.e. \( Y' \) has a finite admissible affinoid covering), \( f^{-1}(Y') \) is also quasi-compact.

2. \( f \) is separated (resp. quasi-separated) if the diagonal \( \Delta : X \to X \times_Y X \) is a closed immersion (resp. quasi-compact).

3. Say \( X \) is separated if \( X \to \text{Sp} \mathbb{K} \) is separated.
In alg. geom, have $X \rightarrow Y$ is separated iff $\text{Im}(\Delta) \subset X \times_Y X$ is closed. Here:

Propn: $\psi: X \rightarrow Y$ is separated iff:

(i) $\psi$ is quasi-separated; and

(ii) the image of $\Delta$ is a closed analytic subset in $X \times_Y X$ i.e. locally on open affinoids it is Zariski closed.

To define proper, need a notion of relative compactness over a base
Defn: \( f: X \to Y \) morphism with \( Y \) affinoid, 
\( U \subset U' \subset X \) open affinoids. Say \( U \) is 
relatively compact in \( U' \), denoted \( U \subset_{r} U' \), 
if \( \exists f_1, \ldots, f_r \in \mathcal{O}_x(U') \) s.t.
\[
\mathcal{O}_y(Y) \langle f_1, \ldots, f_r \rangle \longrightarrow \mathcal{O}_x(U')
\]
\[
f_i \mapsto f_i
\]
and s.t. \( U \subset \{ x \in U' \mid |f_i(x)| < 1 \} \).

This property behaves well under intersection 
and fibre products.
Def. $\varphi: X \to Y$ is proper if:

(i) $\varphi$ is separated; and

(ii) $\exists$ admissible affinoid covering $(Y_i)_{i \in I}$ of $Y$ and, for each $i \in I$, two finite admissible affinoid coverings $(X_{ij})_{j=1}^{n_i}$ and $(X'_{ij})_{j=1}^{n'_i}$ of $\varphi^{-1}(Y_i)$ s.t. $X_{ij} \subset X'_{ij}$ for $i,j$.

This def. is inspired by the theory of compact Riemann surfaces.

Ex. - Finite/projective morphisms are proper.
- If $X$ proper $k$-scheme $\to X^{an}$ is proper.
Thm: (Kiehl) Let \( f : X \to Y \) be a proper morphism of rigid analytic \( \mathbb{K} \)-spaces, and let \( F \) be a coherent \( \mathcal{O}_X \)-module. Then \( R^i f_* (F) \) is a coherent \( \mathcal{O}_Y \)-module for all \( i \geq 0 \).

Application: Stein factorisation

\( f : X \to Y \) proper. Then \( f_* (\mathcal{O}_X) \) is a coherent \( \mathcal{O}_Y \)-module.

\( \Rightarrow \) rigid \( \mathbb{K} \)-space \( Y' \) that is finite over \( Y \).

\( \Rightarrow \) factorise \( f = X \to Y' \to Y \) proper finite with connected fibres.
Some GAGA result:

Theorem: Let $X$ be a proper $k$-scheme, $\mathcal{F}$, $\mathcal{G}$ coherent $\mathcal{O}_X$-modules. Then:

(i) $H^i(X, \mathcal{F}) \cong H^i(X_{\text{an}}, \mathcal{F}_{\text{an}})$ $\forall i \geq 0$

(ii) $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \cong \text{Hom}_{\mathcal{O}_{X_{\text{an}}}}(\mathcal{F}_{\text{an}}, \mathcal{G}_{\text{an}})$.

Moreover, if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module, then there exists a unique coherent $\mathcal{O}_{X_{\text{an}}}$-module $\mathcal{F}$, up to isomorphism, $\mathcal{F}' = \mathcal{F}_{\text{an}}$. In particular

\[
\left\{ \text{coherent } \mathcal{O}_X-\text{mod} \right\} \rightarrow \left\{ \text{coherent } \mathcal{O}_{X_{\text{an}}}-\text{mod} \right\}
\]

$\mathcal{F} \mapsto \mathcal{F}_{\text{an}}$

is an equivalence.
e.g. \( X = \mathbb{P}^n_k \), \( F' = \text{coherent ideal } \mathfrak{I}' \subset \mathcal{O}_X \), then \( F' = \text{analytification of ideal } \mathfrak{I}' \subset \mathcal{O}_X \rightarrow \text{nonarchimedean Chow's thm: closed analytic subsets of } \mathbb{P}^n_k \text{ are algebraic.} \)

Remark: Analogous to Serre's original results on complex projective varieties in GAGA

Few words on proof: (proof ~ 10-15 pages)

Work locally so WLOG \( Y \) is affinoid.

