$p$-adic representations and arithmetic $D$-modules

Grassmannians

Andrés Sarrazola Alzate

University of Padua

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**Definition**

Plücker embedding

Totally decomposable vectors

Main theorem

Examples

Flag varieties
Main idea.

To parametrize higher dimensional subspaces of a fixed (complex) vector space $V$.

Set-theoretic definition

**Definition.** Let us suppose that $V$ is an $n$-dimensional vector space. We define the Grassmannian $G(k, V)$ as follows:

$$G(k, V) := \{ U \subseteq V \mid \dim(U) = k \}.$$ 

Objective

To realize Grassmannians as projective varieties.
Set-theoretic definition

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To realize Grassmannians as projective varieties.
Dimension of the Grassmannian

**Key observation**

\[ \mathbb{G}_m \text{ acts on } \mathbb{A}^{n+1} \setminus \{0\} \text{ by scalar multiplication and} \]

\[ \mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \mathbb{G}_m. \]

**Geometric description**

Fixing a basis \( \{v_1, \cdots, v_n\} \) of \( V \) we have \( V \cong \mathbb{C}^n \) and

\[ G(k, n) = \{U \subseteq \mathbb{C}^n \mid \dim(U) = k\}. \]

The **Linear group** \( \mathbb{G}L_k \) acts on \( M_{k \times n} \) on the left and

\[ G(k, n) = M_{k \times n}^{\text{rank } k} / \mathbb{G}L_k \]
Dimension of the Grassmannian

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Dimension of the Grassmannian

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\( \mathbb{G}_m \) acts on \( \mathbb{A}^{n+1} \setminus \{0\} \) by scalar multiplication and

\[
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The **Linear group** $GL_k$ acts on $M_{k \times n}$ on the left and

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Dimension of the Grassmannian

$$\begin{pmatrix}
\lambda_{1,1} & \cdots & \lambda_{1,k} \\
\vdots & \ddots & \vdots \\
\lambda_{k,1} & \cdots & \lambda_{k,k}
\end{pmatrix} \leftrightarrow 
\begin{pmatrix}
a_{1,1} & \cdots & a_{k,1} & \cdots & a_{1,n} \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_{k,1} & a_{k,k} & \cdots & a_{k,n}
\end{pmatrix}$$

If the first $k \times k$ minor is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix}
1 & 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,n-k} \\
0 & 1 & \cdots & 0 & b_{2,1} & \cdots & b_{2,n-k} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & b_{k,1} & \cdots & b_{k,n-k}
\end{pmatrix}$$

$$\uparrow \text{bijection}$$

$A^{k(n-k)}$

$G(k, n)$ is covered by $\binom{n}{k}$ affine spaces $A^{k(n-k)}$. 
Dimension of the Grassmannian

\[
\begin{pmatrix}
\lambda_{1,1} & \cdots & \lambda_{1,k} \\
\vdots & \ddots & \vdots \\
\lambda_{k,1} & \cdots & \lambda_{k,k}
\end{pmatrix}
\begin{pmatrix}
a_{1,1} & \cdots & a_{k,1} & \cdots & a_{1,n} \\
a_2 & \cdots & a_k & \cdots & a_n \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
a_k & \cdots & a_{k,n}
\end{pmatrix}
\]

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\( \uparrow \) bijection

\( \mathbb{A}^{k(n-k)} \)

\( G(k, n) \) is covered by \( \binom{n}{k} \) affine spaces \( \mathbb{A}^{k(n-k)} \).
**Dimension of the Grassmannian**

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Dimension:

\(G(k, n)\) is covered by \(\binom{n}{k}\) affine spaces \(\mathbb{A}^{k(n-k)}\).
Definition

Plücker embedding

Totally decomposable vectors

Main theorem

Examples

Flag varieties
Plücker embedding

We have fixed a basis $B := \{v_1, \cdots, v_n\}$ of $V$.

