\[ K = \overline{K} \]

\[ G \leq \text{GL}(V) \]

\[ g = su \quad s \text{ semisimple} \quad u \text{ unipotent} \]

\[ [s, u] = 1 \]

\[ g \in G \Rightarrow s, u \in G \]

**Example** \[ K = \overline{\mathbb{F}_p} \]

\[ \text{GL}_n(K) = \bigcup_{\ell \geq 0} \text{GL}_n(\mathbb{F}_{p^\ell}) \]

\[ \Rightarrow \text{every det. has finite order} \]

\[ u \text{ unipotent} \Rightarrow \]

\[ 0 = (u - 1) = (u - 1)^{p^\ell} = u^{p^\ell} - 1 \]

\[ \Rightarrow u \text{ is a } p\text{-element.} \]

\[ s \text{ semis.} \Rightarrow s \sim \left( \begin{array}{ccc} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{array} \right) \]

\[ \lambda_i \in \mathbb{F}_{p^\ell} \Rightarrow (\lambda_i, 1, p) = 1 \Rightarrow (|s|, p) = 1, \text{ vice versa holds} \]
Let $U_G := \{ g \in G \mid g \text{ is unipotent} \}$ and $S_G := \{ g \in G \mid g \text{ is semisimple} \}$.

For $G \leq \text{GL}_N(k)$

$$U_{\text{GL}_N(k)} = \{ g \in \text{GL}_N(k) \mid (g-1)^N = 0 \}$$

hence

$$U_G = G \cap U_{\text{GL}_N(k)} \text{ is closed}$$

$S_G = ?$

**Example**

$G = \frac{1}{2} \left( \begin{array}{cc} a & c \\ 0 & b \end{array} \right) \ ab \neq 0$

$S_G = \left\{ \left( \begin{array}{cc} a & c \\ 0 & b \end{array} \right) \in G \mid a \neq b \right\} \cup \left\{ \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \in G, \ a = b \right\}$

neither open nor closed, DENSE in $G$
GROUPS WHERE ALL ELEMENTS ARE UNIPOTENT OR SEMISIMPLE

1) \( G_a = (k, +) \cong U_2 = \{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in k \} \)

\( G_a \) IS UNIPOTENT (\(=a \)L \(\)ITS \(\)ELEMENTS \(\)ARE \(\)SO

\( U_n = \{ \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \} \) PROTOTYPE UNIPOTENT

\( U_n = \{ \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \} \)

all \ its \ elements \ are \ ss

\( \downarrow \)

\( G_m \cong D_n \cong (k^*, \times) \)

Def A LAG is a TURUS if it is isomorphic to \( D_n \) for some \( m \).
What is special about TORI?
ABELIAN, ALL ELTS ARE SS

For any \( T \rightarrow GL(V) \) rational repn.
all \( p(t) \), \( t \in T \)
are simultaneously diagonalizable

In particular: if
\( T \leq GL(V) \) is a Torus

\[ \exists g \in GL(V) \text{ s.t. } gTg^{-1} \leq D_n. \]
If \( T \) is maximal we have =

Hence:
Maximal tori (in \( GL_n(k) \))
are all conjugate
to \( D_n \).
1. Any closed connected subgroup of $T$ is a torus.

2. $N_G(T) \big/ C_G(T)$ is finite.

$G = \text{GL}_4(k)$, $T = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : ac \neq 0 \right\}$

$N_G(T) = \left\{ \begin{pmatrix} x & y \\ 0 & A \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 & y \\ 0 & A \end{pmatrix} \right\}$

$C_G(T) = \left\{ \begin{pmatrix} x & y \\ 0 & A \end{pmatrix} \right\}$, $N_G(T) / C_G(T) \cong C_2$

If $T$ is maximal we set $W = N_G(T) / C_G(T)$ called the Weyl group.
3) $T$ maximal in $G$

$\delta_g = \bigcup_{g \in G} g T g^{-1}$
Prop: If \( G \leq GL_n(k) \) is unipotent

\[ \Rightarrow \exists \ g \in GL_n(k) \quad gGg^{-1} \leq U_n \]

(\( = U_n \) is really a prototype)

Proof: By induction on \( n \):

\[ \exists \mathfrak{w} \in k^n, \quad GW = W \]

If not, use induction on the blocks.

\[ \forall h \in GL_n(k), \quad hGh^{-1} \leq \left\{ \begin{array}{ccc} * & * \\ 0 & * \end{array} \right\} \]

\[ \Rightarrow \text{span}_k G = \text{End}(k^n) \]

+ trick on traces

\[ \forall g \in G \quad Tr g = n \]

\[ G = 1 \]

Consequence:

Essentially

\[ G \text{ unipotent} \quad \Rightarrow \quad G \leq U_n \text{ so} \]

\[ G \text{ is nilpotent} \quad \text{why?} \]
Def: Let $H$ be a group, we set
\[ H^i = [H, H^i], \quad H^i = [H, H^{i-1}] \]

$H$ is nilpotent if the series terminates to 1.

