Radon Transform and Cavalieri Condition: a Cohomological Approach

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Abstract

We apply the theory of integral transforms for sheaves and $\mathcal{D}$-modules to the study of the real Radon transform. By identifying abstract adjunction formulas with the explicit integral formulas appearing in the literature, we give, in particular, a cohomological interpretation of Gelfand's Cavalieri condition, and of Helgason's support theorem.

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Contents

Introduction

Let P be a real n-dimensional projective space. For $k \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote by $\mathcal{C}^\infty_P(\varepsilon|k)$ the $\mathcal{C}^\infty$-line-bundle on P whose sections f satisfy the relation:

$$f(\lambda x) = (\operatorname{sgn} \lambda)^\varepsilon \lambda^k f(x) \quad \forall \lambda \in \mathbb{R}^\times,$$

where $[x]$ is a system of homogeneous coordinates. As is well-known (see e.g., [4], [8]), for $-n-1 < k < 0$ the real projective Radon transform interchanges global sections of $\mathcal{C}^\infty_P(\varepsilon|k)$ with global sections of the line-bundle $\mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)$ on the dual projective space P^*, where $k^* = -n-1-k$, $\varepsilon^* \equiv -n-1-\varepsilon$.

With Pierre Schapira, we studied in [2] complex integral transforms within the framework of sheaves and $\mathcal{D}$-modules (note that another approach to integral transforms by $\mathcal{D}$-module theory is announced in [7]). In particular, we obtained general adjunction formulas which, by [13], have their analogue within the framework of tempered and formal cohomology. Then, we investigated in [3] the analytical aspects of the complex projective Radon transform. As an application, we recovered the above mentioned isomorphism related to the real projective Radon transform.

Let $E = P \setminus H$ be the affine chart associated to a hyperplane $H \subset P$. The Schwartz space $\mathcal{S}(E)$ of rapidly decreasing $\mathcal{C}^\infty$-functions on E, is naturally identified with the space of global sections of $\mathcal{C}^\infty_P(\varepsilon|k)$, vanishing up to infinite order in H. The real affine Radon transform is thus obtained by restriction of the projective one. A Paley-Wiener-type theorem gives necessary and sufficient conditions for a section φ of $\mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)$ to be in the image of $\mathcal{S}(E)$ by the Radon transform: depending on the parity of ε^*, either the so-called Cavalieri condition appears, or all nonlocal differentials (in the sense of [5]) have to vanish.

Here, we will use the adjunction formulas of [2], [13], and the analytical results of [3]. By computing the transform of some constant sheaves, we will then recover the Paley-Wiener-type theorem mentioned above, as well as prove other related results such as: a Borel-type theorem for nonlocal differentials, Helgason's support theorem, a description of the conformal Radon transform, or of the affine Radon transform for distributions or hyperfunctions.

As in [2], [3], the main point that we make is that our general adjunction formulas allow one to separate the analytical (i.e., $\mathcal{D}$-module-theoretical) and the topological (i.e., sheaf-theoretical) features of the transform under consideration. Thus, we will see how the different phenomena occurring in the real projective, real affine, or conformal Radon transform reflect different topological configurations attached to the same analytical setting, given by the complex projective Radon transform.

Using this approach, a delicate problem consists in making the link between the isomorphisms that we obtain, and the explicit integral formulas in the literature. We thus give in Appendix A some results on quantized integral transforms. In particular, we will show how the distribution kernel of the real Radon transform is best understood as boundary value of a meromorphic kernel associated to the complex Radon transform.

This paper is organized as follows. In section 1, we state the theorems related to the real Radon transform, using a classical formalism (i.e., without mentioning sheaves or $\mathcal{D}$-modules). In section 2, we collect from [2] and [13] the formalism and the results of the theory of integral transforms for sheaves and $\mathcal{D}$-modules that we need, and we present a brief review of [3]. Then, in section 3, after some geometric preparation (i.e., computation of the transform of some constant sheaves), we prove the theorems stated in section 1. Additional results are obtained in section 4. Finally, we gather in Appendix A some results on quantized adjunction formulas for integral transforms, necessary for identifying the associated distribution kernels, and in Appendix B a description in bi-homogeneous coordinates of the blow-up of a projective space along a point, necessary to give a projective invariant expression of nonlocal differentials.

The results in this paper were announced in [1].

We wish to express our gratitude to Masaki Kashiwara for many useful discussions during the preparation of this work.

1 Statement of the Main Results

1.1 Projective Radon Transform

In this section, using a classical formalism, we will state the results on the real Radon transform that we will discuss later in this article. Most of these results are classical and may be found in [4], [8], for example. References will be made only to [4].

Denote by $\mathbb{R}^\times$ the multiplicative group $\mathbb{R} \setminus \{0\}$. For $k \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we will say that a function f on an $\mathbb{R}^\times$-homogeneous space is $(\varepsilon|k)$-homogeneous, if

$$f(\lambda x) = (\operatorname{sgn} \lambda)^\varepsilon \lambda^k f(x) \quad \forall \lambda \in \mathbb{R}^\times. \tag{1.1}$$

Let $P = P(V)$ be the n-dimensional projective space attached to a real $(n+1)$-dimensional vector space V. Set $\dot{V} = V \setminus \{0\}$, and let $\gamma\colon \dot{V} \to P$ be the natural projection. As usual, if (x) is the system of coordinates associated to a base of V, we denote by $[x]$ the corresponding system of homogeneous coordinates in P. Let $\mathcal{C}^\infty_{\dot{V}}(\varepsilon|k)$ be the sheaf of $(\varepsilon|k)$-homogeneous $\mathcal{C}^\infty$-functions on $\dot{V}$, and set:

$$\mathcal{C}^\infty_P(\varepsilon|k) = \gamma_* \mathcal{C}^\infty_{\dot{V}}(\varepsilon|k). \tag{1.2}$$

This is the $\mathcal{C}^\infty$-line-bundle on P, whose sections satisfy the relation (1.1) when written in homogeneous coordinates. We denote by $f[x]$ the section of $\mathcal{C}^\infty_P(\varepsilon|k)$ on an open subset $U \subset P$ associated with a section $f(x)$ of $\mathcal{C}^\infty_{\dot{V}}(\varepsilon|k)$ on $\gamma^{-1}(U)$.

Remark 1.1. For $l \in \mathbb{Z}$, the map $f[x] \mapsto |x|^l f[x]$ gives an isomorphism from $\mathcal{C}^\infty_P(\varepsilon|k)$ to $\mathcal{C}^\infty_P(\varepsilon + l|k + l)$. Since this isomorphism is not canonical, we prefer to keep k as part of our notation.

Consider the distribution on $\mathbb{R}$:

$$\delta^{(\varepsilon|k)}(t) = \frac{1}{2\pi i}\frac{d^k}{dt^k}\left(\frac{1}{t - i0} - \frac{(-1)^\varepsilon}{t + i0}\right). \tag{1.3}$$

Note that the distribution $\delta^{(\varepsilon|k)}(t)$ is $(-\varepsilon - 1| - k - 1)$-homogeneous, $\delta^{(0|0)}(t)$ is the classical Dirac delta function, and $\delta^{(1|0)}(t)$ equals $\mathrm{pv}(1/t)$, the principal value of $1/t$. The n-form on V:

$$\omega(x) = \sum_{j=0}^{n}(-1)^j x_j dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

is $(n+1|n+1)$-homogeneous. We denote by $\omega[x]$ the associated $(n+1|n+1)$-density on P, usually called (real) Leray form. Let us set:

$$k^* = -n - 1 - k, \quad \varepsilon^* \equiv -n - 1 - \varepsilon, \quad \overline{\varepsilon} \equiv \varepsilon + 1. \tag{1.4}$$

Definition 1.2. Let $P^* = P(V^*)$ denote the dual projective space to P. We denote by $R^{(\varepsilon|k)}_P$ the real projective Radon transform:

$$\begin{aligned} R^{(\varepsilon|k)}_P \colon \Gamma(P; \mathcal{C}^\infty_P(\varepsilon|k)) &\to \Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)) \\ f[x] &\mapsto \int f[x]\delta^{(n+\varepsilon|n+k)}(\langle x, \xi\rangle)\omega[x]. \end{aligned}$$

Concerning the Radon transform of homogeneous $\mathcal{C}^\infty$-functions, the following result is known (cf e.g., [4, end of page 73]):

Theorem 1.3. *For $-n - 1 < k < 0$, the transform $R^{(\varepsilon|k)}_P$ introduced above is an isomorphism of inverse $R^{(\varepsilon^*|k^*)}_{P^*}$.*

1.2 Nonlocal Differentials

For $U \subset P$ a sub-analytic open subset, let us denote by

$$\Gamma_{\mathrm{w}}(U; \mathcal{C}^\infty_P(\varepsilon|k)) \subset \Gamma(P; \mathcal{C}^\infty_P(\varepsilon|k))$$

the subspace of functions vanishing up to infinite order in $P \setminus U$.

Let $H \subset P$ be a hyperplane, $\xi_\circ \in P^*$ its dual point, and set $E = P \setminus H$, $P^*_{\xi_\circ} = P^* \setminus \{\xi_\circ\}$. Let $H^* = \dot{T}_{\xi_\circ}P^*/\mathbb{R}^\times$ be the projective tangent bundle to P^* at $\xi_\circ$, and identify it with a hyperplane of P^* not containing $\xi_\circ$.

Definition 1.4. (i) For $\varphi \in \Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$ and $\xi' \in H^*$, set:

$$d^{(\omega|m)}_{\xi_\circ}\varphi[\xi'] = \int_{-\infty}^{+\infty} \varphi(\xi_\circ + t\xi')\delta^{(\omega|m)}(t)dt.$$

This is a section of $\Gamma(H^*; \mathcal{C}^\infty_{H^*}(\omega|m))$.

(ii) For $\varphi \in \Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$ and $\xi' \in H^*$, set:

$$c^{(\omega|m)}_{\xi_\circ}\varphi[\xi'] = \int_{-\infty}^{+\infty} \varphi(s\xi_\circ + \xi')\,\mathrm{sgn}(s)^\omega s^m ds.$$

This is a section of $\Gamma(H^*; \mathcal{C}^\infty_{H^*}(\omega + \varepsilon^* + 1|m + k^* + 1))$.

Remark 1.5. (a) For $t, s \in \mathbb{R}$, the points of homogeneous coordinates $[\xi_\circ + t\xi']$, $[s\xi_\circ + \xi']$ describe two affine charts of the projective line issued from $\xi_\circ$ with tangent direction ξ'. This description is not projectively invariant, since it depends on the non canonical identification of H^* with a hyperplane of P^*. That is why we had to write $\varphi(\cdot)$ instead of $\varphi[\cdot]$ as integrand in the above definitions. For an invariant expression of the above functionals, refer to Appendix B.

(b) Note that $d^{(0|1)}_{\xi_\circ}\varphi$ is the usual differential of φ at $\xi_\circ$, and that

$$d^{(1|1)}_{\xi_\circ}\varphi[\xi'] = \mathrm{pv}\int_{-\infty}^{+\infty} \frac{\varphi(\xi_\circ + t\xi')}{t^2}dt$$

is the nonlocal differential of φ at $\xi_\circ$ in the sense of [5].

(c) For $\omega \equiv \varepsilon^*$ it is possible to make $c^{(\omega|m)}_{\xi_\circ}$ act on the whole $\Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$, by patching the two charts described in (a). This gives:

$$c^{(\varepsilon^*|m)}_{\xi_\circ}\varphi[\xi'] = d^{(1|m+k^*+1)}_{\xi_\circ}\varphi[\xi'].$$

On the contrary, note that for $\omega \equiv \overline{\varepsilon}^*$ the tempered distribution kernel on $P^*_{\xi_\circ} \times H^*$ associated to $c^{(\overline{\varepsilon}^*|m)}_{\xi_\circ}$ cannot be extended to the whole $P^* \times H^*$ as a kernel sending $\Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$ to $\Gamma(H^*; \mathcal{C}^\infty_{H^*}(0|m + k^* + 1))$.

1.3 Affine Radon transform

The space $\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(\varepsilon|k))$ is naturally identified with the Schwartz space of rapidly decreasing $\mathcal{C}^\infty$-functions on the affine chart E. For $k = -n$, $\varepsilon \equiv -n$, the restriction of $R^{(\varepsilon|k)}_P$ to the space $\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(\varepsilon|k))$ is the classical affine Radon transform, given by integration along hyperplanes. A natural problem is then to describe the image by $R^{(\varepsilon|k)}_P$ of $\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(\varepsilon|k))$ in $\Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$.

Theorem 1.6. *Assume* $-n-1 < k < 0$. *Let* $\varphi \in \Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*))$.

(i) If $\varepsilon^* \equiv 0$, *then there exists a function* $f \in \Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(\overline{n}|k))$ *such that* $\varphi = R_P^{(\overline{n}|k)} f$ *if and only if* φ *satisfies the odd Cavalieri condition: for any non negative integer* m*:*

$$d_{\xi_\circ}^{(1|k^*+m+1)} \varphi = 0. \tag{1.5}$$

(ii) If $\varepsilon^* \equiv 1$, *then there exists a function* $f \in \Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(n|k))$ *such that* $\varphi = R_P^{(n|k)} f$ *if and only if* φ *belongs to* $\Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(1|k^*))$, *and satisfies the (even) Cavalieri condition: for any non negative integer* m *and any* $\xi' \in H^*$, *the integral*

$$c_{\xi_\circ}^{(0|m)} \varphi[\xi'] = \int_{-\infty}^{+\infty} \varphi(s\xi_\circ + \xi') s^m ds \tag{1.6}$$

is a homogeneous polynomial of degree $m + k^* + 1$ *in* ξ'.

Note that part (ii) was obtained e.g., in [4, page 86] for $k = -n$. Of course, part (i) could also be proved by the same method, which make use of the inversion formula for the Fourier transform. As remarked by those authors, the exponential kernel of the Fourier transform makes their method of proof violate the projective invariance of the Radon transform. Our approach is different, and will show how Cavalieri condition is of a geometrical nature, related to the complex projective Radon transform.

An old theorem of Borel asserts that any formal series is the Taylor series of some $\mathcal{C}^\infty$ function. As a byproduct of our cohomological proof of Theorem 1.6, we will get the following Borel-type theorem for nonlocal differentials.

Theorem 1.7. *Assume* $-n-1 < k < 0$. *For* $\omega \in \mathbb{Z}/2\mathbb{Z}$, *and for any non negative integer* m, *take* $g_m \in \Gamma(H^*; \mathcal{C}^\infty_{H^*}(\omega|k^* + m + 1))$.

(i) If $\omega \equiv 1$, *then there exists* $f \in \Gamma(P^*; \mathcal{C}^\infty_{P^*}(0|k^*))$ *such that* $d_{\xi_\circ}^{(1|k^*+m+1)} f = g_m$, *for any* m.