The exterior algebra

Expressing $w_1, \cdots, w_k \in V$ in terms of the basis $B$ we have

$$\left( \sum_{i=1}^{n} a_{i,1} v_i \right) \wedge \cdots \wedge \left( \sum_{i=1}^{n} a_{i,k} v_i \right) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det(A(ik)) v_{i_1} \wedge \cdots \wedge v_{i_k}$$

Example. If $A := \{w_1, \cdots, w_n\}$ is another basis then $\text{det}(A(nn))$ is the determinant of the change of basis matrix $A \rightarrow B$.

The Plücker embedding

$$i : \quad G(k, n) \rightarrow \mathbb{P} \left( \wedge^k V \right)$$

$$W \quad \mapsto \quad [w_1 \wedge \cdots \wedge w_k]$$

{$w_1, \cdots, w_k$} is a basis for $W$. The map $i$ is well defined by the preceding relation.
Plücker embedding

We have fixed a basis \( B := \{v_1, \cdots, v_n\} \) of \( V \).

The exterior algebra

Expressing \( w_1, \cdots, w_k \in V \) in terms of the basis \( B \) we have

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\[
W \mapsto [w_1 \wedge \cdots \wedge w_k]
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We have fixed a basis $B := \{v_1, \cdots, v_n\}$ of $V$.

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**Example.** If $A := \{w_1, \cdots, w_n\}$ is another basis then $\det(A(nn))$ is the determinant of the change of basis matrix $A \to B$.

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$\{w_1, \cdots, w_k\}$ is a basis for $W$. The map $i$ is well defined by the preceding relation.
Plücker embedding

Lemma. $i$ is injective.

Remark. The Plücker embedding allows us to view $G(k, n)$ as a subset of the projective space $\mathbb{P}^{n \choose k}$. 

Plücker coordinates

$$I_{k,n} := \{ \bar{i} := (i_1, \cdots, i_k) \mid 1 \leq i_1 < \cdots i_k \leq n \}$$

the set of coordinates of $\mathbb{P}(\wedge^k V)$. We want to compute $\bar{i}(W)$.

We take $B_W := \{ w_1, \cdots, w_k \}$ a basis of $W$. The basis $B$ of $V$ gives rise to a matrix $M_W(a_{ij}) \in M_{n \times d}$. We have

$$w_1 \wedge \cdots \wedge w_k = \sum_{\bar{i} \in I_{k,n}} \sum_{\sigma \in S_k} \text{sig}(\sigma)a_{i_1,\sigma(1)} \cdots a_{i_k,\sigma(k)}v_{\bar{i}}.$$

$\bar{i}(W) = \det(M_{\bar{i}})$, with $M_{\bar{i}}$ the $k \times k$ sub-matrix formed from the $\bar{i}$ rows of $M_W$. 
Plücker embedding

**Lemma.** $i$ is injective.

**Remark.** The Plücker embedding allows us to view $G(k, n)$ as a subset of the projective space $\mathbb{P}(k)^{-1}$.

**Plücker coordinates**

\[ I_{k,n} := \{ \bar{i} := (i_1, \cdots, i_k) \mid 1 \leq i_1 < \cdots i_k \leq n \} \]

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$\bar{i}(W) = det(M_{\bar{i}})$, with $M_{\bar{i}}$ the $k \times k$ sub-matrix formed from the $\bar{i}$ rows of $M_W$. 
**Plücker embedding**

**Lemma.** \( i \) is injective.

**Remark.** The **Plücker embedding** allows us to view \( G(k, n) \) as subset of the projective space \( \mathbb{P}(k)^{n-1} \).

**Plücker coordinates**

\[
I_{k,n} := \{ \bar{i} := (i_1, \cdots, i_k) \mid 1 \leq i_1 < \cdots i_k \leq n \}
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the set of coordinates of \( \mathbb{P}(\wedge^k V) \). We want to compute \( \bar{i}(W) \).

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Totally decomposable vectors

Let us prove know that the Grassmannian is a projective variety.

We will need the following notion.

