Admissible locally analytic representations

Phuong Vu

I) Introduction

Setup: $\mathbb{Q}_p \subset L \subset K$, $[L:\mathbb{Q}_p] < \infty$

K spherically complete w.r.t the NAAV extending from L

(ex: $[K:L] < \infty$)

G: Locally L-analytic gp

V: Hausdorff loc. convex K-vector space

$LA(G,V) = \{ f : G \to V, \text{ locally ana} \}$

Definition: $\rho : G \longrightarrow GL(V)$ is a locally analytic rep. if $\forall\, v \in V$, the orbit map,

$$\rho_v : G \longrightarrow V \quad \text{is locally analytic}$$
$$g \longmapsto gv$$
$$(\rho_v \in LA(G, V)).$$

Examples:

1) $G = \mathbb{Z}_p$ the additive gp of p-adic integers

I $z \in K$, $|z| < 1$

Then $\chi(a) = z^a = \sum_{n=0}^{\infty} (z-1)^n \binom{a}{n}$, $\forall\, a \in \mathbb{Z}_p$.

where $\binom{a}{n} = \dfrac{a(a-1)\dots(a-n+1)}{n!}$

Theorem: (Amice): χ is locally analytic. $(\chi \in Hom(\mathbb{Z}_p, K^\times))$

or 1'

Rmk 1 If z is sufficiently closed to $1 \in K^\times$ then

$$\chi(a) = \exp(a \log(z)).$$

Rmk 2: $\forall\ \chi \in \mathrm{Hom}_{an}(\mathbb{Z}_p, K)$ is of this form!

2) Let $G = \mathbb{Q}_p^\times$ the multiplicative gp.

$\chi \in \mathrm{Hom}_{an}(G, K^\times)$

$\rightsquigarrow \exists\ c(\chi) \in K$ such that

$$\chi(a) = \exp(c(\chi) \log(a))$$

$\forall\ a \in \mathbb{Z}_p^\times$ sufficiently closed to 1.

$\left(\mathbb{Z}_p \xrightarrow{\exp} \mathbb{Z}_p^\times \xrightarrow{\chi} K^\times \right.$ is a character of $\left. \mathbb{Z}_p! \right)$

3) Induction

$B \subset G$ subgp such that G/B is compact

(ex : $G = GL_2(\mathbb{Q}_p)$, B lower triangular Borel)

$\forall\, \rho \in \mathrm{Rep}^{la}_K(B)$, define

$$\mathrm{Ind}_B^G(\rho) = \left\{ f \in LA(G, V) ,\ f(gb) = b \cdot f(g) \ \ \forall\, g \in G, b \in B \right\}$$

is locally analytic.

II) Locally analytic reps vs topological $D(G,K)$ - mod.

From now on, every loc. an. rep is <u>barrelled</u>.

<u>Theorem</u> (Schneider - Teitelbaum)

There is an anti-equivalence of categories

$$\left\{\begin{array}{c}\text{locally ana. representations}\\ \text{of compact type}\end{array}\right\} \longleftrightarrow \underbrace{\left\{\begin{array}{c}\text{separately continuous } D(G,K)\\ \text{modules in nuclear Fréchet}\\ \text{spaces}\end{array}\right\}}_{\text{“}= \mathrm{Mod}^{nF}(D(G,K))\text{”}}$$

$$V \longmapsto V_b'$$

Here, if $V \in \mathrm{Rep}(G)$ is loc. ana., then $V \longrightarrow LA(G,V)$

and $\exists$ canonical morp. $LA(G,V) \longrightarrow \mathcal{L}(D(G,K),V)$

$$LA(G,V) \cong LA(G,K)\hat{\otimes} V$$

$\rightsquigarrow$ Composing these ~~three~~ maps gives a $D(G,K)$-action on V.

Remarks 1) $\mathbb{C}$ study $\mathrm{Rep}^{la}(G)$ by looking at topological $D(G,K)$-modules.

2) The category $\mathrm{Mod}^{nF}(D(G,K))$ is not abelian!

(taking quotient of 2 object in $\mathrm{Mod}^{nF}(D(G,K))$ doesn't give an object of $\mathrm{Mod}^{nF}(D(G,K))$ in general).

III) Admissible locally analytic representations

1. Definition

Theorem: If G is compact $\rightsquigarrow$ $D(G,K)$ is Fréchet-Stein K-algebra
$\big(\underset{\cong}{D(G,K)} \simeq \varprojlim D_n\big)$

& $\{M = \varprojlim M_n$ coadmissible $D(G,K)$-mod$\}$ is abelian!