(ii) If $\omega \equiv 0$, *then there exists* $f \in \Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(1|k^*))$ *such that* $c_{\xi_\circ}^{(0|m)} f \equiv g_m$ *modulo homogeneous polynomials, for any* m.

Other results that can be obtained along the same lines are discussed in section 4. These are for example Helgason's support theorem, and a description of the conformal Radon transform, or of the affine Radon transform for distributions or hyperfunctions.

2 Review on Integral Transforms

2.1 Notations

Let M be a real analytic manifold. If $A \subset M$ is a locally closed subset, we denote by $\mathbb{C}_A$ the sheaf on M which is the constant sheaf on A with stalk $\mathbb{C}$, and zero on $M \setminus A$. Denote by $\mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on M, and by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$ its full triangulated sub-category of objects with $\mathbb{R}$-constructible cohomology groups. We denote by $\otimes$, $Rf_!$, f^{-1}, $R\mathcal{H}om$, Rf_*, and $f^!$ the "six operations" of sheaf theory, and we denote by $\boxtimes$ the exterior tensor product. We set for short $\mathrm{RHom}(\cdot,\cdot) = \mathrm{R}\Gamma(M; R\mathcal{H}om(\cdot,\cdot))$. To $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_M)$ we associate its duals $D'F = R\mathcal{H}om(F, \mathbb{C}_M)$, $DF = R\mathcal{H}om(F, \omega_M)$, where $\omega_M \simeq or_M[\dim^{\mathbb{R}} M]$ denotes the dualizing complex, and or_M the orientation sheaf. We denote by T^*M the cotangent bundle to M, and we set $\dot{T}^*M = T^*M \setminus T^*_M M$, the cotangent bundle with the zero-section removed. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M; \dot{T}^*M)$ the localization of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$ by the null system $\mathcal{N}$ of complexes of constant sheaves in M with finite rank (in the terminology of [12], $\mathcal{N}$ is the null system of objects F whose micro-support $SS(F)$ is contained in the zero-section $T^*_M M$). Recall that the objects of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M; \dot{T}^*M)$ are the same as those of $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$, and that a morphism $F \underset{u}{\to} G$ in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)$ becomes an isomorphism in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M; \dot{T}^*M)$ if and only if the third term H of a distinguished triangle $F \underset{u}{\to} G \to H \underset{+1}{\longrightarrow}$ is in $\mathcal{N}$.

Denote by $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_M)$ the derived category of the category of bounded complexes of left modules over the sheaf of rings $\mathcal{D}_M$ of linear differential operators. Following [10], [13], one considers the functors:

$$
\begin{aligned}
T\mathcal{H}om(\cdot, \mathcal{D}b_M)\colon \quad & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M)^{\mathrm{op}} \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_M),\\
\cdot \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M\colon \quad & \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_M) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_M),
\end{aligned}
$$

where $\mathcal{D}b_M$ denotes the sheaf of Schwartz's distributions on M. These are induced by exact functors from the abelian category of $\mathbb{R}$-constructible sheaves, characterized by the requirement that if Z is a closed sub-analytic subset of M, then $T\mathcal{H}om(\mathbb{C}_Z, \mathcal{D}b_M) = \Gamma_Z \mathcal{D}b_M$ and $\mathbb{C}_{M\setminus Z} \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_M = \mathcal{I}^\infty_{Z,M}$, where $\mathcal{I}^\infty_{Z,M}$ denotes the ideal of $\mathcal{C}^\infty_M$ of functions vanishing to infinite order on Z.

Let X be a complex manifold, and denote by d_X its dimension. If $f\colon S \to X$ is a morphism, we set $d_{S/X} = d_S - d_X$. Denote by $\mathcal{O}_X$ the structural sheaf of X, by Ω_X the holomorphic forms of maximal degree, and by $\mathcal{D}_X$ the sheaf of rings of holomorphic linear differential operators.

We denote by $\underline{f}_*$ and $\underline{f}^{-1}$ the proper direct image and inverse image for $\mathcal{D}$-modules, and we denote by $\underline{\boxtimes}$ the exterior tensor product. To $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ we associate its dual $\underline{D}_X \mathcal{M} = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X])$. We set for short $\mathcal{S}ol(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$. We say that a $\mathcal{D}_X$-module $\mathcal{M}$ is good (resp. quasi-good) if, on every relatively compact open subset of X, it admits a filtration $\{\mathcal{M}_k\}$

by coherent $\mathcal{D}_X$-submodules such that each quotient $\mathcal{M}_k/\mathcal{M}_{k-1}$ admits a good filtration and $\mathcal{M}_k = 0$ for $|k| \gg 0$ (resp. $k \ll 0$). We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ (resp. $\mathbf{D}^{\mathrm{b}}_{\mathrm{q-good}}(\mathcal{D}_X)$) the full triangulated sub-category of $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$ consisting of objects with good (resp. quasi-good) cohomology groups. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{rh}}(\mathcal{D}_X)$ the full triangulated sub-category of $\mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$ of objects with regular holonomic cohomology groups.

For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, the complexes $T\mathcal{H}om(F, \mathcal{O}_X)$ and $F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ are defined in [10], [13] as the Dolbeault complexes with coefficients in $T\mathcal{H}om(F, \mathcal{D}b_X)$ and $F \overset{\mathrm{w}}{\otimes} \mathcal{C}^\infty_X$, respectively. We set for short $\mathrm{THom}\,(\cdot,\cdot) = \mathrm{R}\Gamma(X; T\mathcal{H}om(\cdot,\cdot))$.

2.2 Integral Kernels

Let us be given a correspondence of complex manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

This induces a morphism $h = (f, g)$:

$$S \xrightarrow[h]{} X \times Y.$$

For $G \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_Y)$, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, and kernels $L \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_S)$, $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_S)$, we define:

$$\left\{ \begin{array}{rcl} L \circ G & = & Rf_!(L \otimes g^{-1}G), \\ \mathcal{M} \underline{\circ} \mathcal{L} & = & \underline{g}_*(\underline{f}^{-1}\mathcal{M} \otimes^L_{\mathcal{O}_S} \mathcal{L}), \end{array} \right. \tag{2.1}$$

and we similarly define $F \circ L$ and $\mathcal{L} \underline{\circ} \mathcal{N}$ for $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$.

Remark 2.1. By the projection formula, there are natural isomorphisms:

$$L \circ G \simeq (Rh_!L) \circ G, \quad \mathcal{M} \underline{\circ} \mathcal{L} \simeq \mathcal{M} \underline{\circ} (\underline{h}_*\mathcal{L}), \tag{2.2}$$

where we consider $Rh_!L$ and $\underline{h}_*\mathcal{L}$ as kernels on the product $X \times Y$, endowed with the two natural projections. Recall that if h is proper and L is $\mathbb{C}$-constructible, one has $T\mathcal{H}om(Rh_!L, \mathcal{O}_{X\times Y})[d_X + d_Y - d_S] \simeq \underline{h}_*\mathcal{L}$.

We obtained in [2] general adjunction formulas for the functors $\circ$ and $\underline{\circ}$. Analogue formulas in the framework of formal and temperate cohomology were obtained in [13]:

Theorem 2.2. *([13, Theorem 10.8]) Let $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_Y)$, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{q-good}}(\mathcal{D}_X)$, $L \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_S)$, and set $\mathcal{L} = T\mathcal{H}om(L, \mathcal{O}_S)$. Assume that:*

$$\left\{ \begin{array}{ll} f^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L}) & \textit{is proper over } Y, \\ g^{-1}\operatorname{supp}(G) \cap \operatorname{supp}(L) & \textit{is proper over } X. \end{array} \right.$$

Then, there are natural isomorphisms:

$$\mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{M} \underline{\circ} \mathcal{L}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y) \overset{\sim}{\leftarrow} \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, (L \circ G) \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_{S/Y}], \tag{2.3}$$

$$\mathrm{R}\Gamma_c(Y; T\mathcal{H}om(G, \Omega_Y) \otimes^L_{\mathcal{D}_Y} (\mathcal{M} \underline{\circ} \mathcal{L}))[d_{S/X}] \tag{2.4}$$
$$\overset{\sim}{\to} \mathrm{R}\Gamma_c(X; T\mathcal{H}om(L \circ G, \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}).$$

In order to apply the above general result to specific cases, a topological and an analytical computations have to be performed. The first consists in computing $L \circ G$, or at least in finding an $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$ and a morphism:

$$F \to L \circ G, \tag{2.5}$$

while the second consists in computing $\mathcal{M} \underline{\circ} \mathcal{L}$, or at least in finding an $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$ and a morphism:

$$\mathcal{N} \to \mathcal{M} \underline{\circ} \mathcal{L}. \tag{2.6}$$

Finally, one has to explicitly describe the morphisms:

$$\mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{N}, G \overset{\mathrm{w}}{\otimes} \mathcal{O}_Y) \leftarrow \mathrm{RHom}_{\mathcal{D}_X}(\mathcal{M}, F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X)[d_{S/Y}], \tag{2.7}$$

$$\mathrm{R}\Gamma_c(Y; T\mathcal{H}om(G, \Omega_Y) \otimes^L_{\mathcal{D}_Y} \mathcal{N})[d_{S/X}] \to \mathrm{R}\Gamma_c(X; T\mathcal{H}om(F, \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}), \tag{2.8}$$

obtained by means of (2.5), (2.3) (resp. (2.4)), and (2.6). The following lemma describes the kernels associated to morphisms like (2.6). We discuss in Appendix A how to compute the distribution kernels associated to morphisms like (2.7), (2.8).

Lemma 2.3. *(cf [3, Lemma 3.1]) Let $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{good}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, $\mathcal{L} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_S)$. Assume that $f^{-1}\operatorname{supp}(\mathcal{M}) \cap \operatorname{supp}(\mathcal{L})$ is proper over Y. Then, there is a natural isomorphism:*

$$\alpha\colon \mathrm{Hom}_{\mathcal{D}_{X\times Y}}(\underline{D}\mathcal{M} \boxtimes \mathcal{N}[-d_X], \underline{h}_*\mathcal{L}) \overset{\sim}{\to} \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{N}, \mathcal{M} \underline{\circ} \mathcal{L}).$$

Proof. By (2.2), it is not restrictive to assume $S = X \times Y$, f and g being the natural projections. In this case, one has:

$$\mathcal{M} \underline{\circ} \mathcal{L} \simeq Rg_!(\mathcal{L}^{(d_X,0)} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}),$$

where we set $\mathcal{L}^{(d_X,0)} = f^{-1}\Omega_X \otimes_{f^{-1}\mathcal{O}_X} \mathcal{L}$. To conclude, it is then enough to consider the chain of isomorphisms:

$$\begin{aligned}
\mathrm{RHom}_{\mathcal{D}_S}(\underline{D}\mathcal{M} \boxtimes \mathcal{N}[-d_X], \mathcal{L}) &\simeq \mathrm{RHom}_{g^{-1}\mathcal{D}_Y}(g^{-1}\mathcal{N}, R\mathcal{H}om_{f^{-1}\mathcal{D}_X}(f^{-1}\underline{D}\mathcal{M}[-d_X], \mathcal{L}))\\
&\simeq \mathrm{RHom}_{g^{-1}\mathcal{D}_Y}(g^{-1}\mathcal{N}, f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{f^{-1}\mathcal{D}_X} \mathcal{L})\\
&\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{N}, Rg_*(\mathcal{L}^{(d_X,0)} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M}))\\
&\simeq \mathrm{RHom}_{\mathcal{D}_Y}(\mathcal{N}, Rg_!(\mathcal{L}^{(d_X,0)} \otimes_{f^{-1}\mathcal{D}_X} f^{-1}\mathcal{M})).
\end{aligned}$$

□

2.3 Complex Projective Radon Transform

In this subsection we will review the results of [3] on the complex projective Radon transform. Note that here, following [14], we use a different kernel with respect to loc. cit. Since the proofs do not change significantly, we will not repeat them here.

Let $\mathbb{P}$ be a complex n-dimensional projective space, $\mathbb{P}^*$ be the dual projective space, and consider the diagram:

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook] & \mathbb{P} \times \mathbb{P}^* \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P} & & \mathbb{P}^*, \end{tikzcd}$$

where f, g are the natural projections, and $\mathbb{A} = \{(z,\zeta);\ \langle z,\zeta\rangle = 0\}$ denotes the incidence relation. Note that, denoting by $\Lambda = \dot{T}^*_{\mathbb{A}}(\mathbb{P}\times\mathbb{P}^*)$ the conormal bundle to $\mathbb{A}$ with the zero-section removed, the associated microlocal correspondence:

$$\begin{tikzcd} & \Lambda \arrow[dl] \arrow[dr] & \\ \dot{T}^*\mathbb{P} & & \dot{T}^*\mathbb{P}^* \end{tikzcd}$$

induces a globally defined contact transformation (the Legendre transform):

$$\chi\colon \dot{T}^*\mathbb{P} \xrightarrow{\sim} \dot{T}^*\mathbb{P}^*. \tag{2.9}$$

Set $\Omega = (\mathbb{P}\times\mathbb{P}^*) \setminus \mathbb{A}$, and consider the kernels on $\mathbb{P}\times\mathbb{P}^*$:

$$L = \mathbb{C}_\Omega, \qquad \mathcal{L} = T\mathcal{H}om(L, \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}), \tag{2.10}$$

so that $\mathcal{S}ol(\mathcal{L}) \simeq L$ by the Riemann-Hilbert correspondence of [10]. Note that $\mathcal{L} = \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*}[*\mathbb{A}]$ is the sheaf of meromorphic functions with poles on $\mathbb{A}$.

Remark 2.4. By [3, Lemma 4.6] (see also [12, Exercise III.15]), the functors:

$$\begin{aligned} \cdot \circ L\colon &\qquad \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}}) \to \mathbf{D}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}^*}),\\ \cdot \mathbin{\underline{\circ}} \mathcal{L}\colon &\qquad \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}}) \to \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}^*}), \end{aligned}$$

are equivalences of categories, with quasi-inverse $DL \circ \cdot$ and $\underline{D}\mathcal{L} \mathbin{\underline{\circ}} \cdot$, respectively. Moreover, the first equivalence preserves $\mathbb{R}$- and $\mathbb{C}$-constructibility, while the second preserves goodness. Using (2.9) it follows as in [2, Proposition 3.5] that the transform $\mathcal{M} \mathbin{\underline{\circ}} \mathcal{L}$ of a good $\mathcal{D}_{\mathbb{P}}$-module $\mathcal{M}$ is essentially concentrated in degree zero (precisely, this means that the cohomology groups $H^j(\mathcal{M} \mathbin{\underline{\circ}} \mathcal{L})$ are flat holomorphic connections for $j \neq 0$).

