Semisimple Lie algebras and Jordan decomposition

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Definition

A Lie algebra $L$ is *simple* if it has no nontrivial ideals and it is not abelian.

Example

Consider the Lie algebra $\mathfrak{sl}_2(K)$ of traceless $2 \times 2$ matrices. Define

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Then $\mathfrak{sl}_2(K)$ is generated by $e, f, h$. By a straight computation, one finds that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$ 

Also, $\mathfrak{sl}_2(K)$ is simple.
### Definition
A Lie algebra $L$ is called **semisimple** if $\text{Rad}(L) = 0$.

### Example
- If $L$ is a Lie algebra, then $L/\text{Rad} L$ is always semisimple.
- Simple Lie algebras are semisimple.

### Lemma
A Lie algebra $L$ is semisimple if and only if $L$ has no nonzero abelian ideals.

### Proof
- If $L$ is semisimple, $\text{Rad} L = 0$ and abelian ideals are contained in $\text{Rad} L$.
- Let $j$ be minimal s.t. $(\text{Rad} L)^{(j)} = 0$. Then $(\text{Rad} L)^{(j-1)}$ is an abelian ideal, since $[(\text{Rad} L)^{(j-1)}, (\text{Rad} L)^{(j-1)}] = 0$. 
The killing form

**Definition**

Let $L$ be a Lie algebra. The *killing form* on $L$ is the map

$$
\kappa_L(\cdot, \cdot) : L \times L \to K
$$

$$(x, y) \mapsto \text{Tr}(\text{ad}_L(x)\text{ad}_L(y)).$$

**Proposition**

(a) $\kappa_L$ is symmetric and bilinear.

(b) $\kappa_L(x, [y, z]) = \kappa_L([x, y], z)$ for every $x, y, z \in L$.

(c) If $I$ is an ideal of $L$, then $\kappa_I = \kappa_L|_{I \times I}$.

**Proof of (c)**

Let $x, y \in I$, $\text{ad}_I(x) = \begin{pmatrix} \text{ad}_I(x) & \star \\ 0 & 0 \end{pmatrix}$, $\text{ad}_I(x)\text{ad}_I(y) = \begin{pmatrix} \text{ad}_I(x)\text{ad}_I(y) & \star \\ 0 & 0 \end{pmatrix}$. 
Lemma

Let $I \subseteq L$ be an ideal. Then $I^\perp$ is an ideal in $L$ and $L^\perp \subseteq \text{Rad } L$.

Theorem

A Lie algebra $L$ is semisimple if and only if $\kappa_L$ is nondegenerate.

Proof

- Let $L^\perp = 0$ and consider $I \subseteq L$ an abelian ideal.
- For every $x \in I$, $y \in L$ we have $\text{ad}_L(x) \text{ad}_L(y) : L \rightarrow I$.
- $(\text{ad}_L(x) \text{ad}_L(y))^2 : L \rightarrow 0$ since $I$ is abelian.
- Then $\text{ad}_L(x) \text{ad}_L(y)$ is nilpotent, hence $\text{Tr}(\text{ad}_L(x) \text{ad}_L(y)) = 0$.
- Then $x \in L^\perp = 0$, so $L$ has no nonzero abelian ideals.
Structure of semisimple Lie algebras

**Theorem**

Let $L$ be semisimple. Then there exist $L_1, \ldots, L_r$ simple ideals of $L$ such that $L = L_1 \oplus \cdots \oplus L_r$, $[L_i, L_j] = \delta_{ij} L_i$ and every simple ideal of $L$ is one of these.

**Proof (Idea)**

- Ideals $I$ of $L$ correspond to subrepresentations for $\text{ad}_L$, and $I \oplus I^\perp = L$.
- Taking the irreducible representations, we find the decomposition $L = L_1 \oplus \cdots \oplus L_r$.
- If $I$ is a simple ideal, then $I = [L, I] = \oplus_i [L_i, I] = \oplus_{\text{some } i} L_i = L_j$ for some $j$. 
Corollary

Let $L$ be a semisimple Lie algebra. Then

- $[L, L] = L.$
- Every ideal and homomorphic image of $L$ is semisimple.
- Every ideal of $L$ is a sum of certain simple ideals of $L$.

Theorem (Weyl)

Every finitely dimensional representation of a semisimple Lie algebra is completely reducible.
For any $x \in \mathfrak{gl}_n(K)$ there exist a unique $x_s \in \mathfrak{gl}_n(K)$ semisimple and a unique $x_n \in \mathfrak{gl}_n(K)$ nilpotent such that $x = x_s + x_n$ and $[x_s, x_n] = 0$.

**Proposition**

Let $L$ be a Lie algebra. Then $\text{Der } L$ contains the semisimple and nilpotent parts of its elements.