Definition: A loc. an. G-representation V is called admissible if it is of compact type and V_b' is a coadmissible $D(H,K)$-module for an open compact subgp $H \leq G$.

$$\mathrm{Rep}_K^a(G) := \{\text{admissible loc. ana. } G\text{-reps}/K\}$$

Theorem: The functor

$$\begin{aligned} \mathrm{Rep}_K^a(G) &\longrightarrow \mathcal{C}_{D(G,K)} \\ V &\longmapsto V_b' \end{aligned}$$

is an anti-equivalence of categories.

Rmk: 1) $\mathrm{Rep}_K^a(G)$ is an abelian categories

2) Any map in $\mathrm{Rep}_K^a(G)$ is strict and has dense image.

2) <u>Examples</u>:

1) $G = \mathbb{Z}_p$ $\quad X(\overline{K}) = \{x \in \overline{K}, |x| < 1\}$

$X = X(\overline{K})/Gal(\overline{K}/K)$

<u>Thm</u> (Amice):

$D(\mathbb{Z}_p, K) \simeq \mathcal{O}(X)$ (commutative K-alg)

Via this isomorphism, $\forall$ simple coadmissible $D(\mathbb{Z}_p, K)$ module is of the form $\mathcal{O}(X)/\mathfrak{m}$, $\mathfrak{m}$ max. closed ideal.

$\rightsquigarrow$ finite dimensional !

$\rightsquigarrow$ $\forall$ irred. admissible loc. ana G-rep is finite dimensional & is dual of $\mathcal{O}(X)/\mathfrak{m}$.

2) $G = GL_2(\mathbb{Q}_p)$

$G_0 = GL_2(\mathbb{Z}_p)$

$B = \left\{ \begin{pmatrix} \Delta & 0 \end{pmatrix} \right\} \subset G$ Borel subgp

$T = \left\{ \begin{pmatrix} \diagdown & 0 \\ 0 & \end{pmatrix} \right\} \subset G$ max. torus

$I = \begin{Bmatrix} \mathbb{Z}_p^\times & p\mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{Bmatrix} \subset G_0$: Iwahori

$B_0 = B \cap G_0 = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ \mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \subset I$

$\chi \in \mathrm{Hom}_{an}(T, K^\times)$: 1 dimensional rep

$$\mathrm{Ind}_B^G(\chi) := \{ f \in C^{an}(G, K^\times) : f(gb) = \chi(b^{-1}) f(g) \}.$$

Remark: $\exists\, c(\chi) \in K$ such that

$$\chi\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right) = \exp\left(c(\chi)\log(t)\right)$$

for $t \in \mathbb{Z}_p$ sufficiently closed to 1.

Theorem (ST): If $c(\chi) \notin -\mathbb{N}_0$, then

$$M_\chi = \mathrm{Ind}_B^G(\chi) \quad \text{is a irreducible } G\text{-representation}$$

Proof (Sketch):

1) Iwasawa decomp: $G = G_0 B$ $\qquad B_0 = B \cap G_0$

$$\rightsquigarrow \quad \mathrm{Ind}_B^G(\chi) \xrightarrow{\simeq} \mathrm{Ind}_{B_0}^{G_0}(\chi)$$

$S =$ representatives of G_0 in G

$$f \longmapsto f|_{G_0}$$

$$G = \bigsqcup_{b \in S} G_0 b$$

$$\tilde{f} \in \mathrm{Ind}_{B\cap G_0}^{G_0}(\chi) \rightsquigarrow \tilde{f}(\tilde{g}) = \begin{cases} \chi(b^{-1})\,\tilde{f}(g) & \text{if } \tilde{g} = gb,\ b \in S \\ 0 & \text{if not} \end{cases}$$

2) <u>Bruhat decomp</u>: $G_0 = I \sqcup I w B_0$, $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$$\rightsquigarrow \mathrm{Ind}_{B_0}^{G_0}(\chi) = \mathrm{Ind}_{B_0}^{I}(\chi) \oplus \mathrm{Ind}_{B_0^+}^{I}(w\chi)$$

(as I-representations)

$$\left(\begin{array}{l} B_0 = B_0 \cap I = B_0 \cap G \\ B_0^+ = I \cap wBw \end{array} \right.$$

where $w\chi : T \to K^\times$, $t \mapsto \chi(w^{-1} t w)$

<u>Remark</u> $\{1, w\}$ is the set of representatives of double cosets
$$I \backslash G_0 / B_0 = \{ I, I w B_0 \}.$$

3) $c(\chi) \notin \mathbb{N}_0$, $M_\chi^- = \mathrm{Ind}_{B_0^-}^{I} (\chi_1)'$ is a simple $D(I, K)$-mod.