Let $k, l \in \mathbb{Z}$. Denote by $\mathcal{O}_{\mathbb{P}}(k)$ the $-k$-th tensor power of the tautological line bundle, and set $\mathcal{D}_{\mathbb{P}}(k) = \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)$. For $\mathcal{H} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}\times\mathbb{P}^*})$, set for short

$$\mathcal{H}^{(n,0)}(k,l) = f^{-1}(\Omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(k)) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}}} \mathcal{H} \otimes_{g^{-1}\mathcal{O}_{\mathbb{P}^*}} g^{-1}\mathcal{O}_{\mathbb{P}^*}(l).$$

Following Leray [15, p. 94], we set:

$$\begin{aligned} \omega[z] &= \sum_{j=0}^{n} (-1)^j z_j \, dz_0 \wedge \cdots \wedge \widehat{dz_j} \wedge \cdots \wedge dz_n, \\ s_k[z,\zeta] &= \frac{1}{2\pi i} \frac{\omega[z]}{\langle z,\zeta\rangle^{n+1+k}} \in \Gamma(\mathbb{P}\times\mathbb{P}^*; \mathcal{L}^{(n,0)}(-k,k^*)). \end{aligned} \tag{2.11}$$

Theorem 2.5. *([3, Theorem 4.3]) Assume $-n-1 < k < 0$. Then, the morphism*

$$\alpha(s_k)\colon \mathcal{D}_{\mathbb{P}^*}(-k^*) \to \mathcal{D}_{\mathbb{P}}(-k) \underline{\circ} \mathcal{L}$$

(where α has been introduced in Lemma 2.3) is an isomorphism in $\mathbf{D}^{\mathrm{b}}(\mathcal{D}_{\mathbb{P}^*})$. Its inverse is associated to the kernel $s_{k^*}[\zeta, z]$.

Briefly, the idea of the proof is as follows: In view of (2.9), the theory of micro-differential operators of [17] implies that $\alpha(s_k)$ is an isomorphism in $\dot{T}^*\mathbb{P}^*$ for $n+1+k > 0$. We use the hypothesis $k < 0$ to extend the isomorphism across the zero-section.

Applying Theorems 2.5 and 2.2, we get the following corollary.

Corollary 2.6. *([3, Corollary 4.5]) Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}})$. Then, for $-n-1 < k < 0$ the section s_k induces isomorphisms:*

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{P}; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)) &\overset{\sim}{\leftarrow} \mathrm{R}\Gamma(\mathbb{P}^*; (F\circ L) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[n], \\ \mathrm{THom}\,(F, \mathcal{O}_{\mathbb{P}}(k))[n] &\overset{\sim}{\to} \mathrm{THom}\,(F\circ L, \mathcal{O}_{\mathbb{P}^*}(k^*)). \end{aligned}$$

3 Proof of the Main Results

3.1 Geometrical Preliminaries

As in (2.10), consider the kernel $L = \mathbb{C}_\Omega$ on $\mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{P}\times\mathbb{P}^*})$. Let P and P^* be the real projective spaces of which $\mathbb{P}$ and $\mathbb{P}^*$ are the respective complexifications (compatible with the embedding of affine charts $\mathbb{R}^n \to \mathbb{C}^n$). Assume for simplicity that $n > 2$. Since $\pi_1(P) = \mathbb{Z}/2\mathbb{Z}$, there are essentially two locally constant sheaves of rank 1 on P: the constant sheaf $\mathbb{C}_P$, that we also denote by $\mathbb{C}_P(0)$, and the canonical line bundle, that we denote by $\mathbb{C}_P(1)$. Recall notations (1.4).

Lemma 3.1. *([3, Proposition 5.16]) For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, set:*

$$\mathbb{C}'_{P^*}(\varepsilon^*)[-n] = \mathbb{C}_P(\varepsilon) \circ L.$$

Then, for $\varepsilon^* \equiv 1$ we have $\mathbb{C}'_{P^*}(1) \simeq \mathbb{C}_{P^*}(1)$, and for $\varepsilon^* \equiv 0$ we have a distinguished triangle (d.t. for short):

$$\mathbb{C}_{\mathbb{P}^* \setminus P^*}[1] \to \mathbb{C}'_{P^*}(0) \to \mathbb{C}_{\mathbb{P}^*} \underset{+1}{\longrightarrow} .$$

In particular, there are natural morphisms:

$$\beta_\varepsilon \in \mathrm{Hom}\,(\mathbb{C}_{P^*}(\varepsilon^*)[-n], \mathbb{C}_P(\varepsilon) \circ L), \tag{3.1}$$

which become isomorphisms in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$.

Proof. For the description of $\mathbb{C}_P(\varepsilon) \circ L$ we refer to loc. cit. Here, we just point out that β_{1^*} is given by the isomorphism $\mathbb{C}'_{P^*}(1) \simeq \mathbb{C}_{P^*}(1)$, and that β_{0^*} is given by the natural morphisms:

$$\mathbb{C}_{P^*}(0) \to \mathbb{C}_{\mathbb{P}^* \setminus P^*}[1] \to \mathbb{C}'_{P^*}(0).$$

□

Notation 3.2. Let $H \subset P$ be a hyperplane, $\mathbb{H} \subset \mathbb{P}$ its natural complexification, and $\xi_\circ \in P^*$ its dual point. Let $H^* \subset \mathbb{H}^*$ be the dual projective spaces to H and $\mathbb{H}$ respectively. Set $E = P \setminus H$, $\mathbb{P}^*_{\xi_\circ} = \mathbb{P}^* \setminus \{\xi_\circ\}$, $P^*_{\xi_\circ} = P^* \setminus \{\xi_\circ\}$, and consider the maps:

$$\mathbb{P}^* \underset{i}{\leftarrow} \mathbb{P}^*_{\xi_\circ} \underset{q}{\rightarrow} \mathbb{H}^*,$$

where i is the embedding, and q is the natural projection, dual of the embedding $\mathbb{H} \to \mathbb{P}$. Finally, set $Q^* = q^{-1}(H^*) \subset \mathbb{P}^*$.

Using Lemma 3.1, we can easily compute the transform by L of the constant sheaves $\mathbb{C}_H(\varepsilon)$:

Lemma 3.3. *Set $\mathbb{C}'_{Q^*}(\varepsilon) = i_! q^{-1} \mathbb{C}'_{H^*}(\varepsilon)$. Then we have:*

$$\mathbb{C}_H(\varepsilon) \circ L \simeq \mathbb{C}'_{Q^*}(\overline{\varepsilon}^*)[1-n].$$

Proof. Consider the natural maps

$$\mathbb{P} \times \mathbb{P}^* \underset{\widetilde{\imath}}{\leftarrow} \mathbb{H} \times \mathbb{P}^*_{\xi_\circ} \underset{\widetilde{q}}{\rightarrow} \mathbb{H} \times \mathbb{H}^*.$$

Let $\mathbb{B} \subset \mathbb{H} \times \mathbb{H}^*$ be the incidence relation, set $\omega = (\mathbb{H} \times \mathbb{H}^*) \setminus \mathbb{B}$, and consider the kernel

$$K = \mathbb{C}_\omega \in \mathbf{D}^{\mathrm{b}}_{\mathbb{C}-c}(\mathbb{C}_{\mathbb{H} \times \mathbb{H}^*}).$$

Since $\Omega \cap (\mathbb{H} \times \mathbb{P}^*) = \Omega \cap (\mathbb{H} \times \mathbb{P}^*_{\xi_\circ}) = \widetilde{q}^{-1}(\mathbb{B})$, we have:

$$\begin{aligned}
\mathbb{C}_H(\varepsilon) \circ L &= Rg_!(\mathbb{C}_\Omega \otimes f^{-1}\mathbb{C}_H(\varepsilon)) \\
&\simeq Rg_!(\mathbb{C}_{\Omega\cap(\mathbb{H}\times\mathbb{P}^*)} \otimes f^{-1}\mathbb{C}_H(\varepsilon)) \\
&\simeq Rg_! R\widetilde{\imath}_! \widetilde{q}^{-1}(\mathbb{C}_\omega \otimes f^{-1}\mathbb{C}_H(\varepsilon)) \\
&\simeq Ri_! q^{-1} Rg_!(\mathbb{C}_\omega \otimes f^{-1}\mathbb{C}_H(\varepsilon)) \\
&\simeq Ri_! q^{-1}(\mathbb{C}_H(\varepsilon) \circ K) \\
&= Ri_! q^{-1}\mathbb{C}'_{H^*}(\overline{\varepsilon}^*)[1-n],
\end{aligned}$$

where the last equality follows from Lemma 3.1. (By abuse of notation, we denoted by f, g both the projections from $\mathbb{P} \times \mathbb{P}^*$ and from $\mathbb{H} \times \mathbb{H}^*$.) □

Remark 3.4. Let us describe the microlocal geometry underlying Lemma 3.3. Recall that χ denotes the Legendre transform (2.9). By the microlocal theory of sheaves of [12], since the sheaf $\mathbb{C}_H(\varepsilon)$ is simple along the conormal bundle $\dot{T}^*_H\mathbb{P}$, one knows *a priori* that its transform by $\cdot \circ L$ is again a simple sheaf along $\chi(\dot{T}^*_H\mathbb{P})$. In this sense, the complex lines in Q^* (which are the fibers of q) correspond to the complex conormal directions to H in $\mathbb{P}$.

Let us define $\mathbb{C}_E(\varepsilon)$ by the natural short exact sequence:

$$0 \to \mathbb{C}_E(\varepsilon) \to \mathbb{C}_P(\varepsilon) \to \mathbb{C}_H(\varepsilon) \to 0.$$

Of course, $\mathbb{C}_E(\varepsilon) \simeq \mathbb{C}_E$, but since the morphism $\mathbb{C}_P(\varepsilon) \to \mathbb{C}_H(\varepsilon)$ depends on ε, we keep the twist as part of our notation.

Lemma 3.5. *In the triangulated category $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$, there are the following d.t.s:*

$$\mathbb{C}_E(0^*) \circ L \to \mathbb{C}_{P^*}(0)[-n] \to \mathbb{C}_{Q^*}(1)[1-n] \xrightarrow[+1]{}, \tag{3.2}$$

$$\mathbb{C}_E(1^*) \circ L \to \mathbb{C}_{P^*_{\xi_\circ}}(1)[-n] \to \mathbb{C}_{\overline{Q^*}}[1-n] \xrightarrow[+1]{} . \tag{3.3}$$

Proof. By Lemmas 3.1 and 3.3, we have a d.t.:

$$\mathbb{C}_E(\varepsilon^*) \circ L \to \mathbb{C}'_{P^*}(\varepsilon)[-n] \xrightarrow[a]{} \mathbb{C}'_{Q^*}(\overline{\varepsilon})[1-n] \xrightarrow[+1]{} . \tag{3.4}$$

Since $\mathbb{C}'_{P^*}(0) \simeq \mathbb{C}_{P^*}(0)$ in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$, (3.2) follows. Since $\mathbb{C}_{\mathbb{P}^*_{\xi_\circ}\setminus Q^*}[1] \simeq \mathbb{C}_{\overline{Q^*}}$ and $\mathbb{C}_{\mathbb{P}^*_{\xi_\circ}}[1] \simeq \mathbb{C}_{\xi_\circ}$ in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$, by definition of $\mathbb{C}'_{Q^*}(0)$ we have a d.t.:

$$\mathbb{C}_{\overline{Q^*}}[1-n] \to \mathbb{C}'_{Q^*}(0)[1-n] \xrightarrow[b]{} \mathbb{C}_{\xi_\circ}[-n] \xrightarrow[+1]{} . \tag{3.5}$$

Applying the octahedral axiom to (3.4), (3.5), and to the d.t.:

$$\mathbb{C}_{P^*_{\xi_\circ}}(1)[-n] \to \mathbb{C}_{P^*}(1)[-n] \xrightarrow[b\circ a]{} \mathbb{C}_{\xi_\circ}[-n] \xrightarrow[+1]{},$$

we get (3.3). □

3.2 Real Projective Radon Transform

As we noticed in [3], the space of $(\varepsilon|k)$-homogeneous $\mathcal{C}^\infty$ functions or distributions, defined as in (1.2), may be described in terms of the functors $\overset{\mathrm{w}}{\otimes}$ and $T\mathcal{H}om$:

Lemma 3.6. *For $k \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ there are natural identifications:*

$$\mathcal{C}^\infty_P(\varepsilon|k) \simeq \mathbb{C}_P(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k), \tag{3.6}$$

$$\mathcal{D}b_P(\varepsilon|k) \simeq T\mathcal{H}om(D'(\mathbb{C}_P(\varepsilon)), \mathcal{O}_{\mathbb{P}}(k)). \tag{3.7}$$

Proof. Since the arguments are similar, we will just consider (3.7). Recall that $P = P(V)$ for an $(n+1)$-dimensional real vector space V, so that $\mathbb{P} = \mathbb{P}(W)$ for a complexification W of V. Set $\dot{W} = W \setminus \{0\}$, and denote by $\mathcal{M}_k$ the left $\mathcal{D}_{\dot{W}}$-module associated with the differential operator $(\sum_{j=0}^n z_j\partial_j) - k$. Then $R\mathcal{H}om_{\mathcal{D}_{\dot{W}}}(\mathcal{M}_k, \mathcal{D}b_{\dot{V}})$ is the sheaf of $\mathbb{R}_{>0}$-homogeneous distributions of degree k on $\dot{V}$. Since any $\mathbb{R}_{>0}$-homogeneous function decomposes into the sum of an even and an odd $\mathbb{R}^\times$-homogeneous function, to prove (3.7) we have to establish the isomorphism:

$$R\gamma_* R\mathcal{H}om_{\mathcal{D}_{\dot{W}}}(\mathcal{M}_k, \mathcal{D}b_{\dot{V}}) \simeq T\mathcal{H}om(D'(\mathbb{C}_P(0) \oplus \mathbb{C}_P(1)), \mathcal{O}_{\mathbb{P}}(k)),$$

where $\gamma\colon \dot{W} \to \mathbb{P}$ is the natural projection. One has $\mathcal{M}_k \simeq \underline{\gamma}^{-1}\mathcal{D}_{\mathbb{P}}(-k)$, and hence:

$$\begin{aligned}
&R\gamma_* R\mathcal{H}om_{\mathcal{D}_{\dot{W}}}(\mathcal{M}_k, \mathcal{D}b_{\dot{V}}) \\
&\quad\simeq R\gamma_* R\mathcal{H}om_{\mathcal{D}_{\dot{W}}}(\underline{\gamma}^{-1}\mathcal{D}_{\mathbb{P}}(-k), T\mathcal{H}om(D'\mathbb{C}_{\dot{V}}, \mathcal{O}_{\dot{W}})) \\
&\quad\simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}_{\mathbb{P}}(-k), \underline{\gamma}_* T\mathcal{H}om(D'\mathbb{C}_{\dot{V}}, \mathcal{O}_{\dot{W}}))[-1] \\
&\quad\simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}_{\mathbb{P}}(-k), T\mathcal{H}om(R\gamma_! D'\mathbb{C}_{\dot{V}}, \mathcal{O}_{\mathbb{P}}))[-2] \\
&\quad\simeq R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}}}(\mathcal{D}_{\mathbb{P}}(-k), T\mathcal{H}om(R\gamma_! \mathbb{C}_{\dot{V}}[1-n], \mathcal{O}_{\mathbb{P}})),
\end{aligned}$$

where in the second isomorphism we used the fact that γ is smooth, and where the third isomorphism follows from [11, Corollary 9.2.2]. Recall that $D'(\mathbb{C}_P(\varepsilon)) \simeq \mathbb{C}_P(\varepsilon^*)[-n]$. One then concludes using the following lemma. □