- $\text{ad}_L(L) \subseteq \text{Der } L \subseteq \mathfrak{gl}(L)$.
- Let $L \subseteq \mathfrak{gl}_n(K)$. Then $\text{ad}_L(x) = \text{ad}_L(x_s) + \text{ad}_L(x_n)$ is the Jordan decomposition for $\text{ad}_L(x)$.
- $\text{ad}_L(x_s)$ and $\text{ad}_L(x_n)$ are elements of $\text{Der } L$. 
**Lemma**

Let $L$ be a Lie algebra. Then $\text{ad}_L(L)$ is an ideal of $\text{Der} L$ and $\left[\delta, \text{ad}_L(x)\right] = \text{ad}_L(\delta(x))$ for every $\delta \in \text{Der} L$ and $x \in L$.

**Proposition**

If $L$ is a semisimple Lie algebra, then $\text{ad}_L(L) = \text{Der}(L)$.

**Proof**

- $\zeta(L) = 0$, hence $L \cong \text{ad}_L(L)$ is a semisimple ideal of $\text{Der} L$.
- Call $M := \text{ad}_L(L)$ and $D := \text{Der} L$. Then $\kappa_M = \kappa_D|_{M \times M}$.
- Since $\kappa_M$ is nondegenerate, $M \cap M^\perp = 0$.
- Let $\delta \in M^\perp$. Then $0 = \left[\delta, \text{ad}_L(x)\right] = \text{ad}_L(\delta x)$ for every $x \in L$.
- Since $\text{ad}_L$ is an isomorphism, $\delta = 0$, hence $M^\perp = 0$, hence $M = D$. 
Jordan decomposition

Consequences
Let $L$ be a semisimple Lie algebra, $x \in L$.

- $\text{ad}_L(x) = (\text{ad}_L(x))_s + (\text{ad}_L(x))_n$ is the Jordan decomposition for $\text{ad}_L(x)$ in $\text{ad}_L(L) = \text{Der } L$.

- Since $\text{ad}_L$ is an isomorphism, there exist $x_s$ ad-semisimple and $x_n$ ad-nilpotent such that $\text{ad}_L(x_s) = (\text{ad}_L(x))_s$ and $\text{ad}_L(x_n) = (\text{ad}_L(x))_n$.

- $[x_s, x_n] = 0$.

Definition
The decomposition $x = x_s + x_n$ is called the abstract Jordan decomposition of the element $x \in L$. 
Question

If $L \subseteq \mathfrak{gl}(V)$ is a semisimple Lie algebra, what are the relations between usual and abstract Jordan decomposition?

Theorem

Let $L \subseteq \mathfrak{gl}(V)$ be a semisimple Lie algebra, $x \in L$. Then the usual Jordan decomposition for $x$ coincides with the abstract Jordan decomposition.

Proof (Idea)

- Let $x = s + n$ be the usual Jordan decomposition.
- $s, n \in L$.
- $s$ and $n$ are respectively ad-semisimple and ad-nilpotent.
- $x = s + n$ is the abstract Jordan decomposition.
Let $L$ be a semisimple Lie algebra, $\rho : L \rightarrow \mathfrak{gl}(V)$ a finitely dimensional representation. If $x = s + n$ is the abstract Jordan decomposition for $x \in L$, then $\rho(x) = \rho(s) + \rho(n)$ is the Jordan decomposition for $\rho(x)$ in $\mathfrak{gl}(V)$.

Proof

- $\rho(L)$ is semisimple.
- Since $L$ is spanned by the eigenvectors of $\text{ad}_L(s)$, $\rho(L)$ is spanned by the eigenvectors of $\text{ad}_{\rho(L)} \rho(s)$. Then $\rho(s)$ is ad-semisimple.
- Since $\text{ad}_L(n)$ is nilpotent, $\text{ad}_{\rho(L)} \rho(n)$ is nilpotent.
- $[\rho(s), \rho(n)] = \rho([s, n]) = 0$.
- $\rho(x) = \rho(s) + \rho(n)$ is the abstract (and hence usual) Jordan decomposition of $\rho(x)$. 

Toral subalgebras

Lemma

*If* $L$ *is a semisimple Lie algebra, then* $L$ *contains a semisimple element.*

**Proof**

- For every $x \in L$, $x = s + n$ is the Jordan decomposition.
- If every $s$ is zero, then $L$ is nilpotent.

Definition

Let $L$ be a semisimple Lie algebra. A subalgebra $T \subseteq L$ consisting of semisimple elements is called a *toral subalgebra*.

Proposition

*Let* $L$ *be a semisimple Lie algebra. Any toral subalgebra* $T$ *of* $L$ *is abelian.*
Definition
Let $L$ be a Lie algebra and $E$ be a subalgebra. Then the centralizer of $E$ in $L$ is
\[
  c_L(E) := \{ x \in L : [x, e] = 0 \text{ for every } e \in E \}.
\]

Lemma
Let $L$ be a semisimple Lie algebra. Any toral subalgebra $T$ of $L$ such that $c_L(T) = T$ it is a maximal toral subalgebra.

