

# Basic notions of non-archimedean functional analysis

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## Contents

|           |   |           |
|-----------|---|-----------|
| <b>0</b>  | <b>Introduction</b>   | <b>1</b>  |
| <b>1</b>  | <b>Base fields</b>  | <b>2</b>  |
| 1.1       | Generalities of topological algebra . . . . .                       | 2         |
| 1.2       | Non-archimedean fields . . . . .                                    | 3         |
| <b>2</b>  | <b>Locally convex spaces: lattices vs. seminorms</b>                | <b>4</b>  |
| <b>3</b>  | <b>Limits and colimits of locally convex spaces</b>                 | <b>7</b>  |
| 3.1       | The strict inductive limit . . . . .                                | 8         |
| <b>4</b>  | <b>Boundedness and topologies on <math>\mathcal{L}(V, W)</math></b> | <b>9</b>  |
| <b>5</b>  | <b>Various notions of compactness</b>                               | <b>11</b> |
| <b>6</b>  | <b>Spaces of compact type</b>                                       | <b>11</b> |
| <b>7</b>  | <b>Duality</b>  | <b>13</b> |
| <b>8</b>  | <b>Distributions on locally analytic groups</b>                     | <b>13</b> |
| <b>9</b>  | <b>The case of <math>G = \mathbb{Z}_p</math></b>                    | <b>15</b> |
| <b>10</b> | <b>Fréchet-Stein algebras</b>                                       | <b>16</b> |

## 0 Introduction

Functional analysis works with TVS (= Topological Vector Spaces), classically over archimedean fields like  $\mathbb{R}$  and  $\mathbb{C}$ . Classical analytic geometry works with rings of analytic functions on suitable domains in  $\mathbb{C}^n$ , and modules over such rings. Although all of the latter rings are rings in the category of TVS, and modules over them are in particular topological  $\mathbb{C}$ -vector spaces, there is a difference in emphasis between functional analysis and analytic geometry. The latter is, to some extent, a relative form of functional analysis.

In the non-archimedean world functional analysis and analytic geometry are closer than they are in the classical case. This is why we prefer to set some classical definitions of non-archimedean functional analysis in a topological algebra framework.

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Since the purpose of this seminar is the understanding of representations of  $p$ -adic locally analytic groups in non archimedean TVS, we modelled our presentation on the beautiful lectures of Peter Schneider and Jeremy Teitelbaum at Hangzhou [5] with the addition of some basic definitions and results from the book [4].

## 1 Base fields

### 1.1 Generalities of topological algebra

**Definition 1.1.1.** A group topology in a topological abelian group  $G$  is a topology with a fundamental system of neighborhoods of 0 consisting of open subgroups.

All over this talk, rings will be commutative with 1. If  $R$  is a ring, an algebra over  $R$  is expected to be associative and unital, but not necessarily commutative. It will be understood that topological rings, modules, and algebras are equipped with additive group topologies. Moreover, the map

$$\begin{aligned} \mu_R : R \times R &\rightarrow R, (r, s) \mapsto rs, \text{ for } R \text{ a ring} \\ \text{(resp. } \mu_M : R \times M &\rightarrow M, (r, m) \mapsto rm, \text{ for } M \text{ an } R\text{-module,} \\ \text{resp. } \mu_A : A \times A &\rightarrow A, (a, b) \mapsto ab, \text{ for } A \text{ an } R\text{-algebra) ,} \end{aligned}$$

is supposed to be continuous (resp. continuous in the second variable). For any topological ring  $R$ , a topological  $R$ -module or  $R$ -algebra  $X$  is *separately continuous* (resp. *continuous*) if  $\mu_X$  is separately continuous (resp. continuous).

**Definition 1.1.2.** A topological ring  $R$  (commutative with 1) is linearly topologized if a basis of neighborhoods of 0 consists of ideals. (In that case  $\mu_R$  is uniformly continuous.) If  $R$  is a linearly topologized ring, a topological  $R$ -module  $M$  is  $R$ -linearly topologized if a basis of neighborhoods of  $0 \in M$  consists of  $R$ -submodules. (In that case if  $M$  is separately continuous, then it is continuous.)

**Definition 1.1.3.** Let  $R$  be a linearly topologized ring. We let  $\mathcal{LM}_R$  be the category of linearly topologized  $R$ -modules with continuous  $R$ -linear morphisms. We let  $\mathcal{LM}_R^c$  denote the full subcategory of continuous  $R$ -modules.