Lemma 3.7. *Denote by $\gamma\colon \dot{V} \to P$ the natural projection. Then, there is an isomorphism:*

$$R\gamma_! \mathbb{C}_{\dot{V}}[1] \simeq \mathbb{C}_P(0) \oplus \mathbb{C}_P(1).$$

Proof. Denote by $S = \dot{V}/\mathbb{R}_{>0}$ the real sphere, and decompose γ into:

$$\dot{V} \underset{p}{\to} S \underset{q}{\to} P.$$

Clearly, $H^j Rp_! \mathbb{C}_{\dot{V}}$ vanishes for $j \neq 1$, and, for $j = 1$, it is a locally constant sheaf of rank one. Since S is simply connected, we get $Rp_! \mathbb{C}_{\dot{V}}[1] \simeq \mathbb{C}_S$. The trace morphism $q_! \mathbb{C}_S \simeq q_! q^! \mathbb{C}_P \to \mathbb{C}_P$ induces a short exact sequence:

$$0 \to \mathbb{C}_P(1) \to q_! \mathbb{C}_S \underset{tr}{\to} \mathbb{C}_P(0) \to 0.$$

Composing the natural morphism $\mathbb{C}_P(0) \to q_!\mathbb{C}_S$, $1 \mapsto 1$ with the trace morphism tr, we get twice the identity, and hence the above sequence splits. □

Proof of Theorem 1.3. (see [3, proof of Theorem 5.17]) Applying Corollary 2.6 for $F = \mathbb{C}_P(\varepsilon)$, and using Lemma 3.1, we get:

$$\mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_P(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)) \xleftarrow{\sim} \mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}'_{P^*}(\varepsilon^*) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)).$$

Since $-n-1 < k^* < 0$, one has $\mathrm{R}\Gamma(\mathbb{P}^*; \mathcal{O}_{\mathbb{P}^*}(k^*)) = 0$, and hence the functor $\mathrm{R}\Gamma(\mathbb{P}^*; \cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))$ is well defined in the localized category $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$. By Lemma 3.1, we may then replace $\mathbb{C}'_{P^*}(\varepsilon^*)$ by $\mathbb{C}_{P^*}(\varepsilon^*)$ in the above isomorphism. In view of (3.6) we then get:

$$\Gamma(P; \mathcal{C}^\infty_P(\varepsilon|k)) \xleftarrow{\sim} \Gamma(P^*; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)). \tag{3.8}$$

Theorem A.9 implies that the inverse of the isomorphism (3.8) is the integral transform $R_P^{(\varepsilon|k)}$ of Definition 1.2. □

Using (3.6), if $U \subset P$ is a sub-analytic open subset, one easily checks that:

$$\Gamma_{\mathrm{w}}(U; \mathcal{C}^\infty_P(\varepsilon|k)) \simeq \mathrm{R}\Gamma(\mathbb{P}; \mathbb{C}_U(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k)). \tag{3.9}$$

In view of (3.8), Corollary 2.6 for $F = \mathbb{C}_E(\varepsilon)$ gives:

$$R_P^{(\varepsilon|k)}\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(\varepsilon|k)) \simeq \mathrm{R}\Gamma(\mathbb{P}^*; (\mathbb{C}_E(\varepsilon) \circ L) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)).$$

By Lemma 3.5, we obtain the d.t.:

$$R_P^{(0^*|k)}\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(0^*|k)) \to \Gamma(P^*; \mathcal{C}^\infty_{P^*}(0|k)) \to \mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}_{Q^*}(1) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k^*))[1] \xrightarrow[+1]{}, \tag{3.10}$$

$$R_P^{(1^*|k)}\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(1^*|k)) \to \Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(1|k)) \to \mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}_{\overline{Q^*}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[1] \xrightarrow[+1]{}. \tag{3.11}$$

In the next subsection, in order to describe the above d.t.s (and hence to prove Theorems 1.6, 1.7) we will establish some preliminary results.

3.3 Nonlocal Differentials

Recall Notation 3.2. Denote by $p\colon \widetilde{\mathbb{P}^*_{\xi_\circ}} \to \mathbb{P}^*$ the blow-up of $\mathbb{P}^*$ along $\xi_\circ$, by $\mathbb{D}$ its exceptional divisor, and consider the diagrams:

$$\begin{tikzcd} & \mathbb{P}^*_{\xi_\circ} \arrow[dl, hook', "i"'] \arrow[dr, "q"] & \\ \mathbb{P}^* & & \mathbb{H}^*, \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\mathbb{P}^*_{\xi_\circ}} \arrow[dl, "p"'] \arrow[dr, "\tilde{q}"] & \\ \mathbb{P}^* & & \mathbb{H}^*. \end{tikzcd}$$

For $\zeta' \in \mathbb{H}^*$, $p(\widetilde{q}^{-1}(\zeta')) \subset \mathbb{P}^*$ is the complex projective line issued from $\xi_\circ$ with tangent direction ζ' (recall that $\mathbb{H}^*$ is identified with $\dot{T}_{\xi_\circ}\mathbb{P}^*/\mathbb{C}^\times$). Let $k, l \in \mathbb{Z}$, and let $\mathcal{K}$ be an $\mathcal{O}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}$-module. Set for short

$$\mathcal{K}(k,l) = p^{-1}\mathcal{O}_{\mathbb{P}^*}(k) \otimes_{p^{-1}\mathcal{O}_{\mathbb{P}^*}} \mathcal{K} \otimes_{\widetilde{q}^{-1}\mathcal{O}_{\mathbb{H}^*}} \widetilde{q}^{-1}\mathcal{O}_{\mathbb{H}^*}(l).$$

Consider the kernels on $\widetilde{\mathbb{P}^*_{\xi_\circ}}$:

$$K = \mathbb{C}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}, \qquad \mathcal{K} = \mathcal{O}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}.$$

In the next proposition, although $\widetilde{q}$ is not a trivial bundle over $\mathbb{H}^*$, we will improperly use the term "fiber coordinate" σ for q. This has to be understood in the same sense as the coordinate s in Definition 1.4 (i). For an intrinsic expression of the section γ_m below, refer to Appendix B.

Proposition 3.8. *Denoting by σ a "fiber coordinate" of q, the sections*

$$\gamma_m = \sigma^m d\sigma \in \Gamma(\widetilde{\mathbb{P}^*_{\xi_\circ}}; \Omega_{\widetilde{\mathbb{P}^*_{\xi_\circ}}/\mathbb{H}^*}[*\mathbb{D}](-k^*, k^*+m+1)),$$

for m a non negative integer, induce an isomorphism:

$$\sum_{m\geq 0} \alpha(\gamma_m)\colon \bigoplus_{m\geq 0} \mathcal{D}_{\mathbb{H}^*}(-k^*-m-1) \xrightarrow{\sim} \mathcal{D}_{\mathbb{P}^*}(-k^*) \underline{\circ}\, \mathcal{K}.$$

Proof. Let us begin by proving that the complex $\mathcal{D}_{\mathbb{P}^*}(-k^*) \underline{\circ}\, \mathcal{K}$ is isomorphic to $\bigoplus_{m\geq 0} \mathcal{D}_{\mathbb{H}^*}(-k^*-m-1)$. This is shown by the chain of isomorphisms:

$$\begin{aligned}
\mathcal{D}_{\mathbb{P}^*}(-k^*) \underline{\circ}\, \mathcal{K} &\simeq \underline{\widetilde{q}}_* \underline{p}^{-1} \mathcal{D}_{\mathbb{P}^*}(-k^*) \\
&\simeq \underline{\widetilde{q}}_* (\mathcal{D}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}[*\mathbb{D}](-k^*, 0)) \\
&\simeq R\widetilde{q}_!(\mathcal{D}_{\mathbb{H}^* \leftarrow \widetilde{\mathbb{P}^*_{\xi_\circ}}} \otimes^L_{\mathcal{D}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}} \mathcal{D}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}[*\mathbb{D}](-k^*, 0)) \\
&\simeq \mathcal{D}_{\mathbb{H}^*} \otimes_{\mathcal{O}_{\mathbb{H}^*}} R\widetilde{q}_!(\Omega_{\widetilde{\mathbb{P}^*_{\xi_\circ}}/\mathbb{H}^*}[*\mathbb{D}](-k^*, 0)) \\
&\simeq \bigoplus_{m\geq 0} \mathcal{D}_{\mathbb{H}^*}(-k^*-m-1),
\end{aligned}$$

where in the last isomorphism we considered a "fiber coordinate" σ for q (refer to Appendix B), and we used the identification for $k, l \in \mathbb{Z}$:

$$R\widetilde{q}_!(\Omega_{\widetilde{\mathbb{P}^*_{\xi_\circ}}/\mathbb{H}^*}[*\mathbb{D}](k,l)) \simeq \bigoplus_{m\geq 0} \mathcal{O}_{\mathbb{H}^*}(k-l-m-1) \cdot \sigma^m d\sigma. \tag{3.12}$$

By the above identification, we also see that $\sum_{m\geq 0} \gamma_m$ is indeed the kernel of our isomorphism. To this end, it is enough to compose the natural isomorphism:

$$\begin{aligned}
&\Gamma(\widetilde{\mathbb{P}^*_{\xi_\circ}}; \Omega_{\widetilde{\mathbb{P}^*_{\xi_\circ}}/\mathbb{H}^*}[*\mathbb{D}](-k^*, k^*+m+1)) \qquad (3.13)\\
&\quad \simeq \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^* \times \mathbb{H}^*}}(\underline{D}(\mathcal{D}_{\mathbb{P}^*}(-k^*)) \underline{\boxtimes}\, \mathcal{D}_{\mathbb{H}^*}(-k^*-m-1)[-n], \underline{h}_* \mathcal{O}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}})
\end{aligned}$$

with the isomorphism α of Lemma 2.3 (recall that h denotes the map $(p, \widetilde{q})$).

$\square$

Applying formula (2.3) for $L = K$, and noticing that $K \circ G \simeq Ri_! q^{-1} G$, we get:

Corollary 3.9. *For any $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{H}^*})$, the sections $\{\gamma_m\}_{m\geq 0}$ induce an isomorphism:*

$$\mathrm{R}\Gamma(\mathbb{P}^*; Ri_! q^{-1} G \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[1] \xrightarrow{\sim} \prod_{m\geq 0} \mathrm{R}\Gamma(\mathbb{H}^*; G \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{H}^*}(k^* + m + 1)).$$

For $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*})$ and $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{H}^*})$, the data of a morphism $\beta\colon F \to Ri_! q^{-1} G$ and of the sections γ_m induce a morphism:

$$\mathrm{R}\Gamma(\mathbb{P}^*; F \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[1] \to \prod_{m\geq 0} \mathrm{R}\Gamma(\mathbb{H}^*; G \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{H}^*}(k^* + m + 1))$$

obtained by composing $\mathrm{R}\Gamma(\mathbb{P}^*; \beta \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))$ with the isomorphism of Corollary 3.9. In the following proposition, we will explicitly describe some instances of the above morphism for G equal to $\mathbb{C}_{H^*}(0)$, $\mathbb{C}_{H^*}(1)$ or $\mathbb{C}_{\mathbb{H}^*}$. In the first two cases, these morphisms are precisely those given by the functionals introduced in Definition 1.4.

Proposition 3.10. *With the notations of Definition 1.4, one has:*

(i) The natural morphism $\mathbb{C}_{P^*_{\xi_\circ}}(\varepsilon^*) \to Ri_! q^{-1} \mathbb{C}_{H^*}(\overline{\varepsilon}^*)[1]$ and the sections $\{\gamma_m\}_{m\geq 0}$ induce the morphism:

$$\begin{aligned} c^{(0|\cdot)}_{\xi_\circ}\colon \Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)) &\to \prod_{m\geq 0} \Gamma(H^*; \mathcal{C}^\infty_{H^*}(\overline{\varepsilon}^*|k^* + m + 1)) \\ \varphi &\mapsto \sum_{m\geq 0} c^{(0|m)}_{\xi_\circ}\varphi. \end{aligned}$$

Assume $-n - 1 < k^* < 0$.

(ii) The natural morphism $\mathbb{C}_{P^*}(0) \to Ri_! q^{-1} \mathbb{C}_{H^*}(1)[1]$ in $\mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*}; \dot{T}^*\mathbb{P}^*)$ and the sections $\{\gamma_m\}_{m\geq 0}$ induce the morphism:

$$\begin{aligned} d^{(1|k^*+\cdot+1)}_{\xi_\circ}\colon \Gamma(P^*; \mathcal{C}^\infty_{P^*}(0|k^*)) &\to \prod_{m\geq 0} \Gamma(H^*; \mathcal{C}^\infty_{H^*}(1|k^* + m + 1)) \\ \varphi &\mapsto \sum_{m\geq 0} d^{(1|k^*+m+1)}_{\xi_\circ}\varphi. \end{aligned}$$

(iii) The identification $\mathbb{C}_{\mathbb{P}^*_{\xi_\circ}} \xrightarrow{\sim} Ri_! q^{-1}\mathbb{C}_{\mathbb{H}^*}$ *and the sections* $\{\gamma_m\}_{m\geq 0}$ *induce the isomorphism:*

$$\begin{aligned} d^{(k^*+\cdot+1)}_{\xi_\circ} \colon \mathcal{O}_{\mathbb{P}^*}(k^*)\hat{|}_{\xi_\circ} &\to \prod_{m\geq 0} \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k^*+m+1)) \\ \sum_m \psi_m &\mapsto \sum_m \psi_{k^*+m+1}. \end{aligned}$$

Proof. At the level of constant sheaves, the morphism in (i) is the natural adjunction morphism:

$$\begin{aligned} \mathbb{C}_{P^*_{\xi_\circ}}(\varepsilon^*) &\simeq Ri_!\mathbb{C}_{P^*_{\xi_\circ}}(\varepsilon^*) \\ &\to Ri_! q^! Rq_! \mathbb{C}_{P^*_{\xi_\circ}}(\varepsilon^*) \\ &\simeq Ri_! q^{-1}\mathbb{C}_{H^*}(\overline{\varepsilon}^*)[1]. \end{aligned} \tag{3.14}$$

(Here, to obtain the last isomorphism, note that $P^*_{\xi_\circ} \to H^*$ is the global space of the tautological bundle $\mathbb{C}_{H^*}(1)$, so that $Rq_!\mathbb{C}_{P^*_{\xi_\circ}} \simeq \mathbb{C}_{H^*}(1)[-1]$.) The morphism in (iii) is similarly obtained, and the morphism in (ii) is the one appearing in (3.2).