Proof
If $T \subseteq T' \subseteq L$ with $T'$ toral, then $[T, T'] = 0$ and so $T' \subseteq c_L(T) = T$.

Example
The algebra of traceless diagonal matrices is a maximal toral subalgebra inside $\mathfrak{sl}_n(K)$. 
Let $L$ be a semisimple Lie algebra and $H \subseteq L$ a maximal toral subalgebra.

- We study the action of $\text{ad}_L(H)$ on $L$. All elements of $\text{ad}_L(H)$ are commuting and diagonalizable, hence simultaneously diagonalizable.

**Definition**

For every $\alpha \in H^*$, where $H^*$ is the dual of $H$, we call

$$L_\alpha := \{ x \in L : [h, x] = \alpha(h)x \quad \text{for every } h \in H \}.$$ 

We call $\Phi$ the set of $\alpha \in H^* \setminus \{0\}$ such that $L_\alpha$ is nonzero.

- We have the decomposition $L = L_0 \oplus \left( \bigoplus_{\alpha \in \Phi} L_\alpha \right)$.
- $L_0 = H$.
- $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and the elements of $L_\alpha$ are nilpotent if $\alpha \neq 0$. 
\( \mathfrak{sl}_2(K) \)

\[
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

- \( \mathfrak{sl}_2(K) \) is generated by the elements \( e, f, h \) and

\[
[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.
\]

- \( Kh \) is a maximal toral subalgebra, \( e \) and \( f \) are eigenvectors for the adjoint of \( h \).
- \( \mathfrak{sl}_2(K) = \mathfrak{sl}_2(K)_0 \oplus \mathfrak{sl}_2(K)_2 \oplus \mathfrak{sl}_2(K)_{-2} = Kh \oplus Ke \oplus Kf \).
- \( Ke, Kf \) are maximal nilpotent subalgebras of \( \mathfrak{sl}_2(K) \).
- \( Ke \oplus Kh, Kf \oplus Kh \) are maximal solvable Lie algebras.
Representations of $\mathfrak{sl}_2(K)$

We consider now a representation $\rho : \mathfrak{sl}_2(K) \rightarrow \mathfrak{gl}(V)$.

- Using Weyl’s theorem we know that $\rho$ is completely reducible.
- Consider the action of $h$ on $V$: then $V = \bigoplus_{\lambda} V_{\lambda}$ where $\lambda$ varies among the eigenvalues for the action of $h$.

**Definition**

The spaces $V_{\lambda}$ are called weight spaces and the vectors $v \in V_{\lambda}$ are called weight vectors.

**Lemma**

We have that $eV_{\lambda} \subseteq V_{\lambda+2}$ and $fV_{\lambda} \subseteq V_{\lambda-2}$.

**Proof**

Let $v \in V_{\lambda}$. Then $hev = ehv + [h, e]v = \lambda ev + 2ev = (\lambda + 2)ev$. 

Representations of $\mathfrak{sl}_2(K)$

Definition

A weight vector $0 \neq v \in V_\lambda$ is called a maximal vector if $ev = 0$.

Lemma

Let $v_0 \in V_\lambda$ be a maximal vector. Define $v_{-1} := 0$ and $v_i := \frac{1}{i!} f^i v_0$ for $i \geq 0$. Then for every $i \geq 0$ we have that

(a) $hv_i = (\lambda - 2i) v_i$.
(b) $ev_i = (\lambda - i + 1) v_{i-1}$.
(c) $fv_i = (i + 1) v_{i+1}$.

- The $v_i$’s are zero or linearly independent.
- Since $V$ is finitely dimensional, there exists an $m$ such that $v_m = 0$ and $v_{m-1} \neq 0$. The space generated by $v_0, \ldots, v_{m-1}$ is $\mathfrak{sl}_2(K)$-stable.
Representations of $\mathfrak{sl}_2(K)$

Let now $\rho$ be an irreducible representation of $\mathfrak{sl}_2(K)$.

- $V$ has dimension $m$ and is spanned by $v_0, \ldots, v_{m-1}$.
- Since $0 = v_m = ev_m = (\lambda - m + 1)v_{m-1}$, we find that $m = \lambda + 1$.
- The weights are in $\mathbb{Z}$.

**CONCLUSION**

- Given an irreducible representation $V$ for $\mathfrak{sl}_2$ of dimension $m$, we find a maximal vector $v_0$ (unique up to a scalar) of weight $m - 1$.
- The vector $v_i$ has weight $m - 1 - 2i$, the vector $v_{m-1}$ has weight $-(m - 1)$.
- $\{\text{Irreducible representations of } \mathfrak{sl}_2(K)\} \rightarrow \mathbb{N}$ is a 1:1 correspondence.