<u>Lemma</u>: $M_\chi^- \simeq D(\mathbb{Z}_p, K)$

<u>Fact</u>: If the canonical proj $\pi : G \longrightarrow G/B$ is split, i.e

$\exists\ i : G/B \longrightarrow G \ , \ \pi \circ i = id_{G/B}$

then: $\mathrm{Ind}_B^G(\rho) \xrightarrow{\sim} LA(G/B, V) \quad \forall \rho \in \mathrm{Rep}_K^{la}(G)$

<u>proof of Fact</u>:

$$i^* : LA(G,V) \longrightarrow LA(G/B, V)$$
$$f \longmapsto f \circ i$$

and

$$G \xrightarrow{\sim} G/B \times B$$
$$g \longmapsto (\pi(g), (i \circ \pi(g))^{-1} g)$$

$\rightsquigarrow \mathrm{Ind}_B^G(\rho) \simeq \left\{ \begin{array}{l} f \in LA(G/B \times B, V) \\ f(\tilde{g}, b) \mapsto \chi(b^{-1}) f(\tilde{g}, 1) \end{array} \right.$

$\rightsquigarrow$ can define:

$$LA(G/B, K) \longrightarrow LA(G/B \times B, K)$$
$$\tilde{f} \longmapsto (\tilde{g}, b) \longmapsto \rho(b^{-1}) \tilde{f}(\tilde{g}) \quad \square$$

Now, $I/B_0 \simeq \left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right\} \simeq \mathbb{Z}_p$.

has a section $i : \mathbb{Z}_p \longrightarrow I$

$$b \longmapsto \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

$$\rightsquigarrow M_\chi^- \simeq \left(LA\left(I/B_0^- , K \right) \right)' = D(\mathbb{Z}_p, K).$$

Theorem: $c(\chi) \notin \mathbb{N}_0$, $D(\mathbb{Z}_p, K)$ is simple $D(I, K)$-mod.

Let $I \subset D(\mathbb{Z}_p, K)$ non zero $D(I, K)$-submod

Proof: Using Lazard's theory of divisors:

- $I \subset D(\mathbb{Z}_p, K)$ a $D(I, K)$-submodule
- $\mu_\infty = \{ \mathbb{C}_p : (1+z)^{p^n} = 1 \text{ for some } n \in \mathbb{N} \}$.

① $I = \langle F \rangle$, where the divisor (F) is supported on $x_0, x_1, \ldots \in \mu_\infty$ with multiplicity m_k.

② $u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lie}(I)$ then

$$u^{m_k} F \equiv \left[\prod_{i=0}^{m_k - 1} (c(\mathcal{X}) - i) \right] \cdot \left((1+T)\frac{d}{dT} \right)^{m_k} F$$
$$\left(\mathrm{mod}\ \log(1+T)\,\mathcal{O}(\mathcal{X}) \right)$$

$\rightsquigarrow$ we need the condition $c(\mathcal{X}) \notin \mathbb{N}_0$ so that the function $F^c = \left(\sum_{k=0}^{\infty} b_k u^{m_k} \right) F$ is supported outside of μ_∞ ! $\blacksquare$

More generally :

Theorem (Orlik – Strauch) [1]

Let $\mathbb{G}$ be a connected reductive gp$/L$. $\mathbb{P} \subset \mathbb{G}$ parabolic subgp., $\mathbb{T} \subset \mathbb{P}$, max torus

$$G = \mathbb{G}(L), \quad P = \mathbb{P}(L), \quad T = \mathbb{T}(L)$$

<u>Rm</u> : $\forall\, V \in \mathrm{Rep}_K^{la}(T)$ extends to a locally analytic rep of P by Levi decomp.

If $\dim_K(V) < \infty$, then $\mathrm{Ind}_P^G(p)$ is irred $\iff$ $\mathcal{U}(g) \otimes_{\mathcal{U}(p)} V'$ is irred. as $\mathcal{U}(g)$-module.

[1] On the irreducibility of locally analytic principal series representations.