At the level of $\mathcal{C}^\infty$-functions, the morphism in (i) is clearly described by the functionals $c^{(0|m)}_{\xi_\circ}$. The fact that the morphism in (ii) is described by $d^{(1|k^*+m+1)}_{\xi_\circ}$, follows from Remark 1.5 (c). As for (iii), one has:

$$\begin{aligned} \mathcal{O}_{\mathbb{P}^*}(k^*)\hat{|}_{\xi_\circ} &\simeq \mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}_{\xi_\circ} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)) \\ &\simeq \mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}_{\mathbb{P}^*_{\xi_\circ}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[-1], \end{aligned}$$

where the last isomorphism is due to the assumption $-n-1 < k^* < 0$. A section $\psi \in \mathcal{O}_{\mathbb{P}^*}(k^*)\hat{|}_{\xi_\circ}$ is a formal sum $\psi = \sum_{m=0}^{\infty} \psi_m$, where ψ_m is a homogeneous polynomial of degree $m+k^*$ (hence only $m \geq -k^*$ matters). For $\tau = \sigma^{-1}$, we have:

$$\begin{aligned} (d^{(k^*+\cdot+1)}_{\xi_\circ}\psi[\zeta'])_m &= \int_q \psi(\sigma\xi_\circ + \zeta')\sigma^{m+1}\frac{d\sigma}{\sigma} \\ &= \int_q \psi(\tau^{-1}\xi_\circ + \zeta')\tau^{-1-m}\frac{d\tau}{\tau} \\ &= \int_q \psi(\xi_\circ + \tau\zeta')\tau^{-k^*-m-1}\frac{d\tau}{\tau} \\ &= \psi_{k^*+m+1}[\zeta'], \end{aligned}$$

where the last equality is a formal residue computation. □

3.4 Cavalieri Condition and Nonlocal Borel Theorem

We have now the tools to describe the morphisms (3.10) and (3.11). We do this in the following theorem, which gives a cohomological proof of Theorems 1.6, 1.7.

Theorem 3.11. *Assuming $-n-1 < k < 0$, there are natural short exact sequences:*

$$
\begin{aligned}
0 \to R_P^{(0^*|k)}\Gamma_{\mathrm{w}}(E;\mathcal{C}_P^\infty(0^*|k)) &\to \Gamma(P^*;\mathcal{C}_{P^*}^\infty(0|k^*)) \\
&\xrightarrow[d_{\xi_\circ}^{(1|k^*+\cdot+1)}]{} \prod_{m\geq 0}\Gamma(H^*;\mathcal{C}_{H^*}^\infty(1|k^*+m+1)) \to 0,
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
0 \to R_P^{(1^*|k)}\Gamma_{\mathrm{w}}(E;\mathcal{C}_P^\infty(1^*|k)) &\to \Gamma_{\mathrm{w}}(P_{\xi_\circ}^*;\mathcal{C}_{P^*}^\infty(1|k^*)) \\
&\xrightarrow[c]{} H^1(\mathbb{P}^*;\mathbb{C}_{\overline{Q^*}}\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*)) \to 0,
\end{aligned}
\tag{3.16}
$$

where the map c above enters the commutative diagram with exact row:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_{\mathbb{P}^*}\hat{|}_{\xi_\circ} \arrow[r, "d_{\xi_\circ}^{(k^*+\cdot+1)}"] & \prod_{m\geq 0}\Gamma(H^*;\mathcal{C}_{H^*}^\infty(0|k^*+m+1)) \arrow[r] & H^1(\mathbb{P}^*;\mathbb{C}_{\overline{Q^*}}\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*)) \arrow[r] & 0 \\
& & \Gamma_{\mathrm{w}}(P_{\xi_\circ}^*;\mathcal{C}_{P^*}^\infty(1|k^*)) \arrow[u, "c_{\xi_\circ}^{(0|\cdot)}"] \arrow[ur, "c"'] & &
\end{tikzcd}
\tag{3.17}
$$

Here, the morphisms $c_{\xi_\circ}^{(0|\cdot)}$, $d_{\xi_\circ}^{(1|k^*+\cdot+1)}$, and $d_{\xi_\circ}^{(k^*+\cdot+1)}$ are the ones described in Proposition 3.10.

Proof. The first exact sequence is obtained from (3.10) using Proposition 3.10 (ii), while the second exact sequence is obtained from (3.11) noting that, since $k^* < 0$ and $\overline{Q^*}$ contains complex projective lines, $H^0(\mathbb{P}^*;\mathbb{C}_{\overline{Q^*}}\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*)) = 0$. Applying the functor $\mathrm{R}\Gamma(\mathbb{P}^*;\cdot\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*))$ to the commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{C}_{Q^*} \arrow[r] & \mathbb{C}_{\overline{Q^*}} \arrow[r] & \mathbb{C}_{\xi_\circ} \arrow[r] & 0 \\
& \mathbb{C}_{P_{\xi_\circ}^*}(1)[-1] \arrow[u] \arrow[ur] & & &
\end{tikzcd}
$$

(where the vertical arrow is constructed in (3.14)) we get the commutative diagram:

$$
\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^*}\hat{|}_{\xi_\circ} \arrow[r] & \mathrm{R}\Gamma(\mathbb{P}^*;\mathbb{C}_{Q^*}\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*))[1] \arrow[r] & \mathrm{R}\Gamma(\mathbb{P}^*;\mathbb{C}_{\overline{Q^*}}\overset{\mathrm{w}}{\otimes}\mathcal{O}_{\mathbb{P}^*}(k^*))[1] \arrow[r, "+1"'] & {} \\
& \Gamma_{\mathrm{w}}(P_{\xi_\circ}^*;\mathcal{C}_{P^*}^\infty(1|k^*)) \arrow[u] \arrow[ur, "c"] & &
\end{tikzcd}
$$

The diagram (3.17) is obtained from the one above by taking cohomology groups, and by using Proposition 3.10 (i) and (iii). $\square$

Proof of Theorems 1.6, 1.7. Part (i) of Theorems 1.6 and 1.7 immediately follows by looking at (3.15). Let us then consider the case $\varepsilon^* \equiv 1$, $\omega \equiv 0$. The image by $R_P^{(1^*|k)}$ of the space $\Gamma_{\mathrm{w}}(E; \mathcal{C}_P^\infty(1^*|k))$ is described by (3.16) as the kernel of the morphism c. One sees by (3.17) that φ belongs to $\ker(c)$ if and only if $c_{\xi_\circ}^{(0|m)}\varphi$ is in the image of $d_{\xi_\circ}^{(k^*+m+1)}$, which is precisely the space of (k^*+m+1)-homogeneous polynomials. This proves Theorem 1.6 (ii). The exactness of sequence (3.16) also asserts that the morphism c is surjective. One sees by (3.17) that this implies that the morphism $c_{\xi_\circ}^{(0|m)}\varphi$ is surjective, modulo the image of $d_{\xi_\circ}^{(k^*+m+1)}$. This proves Theorem 1.7 (ii). $\square$

4 Other Related Results

4.1 Laplace-Borel Theorem

Our proof of Theorems 1.6 and 1.7 consisted in successively applying the functor $\mathrm{R}\Gamma(\mathbb{P}; \cdot \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}}(k))$ and the isomorphism (2.3) to the exact sequence:

$$0 \to \mathbb{C}_E(\varepsilon) \to \mathbb{C}_P(\varepsilon) \to \mathbb{C}_H(\varepsilon) \to 0,$$

and in describing the result. For $x_\circ \in P$, $P_{x_\circ} = P \setminus \{x_\circ\}$, we can apply the same procedure to the exact sequence:

$$0 \to \mathbb{C}_{P_{x_\circ}}(\varepsilon) \to \mathbb{C}_P(\varepsilon) \to \mathbb{C}_{x_\circ}(\varepsilon) \to 0.$$

Applying $\cdot \circ L$, we get the d.t.:

$$\mathbb{C}_{P_{x_\circ}}(\varepsilon) \circ L \to \mathbb{C}'_{P^*}(\varepsilon^*)[-n] \to \mathbb{C}_{\mathbb{H}^*}[-2n] \underset{+1}{\longrightarrow},$$

where $\mathbb{H}^*$ denotes the hyperplane of $\mathbb{P}^*$ dual to $x_\circ \in \mathbb{P}$. For $-n-1 < k < 0$, we have the identifications:

$$\begin{aligned}
\mathrm{R}\Gamma(\mathbb{P}^*; \mathbb{C}_{\mathbb{H}^*} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*))[-n] &\simeq \mathrm{R}\Gamma(\mathbb{P}^*; \mathcal{O}_{\mathbb{P}^*}(k^*)\hat{|}_{\mathbb{H}^*})[-n] \\
&\simeq \prod_{m \geq 0} \mathrm{R}\Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k^* - m))[-n] \\
&\simeq \prod_{m \geq 0} \mathrm{R}\Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k + m)),
\end{aligned}$$

where the first isomorphism may be obtained using Proposition 3.10 and Corollary 2.6 (interchanging the role of $\mathbb{P}$ and $\mathbb{P}^*$, and for $F = \mathbb{C}_{x_\circ}$), and the last isomorphism is given by Serre duality. From the d.t. above, we thus get the exact sequence:

$$0 \to R_P^{(\varepsilon|k)}\Gamma_{\mathrm{w}}(P_{x_\circ}; \mathcal{C}_P^\infty(\varepsilon|k)) \to \Gamma(P^*; \mathcal{C}_{P^*}^\infty(\varepsilon^*|k^*)) \xrightarrow[e_{\mathbb{H}^*}^{(\varepsilon^*|\cdot)}]{} \prod_{m \geq 0} \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k+m)) \to 0,$$

which gives the following variant of Theorems 1.6 and 1.7 respectively:

(i) The image by $R_P^{(\varepsilon|k)}$ of the space of $(\varepsilon|k)$-homogeneous $\mathcal{C}^\infty$-functions on P with vanishing formal Taylor series at $x_\circ$, is isomorphic to the kernel of $e_{\mathbb{H}^*}^{(\varepsilon^*|\cdot)}$.

(ii) The morphism $e_{\mathbb{H}^*}^{(\varepsilon^*|\cdot)}$ is surjective.

At least in the particular case $\varepsilon \equiv 1$, the m-th component of the morphism $e_{\mathbb{H}^*}^{(\varepsilon^*|\cdot)}$ is obtained as the composite:

$$\begin{aligned} e_{\mathbb{H}^*}^{(1^*|m)} \colon \Gamma(P^*; \mathcal{C}_{P^*}^\infty(1^*|k^*)) &\xrightarrow[\partial_{H^*}^m]{} \Gamma(H^*; \mathcal{C}_{H^*}^\infty(1^*|k^* - m)) \\ &\hookrightarrow \Gamma(H^*; \mathcal{B}_{H^*}(1^*|k^* - m)) \\ &\simeq \operatorname{Hom}(D'(\mathbb{C}_{H^*}(1^*)), \mathcal{O}_{\mathbb{H}^*}(k^* - m)) \\ &\simeq \operatorname{Hom}(\mathbb{C}_{H^*}[1-n], \mathcal{O}_{\mathbb{H}^*}(k^* - m)) \\ &\xrightarrow[f^{(m)}]{} \operatorname{Hom}(\mathbb{C}_{\mathbb{H}^*}[1-n], \mathcal{O}_{\mathbb{H}^*}(k^* - m)) \\ &\simeq H^{n-1}(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k^* - m)) \\ &\simeq \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(k + m)), \end{aligned}$$

where $\partial_{H^*}^m$ is the usual m-th normal derivatives, where the arrow $f^{(m)}$ is induced by the natural morphism $\mathbb{C}_{\mathbb{H}^*} \to \mathbb{C}_{H^*}$, and where the last isomorphism is given by Serre duality. (Note that we used the identification $D'(\mathbb{C}_{H^*}(\omega)) \simeq \mathbb{C}_{H^*}(\overline{\omega}^*)[-n+1]$.)

4.2 Helgason's Support Theorem

Let $A = \{(x, \xi) \in P \times P^*;\ \langle x, \xi \rangle = 0\}$ denote the real incidence relation. For a locally closed subset $D \subset P$, we set

$$\begin{aligned} \widetilde{D} &= g(A \cap (D \times P^*)) = \{\xi \in P^*; \exists x \in D \text{ such that } \langle x, \xi \rangle = 0\}, \\ \widehat{D} &= g(\mathbb{A} \cap f^{-1}(D)) = \{\zeta \in \mathbb{P}^*; \exists x \in D \text{ such that } \langle x, \zeta \rangle = 0\}. \end{aligned}$$

As in [3], we say that D is $\mathbb{A}$-trivial if for any $\zeta \in \widehat{D}$, one has:

$$\mathbb{C} \xrightarrow{\sim} \mathrm{R}\Gamma(\widehat{\zeta} \cap D; \mathbb{C}_D).$$

We then have the following version of Helgason's support theorem (for the sake of brevity, we consider only the transform $R_P^{(1^*|k)}$ because, for $k^* = -1$, it coincides with the classical Radon transform).

Theorem 4.1. *Assume $-n-1 < k < 0$. Let $D \subset E$ be an open $\mathbb{A}$-trivial domain in P, and let $\varphi \in \Gamma(P^*; \mathcal{C}_{P^*}^\infty(1|k^*))$. Then, φ belongs to the image by $R_P^{(1^*|k)}$ of $\Gamma_{\overline{D}}(P; \mathcal{C}_P^\infty(1^*|k))$ if and only if it belongs to $\Gamma_{\overline{\widetilde{D}}}(P^*; \mathcal{C}_{P^*}^\infty(1|k^*))$ and satisfies the Cavalieri condition (1.6).*

Briefly, the proof goes along the following lines. Set

$$Q^*_D = \mathbb{P}^* \setminus \{[\zeta]; [\xi] \in \widetilde{D}, [\eta] \in \widetilde{\widetilde{[\xi]} \cap D}, \text{ for some } \xi, \eta \text{ with } [\xi + i\eta] = [\zeta]\},$$

and note that $Q^*_E = Q^*$. One then checks that there is a natural commutative diagram:

$$\begin{tikzcd}
\mathbb{C}_E(1^*) \circ L \arrow[r] & \mathbb{C}_{P^*_{\xi_\circ}}(1)[-n] \arrow[r] & \mathbb{C}_{\overline{Q^*}}[1-n] \arrow[r, "+1"'] & {} \\
\mathbb{C}_D(1^*) \circ L \arrow[r] & \mathbb{C}_{\widetilde{D}}(1)[-n] \arrow[r] \arrow[u] & \mathbb{C}_{\overline{Q^*_D}}[1-n] \arrow[u] \arrow[r, "+1"'] & {},
\end{tikzcd}$$

from which we get:

$$\begin{tikzcd}
0 \arrow[r] & R_P^{(1^*|k)}\Gamma_{\mathrm{w}}(E; \mathcal{C}^\infty_P(1^*|k)) \arrow[r] & \Gamma_{\mathrm{w}}(P^*_{\xi_\circ}; \mathcal{C}^\infty_{P^*}(1|k^*)) \arrow[r] & H^1(\mathbb{P}^*; \mathbb{C}_{\overline{Q^*}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)) \arrow[r] & 0 \\
0 \arrow[r] & R_P^{(1^*|k)}\Gamma_{\mathrm{w}}(D; \mathcal{C}^\infty_P(1^*|k)) \arrow[r] \arrow[u] & \Gamma_{\mathrm{w}}(\widetilde{D}; \mathcal{C}^\infty_{P^*}(1|k^*)) \arrow[r] \arrow[u] & H^1(\mathbb{P}^*; \mathbb{C}_{\overline{Q^*_D}} \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*}(k^*)) \arrow[u] &
\end{tikzcd}$$

One then concludes using arguments similar to those in the proof of Theorem 3.11, and noticing that

$$\Gamma_{\mathrm{w}}(D; \mathcal{C}^\infty_P(1^*|k)) \simeq \Gamma_{\overline{D}}(P; \mathcal{C}^\infty_P(1^*|k)),$$

since D is locally on one side of its boundary.