**Definition.** Let \( w \in \bigwedge^k V \). We say that \( w \) is **totally decomposable** if we can write \( w = w_1 \wedge \cdots \wedge w_k \), with \( \{w_1, \cdots, w_k\} \subset V \) l.i.

**Preparation**

**Lemma.** \([w] \in \mathbb{P}(\bigwedge^k V)\) lies in the image of the Grassmannian under the Plücker embedding if and only if \( w \) is totally decomposable.

\[
(w \in \bigwedge^k V) \quad L_w := \{v \in V \mid v \wedge w = 0 \text{ in } \bigwedge^{k+1} V\}
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**Lemma.** The space \( L_w \) has dimension at most \( k \). With equality occurring if and only if \( w \) is totally decomposable.
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\[ \varphi : \wedge^k V \rightarrow \text{Hom}(V, \wedge^{k+1} V) \]

\[ w \mapsto \varphi(w) := w \wedge (\bullet) \]

It is a linear map

**Theorem.** \(i(G(k, n)) \subset \mathbb{P}(\wedge^k V)\) is a projective variety.

**Proof.** First of all

\[ [w] \in i(G(k, n)) \iff \text{rank}(\varphi(w)) = n - k \]

- The matrix \(A(w)\) of \(\varphi(w)\) has homogeneous entries of degree 1 in the coordinates
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- The Plücker embedding of $G(1, V)$ maps a linear subspace

$$W = \text{span}(\lambda_1 v_1 + \cdots + \lambda_n v_n) \quad \quad (\lambda_1 : \cdots : \lambda_n) \in \mathbb{P}^{n-1}$$

As expected $G(1, V) = \mathbb{P}^{n-1}$.

- $\dim(V) = 3$ and $W = \text{span}(v_1 + v_2, v_1 + v_3) \in G(2, V)$. Since

$$\begin{align*}
(v_1 + v_2) \wedge (v_1 + v_3) &= -v_1 \wedge v_2 + v_1 \wedge v_3 + v_2 \wedge v_3 \\
\end{align*}$$

the Plücker coordinates of $W$ in $\mathbb{P}^2$ are given by the vector $(-1 : 1 : 1)$. Alternatively,

$$M_W = \begin{pmatrix}
1 & 1 \\
1 & 0 \\
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and $(-1:1:1)$ keeps the $2 \times 2$-minors of $M_W$. 
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Let $\dim(V) = 4$.

The Plücker coordinates are the $2 \times 2$-minors

$$p_{ij} := \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = x_i y_j - x_j y_i$$

of the $4 \times 2$-matrix

$$M = \begin{pmatrix} x_i & y_i \end{pmatrix}_{0 \leq i \leq 3}$$

The Plücker embedding is defined by

$$i : \ G(2, 4) \to \mathbb{P}^5 \quad W \mapsto (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12})$$

and the Plücker coordinates satisfy the **quadratic Plücker relation**

$$p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0.$$
Let \( \dim(V) = 4 \).

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**Plücker quadratic**

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\[
p_{ij} := \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = x_i y_j - x_j y_i
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of the \( 4 \times 2 \)-matrix

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M = \begin{pmatrix} x_i & y_i \end{pmatrix}_{0 \leq i \leq 3}
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The Plücker embedding is defined by

\[
i : \quad G(2, 4) \quad \rightarrow \quad \mathbb{P}^5
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\[
W \quad \mapsto \quad (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12})
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and the Plücker coordinates satisfy the **quadratic Plücker relation**

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p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0.
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Let \((q_{01}:q_{02}:q_{03}:q_{23}:q_{31}:q_{12}) \in \mathbb{P}^5\). If \(q_{01} \neq 0\)

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\]

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**Key point:**

If the coordinates \(q_{ij}\) satisfy the quadratic relation, then they are the coordinates of \(W\).

We have proved

\[
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Definition

Plücker embedding

Totally decomposable vectors

Main theorem

Examples

Flag varieties
**Flags**

\[ \dim_{\mathbb{C}}(V) = n. \]

**Definition.** A flag in \( V \) is a strictly increasing sequence of subspaces

\[ 0 \subset V_1 \subset \cdots \subset V_l \subset V. \]

The **signature** of the flag is \((\dim(V_1), \cdots, \dim(V_l))\).