Example
\[
[U_{n-1}, U_n] = \begin{pmatrix}
1 & 0 & \cdots & \times \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1
\end{pmatrix}
\]

\[ [U_{n-1}, U_n] = \begin{pmatrix}
q & 0 & \cdots & \times \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 1
\end{pmatrix} \]

So $G$ unipotent $\Rightarrow$ nilpotent

$\Rightarrow$ solvable

Def: Let $H$ be a group, we set
\[ H^{(i)} = [H^{(i-1)}, H^{(i-1)}], \quad H^{(i)} = [H^{(i-1)}, H^{(i-1)}] \]

$H$ is solvable if the series terminates to 1.
**SOLVABLE GROUPS**

Prototype

\[ B_n = \begin{pmatrix} \ast & \ast \\ \circ & \ast \end{pmatrix} \]

Indeed

\[ [B_n, B_n] = U_n \Rightarrow \text{the series terminates} \]

\[ B_n \text{ stabilizes the flag in } k^n \]

\[ \text{span } e \subset \text{span}(e_1, e_2) \subset \cdots \]

On the other hand if \( G \leq GL_n(k) \) stabilizes a flag \( 0 = V \subset \cdots \subset k^n \), then \( \exists \ g \in GL_n(k) \) such that

\[ gGg^{-1} \leq B_n \Rightarrow G \text{ is solvable} \]
The converse also holds

**LIE KOLCHIN THEOREM**

If \( G \leq \text{GL}(V) \) is solvable, connected \( \Rightarrow \) it stabilizes a flag.

**Pf (idea)**

Step 1: Enough to show that \( G \) stabilizes a line

\[
\begin{pmatrix}
\ast \\ \vdots
\end{pmatrix}
\]

Then use induction

Step 2: Use induction on the derived length of \( G \), or on \( \dim V \), using \([G, G] \leq G\)

Solvable, closed, connected
[\text{use induction on blocks}]

\[ G \text{-stable} \]

\[ [G, G] \leq 2(G) \]

G-abelian

[\text{use connectedness to show}]

\[ V' = \text{Span}_k \text{stabilizers of G} \]

\[ \dim V = 1 \]
STRUCTURE OF SOLVABLE GROUPS:

G solvable

$$G \leq B_n = \bar{D}_n \times U_n$$

In this case, $U_{B_n} = U_n \leq B_n$

So $U_6 \triangleleft G$, $T = G/U_6 \rightarrow B_n/\overline{U_n} \cong \bar{D}_n$

$(\text{normal})$

$U_n \cap G$ is a subgroup

In fact

$$G \cong T \times U_6$$
IDEAS

$S$ solvable $\rightarrow$ stabilizes a flag $F$

$G \quad G/S$ quasi-projective

$\text{hope: for some } S, G/S \text{ is projective}$

idea want $G/S$ closed

$\overset{\downarrow}{\text{closed } G.F}$

$\overset{\downarrow}{G.F \text{ of minimal dimension}}$

$\overset{\downarrow}{S \text{ maximal dimension}}$

This motivates the following
Def: A Borel subgroup of $G$ is a maximal closed connected solvable subgroups of $G$.

Ex: $G = \text{GL}_n(k)$

If $B$ Borel subgroup $\Rightarrow$

Solvable connected

$\exists g \in \text{GL}_n(k) \quad gBg^{-1} = B_n$

By maximality

All Borel subgps are conj in $\text{GL}_n(k)$

($G/B$ will be our flag variety)
We will need the Borel Fixed Point Theorem.

If $G$ is connected, solvable, $G \leq X$ and $X$ is a projective variety, then it has a fixed point.

Lie Kolchin: $X = P(CV)$.
RADICAL OF G

Radical of G, denoted as $R(G) = \left( \bigcap_{g \in G} (gB^{-1}) \right)^0$, is the maximal among the normal solvable connected subgroups of G.

$U_{R(G)} \triangleleft R(G)$ is characteristic.

Called unipotent radical.

Example:

$R(\text{GL}_n(k)) \leq \{ \begin{pmatrix} * & \cdots & * \\ 0 & \ddots & \cdots \\ \vdots & \ddots & \ddots \\ 0 & \cdots & \ddots \end{pmatrix} \} \cap \{ \begin{pmatrix} * & \cdots & \cdots & * \\ \cdots & * & \cdots & \cdots \\ \vdots & \cdots & * & \cdots \\ \cdots & \cdots & \cdots & * \end{pmatrix} \}$

$\leq \text{ID}_n$ is a torus.

Also, one may prove that $R(\text{GL}_n(k)) = \mathbb{Z}(\text{GL}_n(k))$. 
Groups for which $\text{RCG}(G) = 1$ are called **semisimple** (e.g., $\text{SL}_n(k)$).

In general: $G$ reductive then

$$G = \text{RCG}(G) [G, G] = Z(G) [G, G]$$
LAG $G$

$p : G \rightarrow \text{GL}(V)$

Assume $V$ is **irreducible** (no $G$-stable subspace).

Then $U_{RC(G)}$ fixes a line pointwise (is unipotent).

$\Rightarrow 0 \neq V_{U_{RC(G)}}$ . Also $U_{RC(G)} < G$

$\Rightarrow V_{U_{RC(G)}}$ is $G$-stable

$\Rightarrow V = V_{U_{RC(G)}}$, i.e.,

$U_{RC(G)} \subseteq \ker p$

$\overline{p} : (G / U_{RC(G)}) \rightarrow \text{GL}(V)$

Gared is reductive

i.e. we reduced to a reductive gp

Use $\text{Gared} = Z(\text{Gred})^0 [\text{Gred}, \text{Gred}]^0$

to reduce to semisimple groups.
$G \cdot \text{LAG} , \ B \ \text{Borel sbgp}$

$G/B \ \text{flag variety}$

$\text{RCG} \subset B \implies \frac{G}{\text{RCG}} \to \frac{G/B \cong \frac{G}{\text{RCG}}}{B/\text{RCG}}$ is semisimple

Also at the level of flag variety we can reduce to semisimple (or reductive) groups.