4.3 Conformal Radon Transform

Let $[x] = [x_0, \ldots, x_n]$ be a system of homogeneous coordinates in P, and let $S_\vartheta \subset P$ be the quadric of equation $\Box_\vartheta(x) = 0$, where $\vartheta \in (\mathbb{Z}/2\mathbb{Z})^n$, $\vartheta \neq (0, \ldots, 0)$, and $\Box_\vartheta(x) = x_0^2 + (-1)^{\vartheta_1} x_1^2 + \cdots + (-1)^{\vartheta_n} x_n^2$. Note that if $\vartheta = (1, \ldots, 1)$, the stereographic projection identifies the restriction of the real projective transform to the sphere S_ϑ with the conformal Radon transform.

For $k \in \mathbb{Z}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, let us denote by $\mathcal{C}^\infty_{S_\vartheta}(\varepsilon|k)$ the $\mathcal{C}^\infty$-line-bundle on S_ϑ, whose sections satisfy the homogeneity conditions (1.1). A theorem of [6], asserts that for $-n-1 < k < -2$, the transform $R_P^{(1^*|k)}$ interchanges $\Gamma(S_\vartheta; \mathcal{C}^\infty_{S_\vartheta}(1^*|k+2))$ with the sections $\varphi \in \Gamma(P^*; \mathcal{C}^\infty_{P^*}(1|k^*))$ satisfying the homogeneous differential equation:

$$\Box_\vartheta(\partial_\xi)\varphi(\xi) = 0.$$

Let us explain how it is possible to recover and precise this result in our framework.

Denote by $\mathbb{S}_\vartheta \subset \mathbb{P}$ the complexification of S_ϑ defined by the equation $\Box_\vartheta(z) = 0$. Then, one has:

$$\begin{aligned}
\mathcal{C}^\infty_{S_\vartheta}(\varepsilon|k) &\simeq (\mathbb{C}_{S_\vartheta}(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_\mathbb{P}) \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_{\mathbb{S}_\vartheta}(k) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_\mathbb{P}}(\mathcal{D}\mathcal{O}_{\mathbb{S}_\vartheta}(k)^*, \mathbb{C}_{S_\vartheta}(\varepsilon) \overset{\mathrm{w}}{\otimes} \mathcal{O}_\mathbb{P}),
\end{aligned}$$

where we set for short $\mathcal{DO}_{\mathbb{S}_\vartheta}(k)^* = \mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} R\mathcal{H}om_{\mathcal{O}_{\mathbb{P}}}(\mathcal{O}_{\mathbb{S}_\vartheta}(k), \mathcal{O}_{\mathbb{P}})$. Denoting by $\mathcal{M}_{\square_\vartheta}(-k^*)$ the left coherent $\mathcal{D}_{\mathbb{P}^*}$-module defined be the exact sequence:

$$0 \to \mathcal{D}_{\mathbb{P}^*}(-k^* + 2) \xrightarrow[\cdot\square_\vartheta(\partial_\xi)]{} \mathcal{D}_{\mathbb{P}^*}(-k^*) \to \mathcal{M}_{\square_\vartheta}(-k^*) \to 0, \tag{4.1}$$

we have the following lemma, similar to a remark in [3].

Lemma 4.2. *Assume that $-n-1 < k < -2$, then the sections s_k, s_{k+2} in (2.11) induce an isomorphism:*

$$\mathcal{M}_{\square_\vartheta}(-k^*) \xrightarrow{\sim} \mathcal{DO}_{\mathbb{S}_\vartheta}(k+2)^* \underline{\circ}\, \mathcal{L}.$$

Proof. The complex $\mathcal{DO}_{\mathbb{S}_\vartheta}(k+2)^*$ is concentrated in degree -1, and is defined by the exact sequence:

$$0 \to \mathcal{D}_{\mathbb{P}}(-k-2) \xrightarrow[\square_\vartheta(z)\cdot]{} \mathcal{D}_{\mathbb{P}}(-k) \to \mathcal{DO}_{\mathbb{S}_\vartheta}(k+2)^*[1] \to 0.$$

Since $-n-1 < k, k+2 < 0$, the conclusion follows from Theorem 2.5, noticing that $\square_\vartheta(\partial_\xi)$ is the Radon transform of $\square_\vartheta(z)$. $\square$

Applying Theorem 2.2, we thus get an isomorphism:

$$R_P^{(1^*|k)}\Gamma(S_\vartheta; \mathcal{C}^\infty_{S_\vartheta}(1^*|k)) \simeq \mathrm{RHom}_{\mathcal{D}_{\mathbb{P}^*}}(\mathcal{M}_{\square_\vartheta}(-k^*), (\mathbb{C}_{S_\vartheta}(1^*) \circ L) \overset{\mathrm{w}}{\otimes} \mathcal{O}_{\mathbb{P}^*})[n-1], \tag{4.2}$$

and we are reduced to compute $\mathbb{C}_{S_\vartheta}(1^*) \circ L$. For the sake of brevity, let us restrict to the simplest case $\vartheta = (1, \dots, 1)$. Let $H \subset P$ be the hyperplane of equation $x_0 = 0$, and $D = \{x; \square_\vartheta(x) \geq 0\}$ be the open ball in the affine chart $E = P \setminus H$ with boundary $S = S_\vartheta$. Consider the short exact sequence:

$$0 \to \mathbb{C}_D(1^*) \to \mathbb{C}_{\overline{D}}(1^*) \to \mathbb{C}_S(1^*) \to 0.$$

In the notations of paragraph 4.2, since $\overline{D}$ is closed convex we easily get $\mathbb{C}_{\overline{D}}(1^*) \circ L \simeq \mathbb{C}_{P^* \setminus \hat{\overline{D}}}$. Taking the zero-th cohomology groups in (4.2), we thus get by Theorem 4.1 that $R_P^{(1^*|k)}$ interchanges $\Gamma(S_\vartheta; \mathcal{C}^\infty_{S_\vartheta}(1^*|k))$ with the sections $\varphi \in \Gamma_{\hat{\overline{D}}}(P^*; \mathcal{C}^\infty_{P^*}(1|k^*))$, satisfying the Cavalieri condition (1.6) and the equation $\square_\vartheta(\partial_\xi)\varphi(\xi) = 0$.

4.4 Affine Radon Transform of Other Functional Spaces

The sheaves of $(\varepsilon|k)$-homogeneous distributions, analytic functions, and hyperfunctions on P are given by:

$$\begin{aligned}
\mathcal{D}b_P(\varepsilon|k) &= T\mathcal{H}om(D'\mathbb{C}_P(\varepsilon), \mathcal{O}_{\mathbb{P}}(k)) \\
&\simeq T\mathcal{H}om(\mathbb{C}_P(\varepsilon^*), \mathcal{O}_{\mathbb{P}}(k))[n], \\
\mathcal{A}_P(\varepsilon|k) &= \mathbb{C}_P(\varepsilon) \otimes \mathcal{O}_{\mathbb{P}}(k), \\
\mathcal{B}_P(\varepsilon|k) &\simeq R\mathcal{H}om(\mathbb{C}_P(\varepsilon^*), \mathcal{O}_{\mathbb{P}}(k))[n].
\end{aligned}$$

Applying formula (2.4), or the analogue of Theorem 2.2 for $\overset{\mathrm{w}}{\otimes}$ and $T\mathcal{H}om$, replaced with $\otimes$ and $R\mathcal{H}om$ respectively (see [2], [13]), we can state the results analogue to those in section 1 for $\mathcal{C}^\infty$ replaced with $\mathcal{D}b$, $\mathcal{A}$ or $\mathcal{B}$. In particular, the analogue of Theorems 1.6, 1.7 are deduced from the following analogue of Theorem 3.11. For the sake of brevity, we consider here only one parity for ε in each case.

For $-n-1<k<0$, there are exact sequences:

$$\begin{aligned} 0 \to R_P^{(0|k)}\Gamma_t(E;\mathcal{D}b_P(0|k)) &\to \Gamma(P^*;\mathcal{D}b_{P^*}(0^*|k^*)) \qquad (4.3)\\ &\to \prod_{m\geq 0}\Gamma(H^*;\mathcal{D}b_{H^*}(1^*|k^*+m+1)) \to 0,\\ 0 \to \Gamma(P^*;\mathcal{A}_{P^*}(0|k^*)) &\underset{a}{\to} H^1_c(\mathbb{P}^*_{\xi_\circ};q^*\mathcal{A}_{H^*}(1|k^*)) \qquad (4.4)\\ &\to R_P^{(0^*|k)}H^1_c(E;\mathcal{A}_P(0^*|k)) \to 0,\\ 0 \to \Gamma(P^*;\mathcal{B}_{P^*}(0^*|k^*)) &\underset{b}{\to} H^1_c(\mathbb{P}^*_{\xi_\circ};q^*\mathcal{B}_{H^*}(1^*|k^*)) \qquad (4.5)\\ &\to R_P^{(0|k)}\Gamma(E;\mathcal{B}_P(0|k)) \to 0. \end{aligned}$$

Here $\Gamma_t(E;\mathcal{D}b_P(0|k)) = \mathrm{THom}\,(\mathbb{C}_E(0^*);\mathcal{O}_\mathbb{P}(k))[n]$ denotes the space of tempered distributions on E, and q^* denotes the $\mathcal{O}$-module inverse image (i.e., if $\mathcal{F}$ is a flat $\mathcal{O}_{\mathbb{H}^*}$-module, we set $q^*\mathcal{F} = \mathcal{O}_{P^*_{\xi_\circ}} \otimes_{q^{-1}\mathcal{O}_{\mathbb{H}^*}} q^{-1}\mathcal{F}$).

Note that (4.3) is essentially different from (4.4) or (4.5), in that the image of $R_P^{(\varepsilon|k)}$ appear as a quotient (or as an extension for the parity of ε not considered above) and not as a subspace of the corresponding space on P^*. This is natural, since conditions like (1.5) or (1.6) are meaningless without imposing growth conditions. (See [9] for a study of the Radon transform for some classes of hyperfunctions with tempered growth at infinity.)

Let us briefly sketch how we obtained (4.4) and how the morphism a is described (the arguments for (4.5) and b are similar). Applying the functor $\mathrm{R}\Gamma(\mathbb{P};\cdot\otimes\mathcal{O}_\mathbb{P}(k))$ and the isomorphism (2.3) to the exact sequence:

$$0 \to \mathbb{C}_E(0^*) \to \mathbb{C}_P(0^*) \to \mathbb{C}_H(0^*) \to 0,$$

we get the exact sequence:

$$0 \to \Gamma(P^*;\mathcal{A}_{P^*}(0|k^*)) \to H^1(\mathbb{P}^*;\mathbb{C}_{Q^*}(1)\otimes\mathcal{O}_{\mathbb{P}^*}(k^*)) \to R_P^{(0^*|k)}H^1_c(E;\mathcal{A}_P(0^*|k)) \to 0,$$

and we have

$$\begin{aligned} \mathrm{R}\Gamma(\mathbb{P}^*;\mathbb{C}_{Q^*}(1)\otimes\mathcal{O}_{\mathbb{P}^*}(k^*)) &\simeq \mathrm{R}\Gamma_c(\mathbb{P}^*_{\xi_\circ};\mathbb{C}_{Q^*}(1)\otimes\mathcal{O}_{P^*_{\xi_\circ}}(k^*))\\ &\simeq \mathrm{R}\Gamma_c(\mathbb{P}^*_{\xi_\circ};q^{-1}\mathbb{C}_{H^*}(1)\otimes\mathcal{O}_{P^*_{\xi_\circ}}\otimes_{q^{-1}\mathcal{O}_{\mathbb{H}^*}} q^{-1}\mathcal{O}_{\mathbb{H}^*}(k^*))\\ &\simeq \mathrm{R}\Gamma_c(\mathbb{P}^*_{\xi_\circ};q^*\mathcal{A}_{H^*}(1|k^*)). \end{aligned}$$

In the above short exact sequence we used the fact that $\mathrm{R}\Gamma_c(E;\mathcal{A}_P(0^*|k))$ is concentrated in degree one. (Recall that

$$H^1_c(E;\mathcal{A}_P(0^*|k)) \simeq \varinjlim_K \frac{\Gamma(P\setminus K;\mathcal{A}_P(0^*|k))}{\Gamma(P;\mathcal{A}_P(0^*|k))},$$

where K ranges over the family of compact subsets of E.) The arrow a is described as follows. Denoting by P^1 and $\mathbb{P}^1$ a real and a complex one-dimensional projective space, and choosing $\infty \in P^1$, we have the natural morphisms

$$\mathbb{C}_{P^1} \to \mathbb{C}_\infty \to \mathbb{C}_{\mathbb{P}^1\setminus\{\infty\}}[1]$$

from which we get a morphism

$$\Gamma(P^1;\mathcal{A}_{P^1}) \to H^1_c(\mathbb{P}^1\setminus\{\infty\};\mathcal{O}_{\mathbb{P}^1}) \tag{4.6}$$

whose topological dual:

$$\begin{aligned} \Gamma(\mathbb{P}^1\setminus\{\infty\};\mathcal{O}_{\mathbb{P}^1}) &\to \Gamma_\infty(P^1;\mathcal{B}_{P^1}) \\ &\to \Gamma(P^1;\mathcal{B}_{P^1}) \end{aligned}$$

is easily understood. The arrow a is the analogue of (4.6), with real analytic parameters.