\( \Gamma(Y, R^i\mathcal{F}^\ast) = H^i(X, F) \) so need to show two things:
(1) $H^i(X, F)$ is a finite $O_Y(Y)$-module

(2) $R^i \mathcal{I}_* F = \text{sheaf of } O_Y$-modules associated to $H^i(X, F)$

i.e. if $Y' = \text{Sp } B' \subset Y = \text{Sp } B$, need to show $H^i(X, F) \otimes_B B' \cong H^i(X \times_Y Y', F)$

(1) $B := O_Y(Y)$. Assume $\exists$ two finite admissible coverings $U = (U_i)$ & $\mathcal{V} = (V_i)$ of $X \ni \forall

V_i \subset_Y U_i \forall i$ (replace $Y$ by $Y_i$, $X$ by $\mathcal{V}^{-1}(Y_i)$)

Čech cohomology $\Rightarrow H^i(U, F) \cong H^i(\mathcal{V}, F)$

$\cong H^i(X, F)$. 
Write \( C^*(U, F) = \check{\text{Cech complex}} \) w.r.t. \( U \)
\( Z^*(U, F) = \) cocycles of \( \check{\text{Cech complex}} \)

Need to show that cokernel of
\[ f^i : C^{i-1}(U, F) \xrightarrow{d} Z^i(U, F) \]
is a finite \( B \)-module.

Using \( V_i \subseteq U_i \forall i \), have restriction maps
\[ C^*(U, F) \rightarrow C^*(V, F) \] & so a morphism
\[ r^i : Z^i(U, F) \rightarrow Z^i(V, F) (i \geq 0) \]

Then as \( H^i(U, F) \cong H^i(V, F) \), see that
\[ f^i + r^i : C^{i-1}(U, F) + Z^i(U, F) \rightarrow Z^i(U, F) \]
is surjective. So can "disturb" \( f^i \) by
some map $r$ to get something surjective.

V. roughly: Need a subtle approximation argument. Say a cts $B$-linear morphism $g : M \to N$ is completely cts if $g = \lim g_i$, where $g_i$ is a cts $B$-linear hom with $\text{Im}(g_i) \overset{f}{\rightarrow} \overset{g}{\rightarrow} B$.

Main tool: When $f, g : M \to N$ are cts homs of complete normed $B$-modules with $f$ surjective & $g$ completely continuous, there's a thm of L. Schwarz that says $\text{Coker}(f + g) = f \cdot g / B$. 
Want to apply this thm to \( f = f' + ri \) & \( g = -ri \).

Problem: \( ri \) may not be completely continuous.
But one can show \( ri \) is "part of" a map that is even strictly completely continuous. Then one can reprove L. Schwarz' thm by using this new cond' on \( g \).

(2) Work by induction on \( d = \text{Krull dim of } B \). Let \( X' = X, Y' \). To show \( H^i(X, F) \otimes_B B' \to H^i(X', F) \) is an iso, enough to show the localization at each max' ideal \( m' \) of \( B' \) is an iso.
As $\overrightarrow{B'_{m'}} \to \overrightarrow{B'_{m'}}$ is faithfully flat, suffices to show $H'(X, F) \otimes_{\overrightarrow{B'_{m'}}} \overrightarrow{B'_{m'}} \to H'(X', F) \otimes_{\overrightarrow{B'_{m'}}} \overrightarrow{B'_{m'}}$ are all isomorphisms.

Idea: work instead with $b$-adic completions for a well-chosen $b \in m' \cap B$.

Can use induction hypothesis on $B/(b)$ and pass to the limit to get the result.

There again, the details involve subtle arguments.