For every sequence of integers

\[ a := 0 < a_1 < \cdots < a_l < n \]

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Flag varieties

**Remark.** $\mathbb{F}(a_1; n) = G(a_1, n)$ is a projective variety.

**Proposition.** $\mathbb{F}(a; n)$ is a Zariski closed subset of $\prod_{i=1}^{l} G(a_i, n)$. 

**Proof.**
- The case $a \in \mathbb{Z}_{>0}$ is the preceding remark.

  **Reduction to** $a \in \mathbb{Z}_{>0}^2$.

- Let

  $$\pi_{ij} : G(a_1, n) \times \cdots G(a_l) \to G(a_i, n) \times G(a_j, n) \times G(a_j, n)$$

  be the projection.

  $$\mathbb{F}(a; n) = \bigcap_{1 \leq i < j \leq l < n} \pi_{ij}^{-1}(\mathbb{F}(a_i, a_j; n))$$

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Flag varieties

Let $r < s$ and

$$(\text{span}(u_1, \cdots, u_r), \text{span}(w_1, \cdots, w_s)) \in G(r, n) \times G(s, n).$$

If $u := u_1 \wedge \cdots \wedge u_r$, $w := w_1 \wedge \cdots \wedge w_s$, and

$$\varphi \oplus \varphi : \bigwedge^r V \oplus \bigwedge^s W \rightarrow \text{Hom} \left( V, \bigwedge^{r+1} V \oplus \bigwedge^{s+1} V \right),$$

then $\ker(\varphi_u \oplus \varphi_w) = U \cap W$

$U \subset W \Leftrightarrow \text{rank}(\varphi_u \oplus \varphi_w) = n - r$

- The $(n - r + 1) \times (n - r + 1)$-minors give polynomials conditions for $\mathbb{F}(r, s; n)$.
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$\diamondsuit$
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Example

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\dim(V = \text{span}(v_1, \cdots, v_4)) = 4.
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Let

\[(\text{span}(u), \text{span}(w_1, w_2)) \in \mathbb{F}(1, 2; 4)\]

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u = \sum_{i=1}^{4} a_i v_i \quad \text{and} \quad w_1 \wedge w_2 = \sum_{i<j} b_{ij} v_i \wedge v_j.
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By definition

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Complete flag varieties

**Definition.** A flag variety $\mathbb{F}(1, \cdots, n - 1; n)$ is called **complete**.

Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{C}^n$. We have a full flag

$$\mathcal{F} := 0 \subsetneq \mathbb{C} \cdot e_1 \subsetneq \cdots \subsetneq \bigoplus_{i=1}^{n-1} \mathbb{C} \cdot e_i \subsetneq \mathbb{C}^n$$

**Facts.**

- $\text{GL}_n$ acts transitively on complete flag varieties.

- The stabilizer $\mathbb{B} := \text{stab}_{\text{GL}_n}(\mathcal{F})$ is the (Borel) subgroup of upper triangular matrices in $\text{GL}_n$, and

$$\text{GL}_n/\mathbb{B} = \mathbb{F}(1, \cdots, n - 1; n).$$

In particular, $\mathbb{P}^1 = \mathbb{F}(1, 2) = G(1, 2) = \text{GL}_2/\left(\begin{array}{cc} * & * \\ 0 & * \end{array}\right)$. 
Complete flag varieties

**Definition.** A flag variety $F(1, \cdots, n-1; n)$ is called **complete**.

Let $\{e_1, \cdots, e_n\}$ be the standard basis of $\mathbb{C}^n$. We have a **full flag**

$$F := 0 \subset \mathbb{C} \cdot e_1 \subset \cdots \subset \bigoplus_{i=1}^{n-1} \mathbb{C} \cdot e_i \subset \mathbb{C}^n$$

**Facts.**

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