A Quantization of Integral transforms

A.1 Distribution Kernels

Let us consider a general correspondence of complex manifolds:

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X & & Y. \end{tikzcd}$$

Let $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$, $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_Y)$, $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_X)$, $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}(\mathcal{D}_Y)$, $L \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_S)$, and set $\mathcal{L} = T\mathcal{H}om(L,\mathcal{O}_S)$. Set:

$$L \overset{*}{\circ} G = Rf_*(L\otimes g^{-1}G).$$

Assume to be given morphisms:

$$\alpha \in \mathrm{Hom}_{\mathcal{D}_Y}(\mathcal{N},\mathcal{M} \underline{\circ} \mathcal{L}), \qquad \beta \in \mathrm{Hom}\,(F, L\overset{*}{\circ} G) \simeq \mathrm{Hom}\,(f^{-1}F, L\otimes g^{-1}G).$$

Then, there is a natural morphism:

$$\mathrm{R}\Gamma_c(Y;T\mathcal{H}om(G,\Omega_Y)\otimes^L_{\mathcal{D}_Y}\mathcal{N})\to \mathrm{R}\Gamma_c(X;T\mathcal{H}om(F,\Omega_X)\otimes^L_{\mathcal{D}_X}\mathcal{M})[-d_{S/X}],\quad (\mathrm{A}.1)$$

obtained as the composite of:

$$\begin{aligned}
&\mathrm{R}\Gamma_c(Y;T\mathcal{H}om(G,\Omega_Y)\otimes^L_{\mathcal{D}_Y}\mathcal{N})\\
&\qquad\underset{\alpha}{\to}\mathrm{R}\Gamma_c(Y;T\mathcal{H}om(G,\Omega_Y)\otimes^L_{\mathcal{D}_Y}(\mathcal{M}\underline{\circ}\mathcal{L}))\\
&\qquad=\mathrm{R}\Gamma_c(Y;T\mathcal{H}om(G,\Omega_Y)\otimes^L_{\mathcal{D}_Y}\underline{g}_*(\underline{f}^{-1}\mathcal{M}\otimes^L_{\mathcal{O}_S}\mathcal{L}))\\
&\qquad\xrightarrow[g^{-1}]{}\mathrm{R}\Gamma_c(S;T\mathcal{H}om(g^{-1}G,\Omega_S)\otimes^L_{\mathcal{D}_S}(\underline{f}^{-1}\mathcal{M}\otimes^L_{\mathcal{O}_S}\mathcal{L}))\\
&\qquad\simeq\mathrm{R}\Gamma_c(S;(T\mathcal{H}om(g^{-1}G,\Omega_S)\otimes^L_{\mathcal{O}_S}T\mathcal{H}om(L,\mathcal{O}_S))\otimes^L_{\mathcal{D}_S}\underline{f}^{-1}\mathcal{M})\\
&\qquad\underset{\bullet}{\to}\mathrm{R}\Gamma_c(S;T\mathcal{H}om(L\otimes g^{-1}G,\Omega_S)\otimes^L_{\mathcal{D}_S}\underline{f}^{-1}\mathcal{M})\\
&\qquad\xrightarrow[\int_f]{}\mathrm{R}\Gamma_c(X;T\mathcal{H}om(Rf_*(L\otimes g^{-1}G),\Omega_X)\otimes^L_{\mathcal{D}_X}\mathcal{M})[-d_{S/X}]\\
&\qquad=\mathrm{R}\Gamma_c(X;T\mathcal{H}om(L\overset{*}{\circ}G,\Omega_X)\otimes^L_{\mathcal{D}_X}\mathcal{M})[-d_{S/X}]\\
&\qquad\underset{\beta}{\to}\mathrm{R}\Gamma_c(X;T\mathcal{H}om(F,\Omega_X)\otimes^L_{\mathcal{D}_X}\mathcal{M})[-d_{S/X}].
\end{aligned}\quad (\mathrm{A}.2)$$

Here, the first and last morphisms are induced by α and β respectively. The morphisms g^{-1} and $\int_f$ are obtained from formulas (5.20) and (5.10) of [13], respectively. Recalling that the interior product is the restriction to the diagonal of the exterior product, the morphism $\bullet$ is obtained from formulas (5.20) and (5.2) of loc. cit.

Remark A.1. By tracing back the proof of [13, Theorem 10.8], it is possible to check that, under the hypotheses of Theorem 2.2, (A.1) coincides with (2.7).

The morphism (A.1) may be considered as an integral transform, in that it consists in: (1) pulling back a "function" from Y to S, (2) taking its product with a kernel on $T\mathcal{H}om(L,\mathcal{O}_S)$ induced by α, (3) integrating along the fibers of f, and using β to recognize the result as a "function" on X. Here, we use the term "quantization" to refer to the fact that such a morphism depends on the choice of α and β.

As we saw in section 3.2 (where we dealt with the functor $\overset{\mathrm{w}}{\otimes}$ instead of $T\mathcal{H}om$), in the framework of the complex Radon transform, for a suitable choice of F, G, $\mathcal{M}$ and $\mathcal{N}$, (A.1) reads:

$$\Gamma(P^*;\mathcal{D}b_{P^*}(\varepsilon^*|k^*))\xrightarrow{\sim}\Gamma(P;\mathcal{D}b_P(\varepsilon|k)),\quad (\mathrm{A}.3)$$

and one would like to check that this integral transform coincides with the Radon transform:

$$\varphi[\xi]\mapsto\int\varphi[\xi]\delta^{(n+\varepsilon^*|n+k^*)}(\langle x,\xi\rangle)\omega[\xi].\quad (\mathrm{A}.4)$$

In terms of (A.2), this means that we need to rewrite the composition of the last two morphisms $\int_f$ and β:

$$\begin{aligned} &\mathrm{R}\Gamma_c(S; T\mathcal{H}om(L \otimes g^{-1}G, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M}) \\ &\qquad \xrightarrow[\beta\circ\int_f]{} \mathrm{R}\Gamma_c(X; T\mathcal{H}om(F, \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M})[-d_{S/X}], \end{aligned} \tag{A.5}$$

so that a distribution kernel appears. We begin with a technical lemma.

Lemma A.2. *The morphism (A.5) decomposes into:*

$$\begin{aligned} &\mathrm{R}\Gamma_c(S; T\mathcal{H}om(L \otimes g^{-1}G, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M}) \\ &\qquad \xrightarrow[\beta]{} \mathrm{R}\Gamma_c(S; T\mathcal{H}om(f^{-1}F, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M}) \\ &\qquad \xrightarrow[\int_f]{} \mathrm{R}\Gamma_c(X; T\mathcal{H}om(Rf_*f^{-1}F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}) \\ &\qquad \to \mathrm{R}\Gamma_c(X; T\mathcal{H}om(F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}), \end{aligned}$$

where the last morphism is induced by the adjunction morphism id $\to Rf_*f^{-1}$.

Proof. It is easy to check the commutativity of the following diagram, where, for lack of space, we omit to write $\mathrm{R}\Gamma_c$:

$$\begin{tikzcd} T\mathcal{H}om(L \otimes g^{-1}G, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M} \arrow[r] \arrow[d] & T\mathcal{H}om(f^{-1}F, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M} \arrow[d] \\ T\mathcal{H}om(Rf_*(L \otimes g^{-1}G)[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M} \arrow[r] \arrow[d, "\beta"'] & T\mathcal{H}om(Rf_*f^{-1}F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M} \arrow[dl] \\ T\mathcal{H}om(F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M} & \end{tikzcd}$$

□

Note that if supp(G) is contained in a submanifold $N \subset Y$, then the morphism $f^{-1}F \to L \otimes g^{-1}G$ induced by β factorizes in:

$$\begin{tikzcd} f^{-1}F \arrow[rr, "\beta"] \arrow[dr] & & L \otimes g^{-1}G \\ & F \boxtimes_S \mathbb{C}_N \arrow[ur, "b"'] & \end{tikzcd} \tag{A.6}$$

where we set $F \boxtimes_S \mathbb{C}_N = f^{-1}F \otimes g^{-1}\mathbb{C}_N$.

Proposition A.3. *Assume that* $\mathrm{supp}(G)$ *is contained in a submanifold* $N \subset Y$. *Then, (A.5) factorizes in:*

$$\begin{aligned}
&\mathrm{R}\Gamma_c(S; T\mathcal{H}om(L \otimes g^{-1}G, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M}) \\
&\qquad \underset{b}{\to} \mathrm{R}\Gamma_c(S; T\mathcal{H}om(F \boxtimes_S \mathbb{C}_N, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M}) \\
&\qquad \underset{\int_f}{\longrightarrow} \mathrm{R}\Gamma_c(X; T\mathcal{H}om(Rf_*(F \boxtimes_S \mathbb{C}_N)[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}) \\
&\qquad \to \mathrm{R}\Gamma_c(X; T\mathcal{H}om(F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}),
\end{aligned}$$

where the first map is the "boundary value" morphism induced by the arrow b *in (A.6), and the last morphism is induced by the natural morphisms* $F \to Rf_*f^{-1}F \simeq Rf_*(F \boxtimes_S \mathbb{C}_Y)$ *and* $\mathbb{C}_Y \to \mathbb{C}_N$.

Proof. In view of Lemma A.2, the statement follows from the commutativity of the following diagram, where again, for lack of space, we omit to write $\mathrm{R}\Gamma_c$:

$$\begin{tikzcd}
T\mathcal{H}om(L \otimes g^{-1}G, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M} \arrow[d, "\beta"'] \arrow[dr, "b"] & \\
T\mathcal{H}om(f^{-1}F, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M} \arrow[d, "\int_f"'] & T\mathcal{H}om(F \boxtimes_S \mathbb{C}_N, \Omega_S) \otimes^L_{\mathcal{D}_S} \underline{f}^{-1}\mathcal{M} \arrow[l] \arrow[d, "\int_f"'] \\
T\mathcal{H}om(Rf_*f^{-1}F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M} \arrow[d] & T\mathcal{H}om(Rf_*(F \boxtimes_S \mathbb{C}_N)[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M} \arrow[l] \arrow[dl] \\
T\mathcal{H}om(F[d_{S/X}], \Omega_X) \otimes^L_{\mathcal{D}_X} \mathcal{M}. &
\end{tikzcd}$$

Here, the top triangle is induced by (A.6). □

In order to apply the above proposition to check that, in the case of the Radon transform, (A.3) is described by (A.4), we will give in Proposition A.8 a construction of the distribution $\delta^{(\varepsilon|k)}(\langle x, \xi \rangle)$ as boundary value of the Leray section (2.11).

A.2 Boundary Values

We begin with a topological lemma.

Lemma A.4. *Let* M *be a connected real analytic manifold, and let* X *be a complexification of* M. *For* $\varphi \in \Gamma(X; \mathcal{O}_X)$, *let* $U = \{z \in X : \varphi(z) \neq 0\}$, $Y = X \setminus U$, $\omega_\pm = \{z \in X : \pm \operatorname{Im} \varphi(z) > 0\}$, $\omega = \omega_+ \cup \omega_-$. *Assume that* φ *satisfies:*

(i) $\operatorname{Im} \varphi|_M = 0$,

(ii) $d\varphi \neq 0$.

Then, $\mathrm{Hom}\,(D'\mathbb{C}_M, \mathbb{C}_{\omega_\pm}) \simeq \mathbb{C}$*, and it has a canonical generator* $\gamma_\pm$*. Assume moreover that* $Y \cap M$ *is connected, and that:*

(iii) $H^1(M; \mathbb{C}_M) = 0$.

Then, the natural morphism $\mathbb{C}_\omega \to \mathbb{C}_U$ *(given by the inclusion* $\omega \subset U$*), induces an isomorphism:*

$$\mathrm{Hom}\,(D'\mathbb{C}_M, \mathbb{C}_\omega) \xrightarrow{\sim} \mathrm{Hom}\,(D'\mathbb{C}_M, \mathbb{C}_U).$$

Proof. Hypothesis (i) implies that $\overline{\omega}_\pm \supset M$, and (ii) implies that $D'\mathbb{C}_{\omega_\pm} \simeq \mathbb{C}_{\overline{\omega}_\pm}$. We then have:

$$\begin{aligned} R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_{\omega_\pm}) &\simeq R\mathcal{H}om\,(D'\mathbb{C}_{\omega_\pm}, D'D'\mathbb{C}_M) \\ &\simeq R\mathcal{H}om\,(\mathbb{C}_{\overline{\omega}_\pm}, \mathbb{C}_M) \\ &\simeq \mathbb{C}_M. \end{aligned} \tag{A.7}$$

Then, $\gamma_\pm$ corresponds to $1 \in \Gamma(M; \mathbb{C}_M)$, or, equivalently, to the natural morphism $\mathbb{C}_{\overline{\omega}_\pm} \to \mathbb{C}_M$, $1 \mapsto 1$.

Setting $N = X \setminus \omega$, the exact sequences:

$$\begin{gathered} 0 \to \mathbb{C}_\omega \to \mathbb{C}_U \to \mathbb{C}_{U\setminus\omega} \to 0, \\ 0 \to \mathbb{C}_{U\setminus\omega} \to \mathbb{C}_N \to \mathbb{C}_Y \to 0, \end{gathered}$$

induce the d.t.s:

$$R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_\omega) \to R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_U) \to R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_{U\setminus\omega}) \underset{+1}{\longrightarrow}, \tag{A.8}$$

$$R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_{U\setminus\omega}) \to R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_N) \to R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_Y) \underset{+1}{\longrightarrow}. \tag{A.9}$$

By (A.8), to prove the second part of the statement we are reduced to show that:

$$F^j := H^j \mathrm{RHom}\,(D'\mathbb{C}_M, \mathbb{C}_{U\setminus\omega}) = 0, \qquad \text{for } j = -1, 0. \tag{A.10}$$

One has:

$$\begin{aligned} R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_Y) &\simeq R\mathcal{H}om\,(D'\mathbb{C}_Y, \mathbb{C}_M) \\ &\simeq R\mathcal{H}om\,(\mathbb{C}_Y[-2], \mathbb{C}_M) \\ &\simeq R\mathcal{H}om\,(\mathbb{C}_{Y\cap M}, \mathbb{C}_M)[2] \\ &\simeq or_{Y\cap M/M}[1] \\ &\simeq \mathbb{C}_{Y\cap M/M}[1], \end{aligned}$$

an orientation of $Y \cap M$ in M being given by $\operatorname{Re}\varphi|_M > 0$. Similarly, noticing that $N \supset M$ and that $or_{N/X}$ is trivial, one has:

$$R\mathcal{H}om\,(D'\mathbb{C}_M, \mathbb{C}_N) \simeq \mathbb{C}_M[1].$$

Applying the functor $\mathrm{R}\Gamma(X;\cdot)$ to (A.9), we thus get the exact sequence:

$$0 \to F^{-1} \to \Gamma(M;\mathbb{C}_M) \underset{j}{\to} \Gamma(Y \cap M;\mathbb{C}_{Y\cap M/M}) \to F^0 \to H^1(M;\mathbb{C}_M) = 0,$$

where the last equality is due to hypothesis (iii). Since M and $Y \cap M$ are connected, j is an isomorphism, and (A.10) follows. □

Following [16], to $u \in \operatorname{Hom}(D'\mathbb{C}_M, \mathbb{C}_U)$ we associate the "boundary value map":

$$b_u\colon \mathrm{THom}\,(\mathbb{C}_U, \mathcal{O}_X) \to \Gamma(M;\mathcal{D}b_M), \tag{A.11}$$

given by $b_u = \mathrm{THom}\,(u, \mathcal{O}_X)$. Under the hypotheses of the above lemma, any morphism $u\colon D'\mathbb{C}_M \to \mathbb{C}_U$ factorizes into:

$$\begin{tikzcd} D'\mathbb{C}_M \arrow[rr, "u"'] \arrow[dr] & & \mathbb{C}_U \\ & \mathbb{C}_\omega \arrow[ur] & \end{tikzcd}$$

In particular, we get the following result:

Proposition A.5. *With the notations and the hypotheses of Lemma A.4, for any* $u \in \operatorname{Hom}(D'\mathbb{C}_M, \mathbb{C}_U)$ *there exist unique constants* $c_\pm \in \mathbb{C}$*, such that:*

$$b_u(f) = c_+ b_{\gamma_+}(f|_{\omega_+}) + c_- b_{\gamma_-}(f|_{\omega_-}). \tag{A.12}$$

A.3 Quantization of Real Projective Duality

Let us now consider the Radon transform, associated to the correspondence:

$$\begin{tikzcd} \mathbb{A} \arrow[r, hook] & \mathbb{P} \times \mathbb{P}^* \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathbb{P} & & \mathbb{P}^*. \end{tikzcd}$$

Recall that $\Omega = (\mathbb{P} \times \mathbb{P}^*) \setminus \mathbb{A}$.

Lemma A.6. *For* $F \in \mathbf{D}^{\mathrm{b}}(\mathbb{C}_\mathbb{P})$, $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_{\mathbb{P}^*})$, *there is a natural isomorphism:*

$$\alpha\colon \operatorname{Hom}(F \boxtimes D'G, \mathbb{C}_\Omega) \simeq \operatorname{Hom}(F, \mathbb{C}_\Omega \circ G).$$

Proof. One has the chain of isomorphisms:

$$\begin{aligned}\mathrm{Hom}\,(F, \mathbb{C}_\Omega \circ G) &\simeq \mathrm{Hom}\,(f^{-1}F, \mathbb{C}_\Omega \otimes g^{-1}G)\\ &\simeq \mathrm{Hom}\,(f^{-1}F, R\mathcal{H}om\,(g^{-1}D'G, \mathbb{C}_\Omega))\\ &\simeq \mathrm{Hom}\,(F \boxtimes D'G, \mathbb{C}_\Omega),\end{aligned}$$

where the last isomorphism follows from [12, Proposition 5.4.14], using the fact that $T^*_{\mathbb{A}}(\mathbb{P} \times \mathbb{P}^*) \cap (T^*_{\mathbb{P}}\mathbb{P} \times T^*\mathbb{P}^*) \subset T^*_{\mathbb{P}\times\mathbb{P}^*}\mathbb{P} \times \mathbb{P}^*$. □

Recall that $P = P(V)$ for an $(n + 1)$-dimensional real vector space V, so that $\mathbb{P} = \mathbb{P}(W)$ for a complexification W of V. Set $\widetilde{\Omega} = \{(z, \zeta); \langle z, \zeta\rangle \neq 0\} \subset \dot{W} \times \dot{W}^*$.

Lemma A.7. *For $F_\varepsilon = D'(\mathbb{C}_P(\varepsilon))[-n]$, $G_{\varepsilon^*} = D'(\mathbb{C}_{P^*}(\varepsilon^*))$, there is a natural isomorphism:*

$$\bigoplus_{\varepsilon\in\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}\,(F_\varepsilon, \mathbb{C}_\Omega \circ G_{\varepsilon^*}) \simeq \mathrm{Hom}\,(D'\mathbb{C}_{\dot{V}\times\dot{V}^*}, \mathbb{C}_{\widetilde{\Omega}}).$$

Proof. Denote by $\gamma : \dot{W} \times \dot{W}^* \to \mathbb{P} \times \mathbb{P}^*$ the natural projection. One has the isomorphisms:

$$\begin{aligned}\mathrm{Hom}\,(D'\mathbb{C}_{\dot{V}\times\dot{V}^*}, \mathbb{C}_{\widetilde{\Omega}}) &\simeq \mathrm{Hom}\,(D'\mathbb{C}_{\dot{V}\times\dot{V}^*}, \gamma^{-1}\mathbb{C}_\Omega)\\ &\simeq \mathrm{Hom}\,(\mathbb{C}_{\dot{V}\times\dot{V}^*}[-2n-2], \gamma^!\mathbb{C}_\Omega[-4])\\ &\simeq \mathrm{Hom}\,(R\gamma_!\mathbb{C}_{\dot{V}\times\dot{V}^*}[2-2n], \mathbb{C}_\Omega)\\ &\simeq \bigoplus_{\varepsilon,\eta\in\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}\,(\mathbb{C}_P(\varepsilon) \boxtimes \mathbb{C}_{P^*}(\eta)[-2n], \mathbb{C}_\Omega)\\ &\simeq \bigoplus_{\varepsilon,\eta\in\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}\,(F_\varepsilon \boxtimes G_\eta, \mathbb{C}_\Omega)[-n]\\ &\simeq \bigoplus_{\varepsilon,\eta\in\mathbb{Z}/2\mathbb{Z}} \mathrm{Hom}\,(F_\varepsilon, \mathbb{C}_\Omega \circ G_{\eta^*}),\end{aligned}$$

where the fourth isomorphism follows from Lemma 3.7, and the last isomorphism from Lemma A.6. By Lemma A.4, the first term in the above chain of isomorphisms is isomorphic to $\mathbb{C}^2$. One then concludes by noting that direct summands in the last line are different from zero for $\varepsilon = \eta$, due to Lemma 3.1. □

Note that the hypotheses of Lemma A.4 are satisfied for $M = \dot{V} \times \dot{V}^*$, $X = \dot{W} \times \dot{W}^*$, $\varphi(z, \zeta) = \langle z, \zeta\rangle$. By the identification in the above lemma, the section β_ε defined in (3.1) induces a boundary value morphism:

$$b_{\beta_\varepsilon} \colon \mathrm{THom}\,(\mathbb{C}_{\widetilde{\Omega}}, \mathcal{O}_{\dot{W}\times\dot{W}^*}) \to \Gamma(\dot{V} \times \dot{V}^*; \mathcal{D}b_{\dot{V}\times\dot{V}^*}).$$

Proposition A.8. *In the above notations, up to a non-zero multiplicative constant $c \in \mathbb{C}^\times$, one has:*

$$b_{\beta_\varepsilon}\left(\frac{1}{2\pi i\langle z,\zeta\rangle^{k+1}}\right) = c\,\delta^{(\varepsilon|k)}(\langle x,\xi\rangle).$$

Proof. By Proposition A.5, $b_{\beta_\varepsilon}(1/2\pi i\langle z,\zeta\rangle^{k+1})$ is a linear combination of $\delta^{(0|k)}(\langle x,\xi\rangle)$ and $\delta^{(1|k)}(\langle x,\xi\rangle)$. One then concludes by a parity argument. □

Theorem A.9. *Let $F = D'(\mathbb{C}_P(\varepsilon))[-n]$, $G = D'(\mathbb{C}_{P^*}(\varepsilon^*))$, $\mathcal{M} = \mathcal{D}_{\mathbb{P}}(k)$, $\mathcal{N} = \mathcal{D}_{\mathbb{P}^*}(k^*)$, $L = \mathbb{C}_\Omega$, $\mathcal{L} = T\mathcal{H}om(L, \mathcal{O}_{\mathbb{P}\times\mathbb{P}^*})$. Let:*

$$\alpha(s_{k^*}) \in \mathrm{Hom}_{\mathcal{D}_{\mathbb{P}^*}}(\mathcal{N}, \mathcal{M} \underline{\circ} \mathcal{L}), \qquad \beta_\varepsilon \in \mathrm{Hom}\,(F, L\circ G),$$

be the sections defined by (2.11) and (3.1) respectively. In this case, (2.8) is given by:

$$\begin{aligned} R^{(\varepsilon^*|k^*)}_{P^*}\colon \Gamma(P^*;\mathcal{D}b_{P^*}(\varepsilon^*|k^*)) &\to \Gamma(P;\mathcal{D}b_P(\varepsilon|k)) \\ \varphi[\xi] &\mapsto \int \varphi[\xi]\delta^{(n+\varepsilon^*|n+k^*)}(\langle x,\xi\rangle)\omega[\xi]. \end{aligned} \tag{A.13}$$

Moreover, (2.7) enters a commutative diagram:

$$\begin{tikzcd} \Gamma(P^*;\mathcal{D}b_{P^*}(\varepsilon^*|k^*)) \arrow[r, "R^{(\varepsilon^*|k^*)}_{P^*}", "\sim"'] & \Gamma(P;\mathcal{D}b_P(\varepsilon|k)) \\ \Gamma(P^*;\mathcal{C}^\infty_{P^*}(\varepsilon^*|k^*)) \arrow[u, hook] & \Gamma(P;\mathcal{C}^\infty_P(\varepsilon|k)). \arrow[l, "(2.7)", "\sim"'] \arrow[u, hook] \end{tikzcd}$$

Proof. To prove (A.13), with the notations of Proposition A.3, we have to check that the boundary value map b induced by β_ε sends $\varphi[\xi]s_{k^*}[z,\xi]\omega[\xi]$ to the distribution $\delta^{(n+\varepsilon^*|n+k^*)}(\langle x,\xi\rangle)\omega[\xi]$. This is clear from Proposition A.8. The commutativity of the diagram follows from the functoriality of the constructions. □

B Homogeneous Coordinates for the Blow-up

Denote by $p\colon \widetilde{\mathbb{P}^*_{\xi_\circ}} \to \mathbb{P}^*$ the blow-up of $\mathbb{P}^*$ along $\xi_\circ$. In terms of microlocal geometry, $\widetilde{\mathbb{P}^*_{\xi_\circ}}$ is identified with the projective normal deformation of $\xi_\circ$ in $\mathbb{P}^*$. In fact, there are natural maps (cf e.g., [12, §4.1], where the analogous construction is performed within the framework of real manifolds):

$$\widetilde{\mathbb{P}^*_{\xi_\circ}} \xrightarrow[t]{} \mathbb{C}/\mathbb{C}^\times \simeq \{0,1\} \qquad \begin{tikzcd} \mathbb{H}^* \arrow[r, hook, "s"] \arrow[d] & \widetilde{\mathbb{P}^*_{\xi_\circ}} \arrow[d, "p"] & \mathbb{P}^*_{\xi_\circ} \arrow[l, hook', "\tilde{\jmath}"'] \arrow[dl, hook] \\ \{\xi_\circ\} \arrow[r, hook] & \mathbb{P}^* & \end{tikzcd}$$

with $\mathbb{P}^*_{\xi_\circ} = t^{-1}(0)$. Moreover, as we noted in subsection 3.3, denoting by $\mathbb{H}^*$ the projective tangent space to $\mathbb{P}^*$ at $\xi_\circ$ there are natural correspondences

$$\begin{tikzcd} & \mathbb{P}^*_{\xi_\circ} \arrow[dl, hook', "i"'] \arrow[dr, "q"] & \\ \mathbb{P}^* & & \mathbb{H}^*, \end{tikzcd} \qquad \begin{tikzcd} & \widetilde{\mathbb{P}^*_{\xi_\circ}} \arrow[dl, "p"'] \arrow[dr, "\tilde{q}"] & \\ \mathbb{P}^* & & \mathbb{H}^*. \end{tikzcd}$$

For $\xi_\circ = [1, 0, \ldots, 0]$, $\mathbb{P}^*_{\xi_\circ}$ may be realized as the quotient space of $\mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ by the relation $(\sigma, \zeta') = (\lambda\sigma, \lambda\zeta')$ for $\lambda \in \mathbb{C}^\times$. Similarly, $\widetilde{\mathbb{P}^*_{\xi_\circ}}$ may be realized as the quotient space of $(\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^n \setminus \{0\})$ by the relation

$$(\upsilon_0, \upsilon_1, \zeta') = (\lambda\mu\upsilon_0, \mu\upsilon_1, \lambda\zeta') \qquad \text{for } \lambda, \mu \in \mathbb{C}^\times. \tag{B.1}$$

We denote by $[\![\upsilon, \zeta']\!]$ the bi-homogeneous coordinate system on $\widetilde{\mathbb{P}^*_{\xi_\circ}}$ associated to (υ, ζ'). In these coordinates, the above maps read:

$$\begin{aligned} &p([\![\upsilon, \zeta']\!]) = [\upsilon_0, \upsilon_1\zeta'], \\ &t([\![\upsilon, \zeta']\!]) = [\upsilon_1], \\ &\tilde{\jmath}([\sigma, \zeta']) = [\![\sigma, 1, \zeta']\!], \\ &q([\sigma, \zeta']) = [\zeta'], \\ &\tilde{q}([\![\upsilon, \zeta']\!]) = [\zeta']. \end{aligned} \tag{B.2}$$

This allows us to give a precise meaning to the sections γ_m of Proposition 3.8 as:

$$\gamma_m[\![\upsilon, \zeta']\!] = \frac{\upsilon_1^m}{\upsilon_0^{m+2+k^*}}\omega[\upsilon],$$

for $\sigma = \upsilon_1/\upsilon_0$. The isomorphisms (3.12) and (3.13) used in the proof of Proposition 3.8 also follow from (B.1) and (B.2). In particular, (3.12) uses the identification

$$\tilde{\jmath}^{-1}\mathcal{O}_{\widetilde{\mathbb{P}^*_{\xi_\circ}}}(k, l) \simeq \mathcal{O}_{\mathbb{P}^*_{\xi_\circ}}(k - l).$$

Of course, the same description holds in the real case. Denoting by $[\![u, \xi']\!]$ a system of bi-homogeneous coordinates on the blow-up of P^* along $\xi_\circ$, we may then intrinsically rewrite the functionals of Definition 1.4 as follows:

$$\begin{aligned} d^{(\omega|m)}_{\xi_\circ}\varphi[\xi'] &= \int_{\tilde{q}} \varphi[u_0, u_1\xi'] \frac{\delta^{(\omega|m)}(u_0/u_1)}{u_0^{2+k^*}}\omega[u], \\ c^{(\omega|m)}_{\xi_\circ}\varphi[\xi'] &= \int_{\tilde{q}} \varphi[u_0, u_1\xi'] \frac{\operatorname{sgn}(u_1/u_0)^\omega (u_1/u_0)^m}{u_0^{2+k^*}}\omega[u], \end{aligned}$$

for $t = u_0/u_1$, $s = u_1/u_0$.

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