**Definition 1.1.4.** Let  $M$  be an object of  $\mathcal{LM}_R$ . An  $R$ -submodule  $P$  of  $M$  is a sponge if, for any  $m \in M$  there exists an open ideal  $J_m$  of  $R$  such that  $J_m m \subset P$ .

An object  $M$  of  $\mathcal{LM}_R$  is continuous iff it admits a fundamental system of open  $R$ -submodules which are sponges. It follows easily that the inclusion of categories  $\mathcal{LM}_R^c \hookrightarrow \mathcal{LM}_R$  admits a left adjoint  $M \mapsto M^c$ .

**Proposition 1.1.5.** Let  $R$  be a linearly topologized ring. The category  $\mathcal{LM}_R$  admits all limits and colimits.

*Proof.* Limits (resp. colimits) are calculated in the category  $\text{Mod}_R$  of  $R$ -modules, and are given the weak *i.e.* the initial (resp. the strong *i.e.* the final)  $R$ -linear topology of the canonical morphisms.  $\square$

**limcolim**

**Corollary 1.1.6.** The category  $\mathcal{LM}_R^c$  admits all limits and colimits.

*Proof.* The subcategory  $\mathcal{LM}_R^c$  is stable by limits and finite colimits in  $\mathcal{LM}_R$ . It is not stable by infinite colimits, though. By general properties of adjunction, for an inductive system  $(M_\alpha)_{\alpha \in A}$  in  $\mathcal{LM}_R^c$ , we have

**lccolimit**

$$(1.1.6.1) \quad \mathcal{LM}_R^c - \varinjlim_{\alpha} M_{\alpha} = (\mathcal{LM}_R - \varinjlim_{\alpha} M_{\alpha})^c .$$

$\square$

**Remark 1.1.7.** It is not difficult to show that  $\mathcal{LM}_R$  is quasi-abelian [6]. Its full subcategory  $\mathcal{LM}_R^c$ , stable by finite limits and finite colimits, is then quasi-abelian, as well.

For  $R$  a linearly topologized ring and  $M, N, X$  objects of  $\mathcal{LM}_R^c$ , let

$$\text{Bil}_R^s(M \times N, X) = \{\varphi : M \times N \longrightarrow X \mid \varphi \text{ is } R\text{-bilinear and separately continuous}\}$$

(resp.

$$\text{Bil}_R^c(M \times N, X) = \{\varphi : M \times N \longrightarrow X \mid \varphi \text{ is } R\text{-bilinear and continuous}\} ).$$

The functor  $X \longmapsto \text{Bil}_R^s(M \times N, X)$  (resp.  $X \longmapsto \text{Bil}_R^c(M \times N, X)$ ),  $\mathcal{LM}_R^c \longrightarrow \text{Mod}_R$  is representable by an object  $M \otimes_R^s N$  (resp.  $M \otimes_R^c N$ ) of  $\mathcal{LM}_R^c$ , called the *inductive* (resp. *projective*) *tensor product* of  $M$  and  $N$ . In both cases the underlying  $R$ -module is  $M \otimes_R N$ . A basis of open  $R$ -submodules of  $M \otimes_R^s N$  (resp.  $M \otimes_R^c N$ ) consists of  $\{F_1 \otimes P_2 + P_1 \otimes F_2\}$  (resp. of  $\{P_1 \otimes P_2\}$ ), for  $F_1 \leq M$  and  $F_2 \leq N$ , finitely generated  $R$ -submodules, and  $P_1 \leq M, P_2 \leq N$ , open  $R$ -submodules. Then  $(\mathcal{LM}_R^c, \otimes_R^s)$  (resp.  $(\mathcal{LM}_R^c, \otimes_R^c)$ ) is a symmetric monoidal category with unit object  $R$ .

## 1.2 Non-archimedean fields

**Definition 1.2.1.** A non-archimedean (n.a. for short) field is a field  $K$  equipped with an absolute value  $|\cdot|$  which satisfies the strong triangle inequality

$$|a + b| \leq \max(|a|, |b|) \quad , \quad \forall a, b \in K .$$

The metric  $d(x, y) = |x - y|$  makes  $K$  a *topological field*. Notice that  $d$  is an *ultrametric* in the sense that

$$d(x, z) \leq \max(d(x, y), d(y, z)) \quad \forall x, y, z \in K .$$

All triangles are isosceles!

A *closed* (resp. *open*) *ball* of radius  $\varepsilon \in |K^\times|$  is

$$B_\varepsilon(a) = \{x \in K \mid |x - a| \leq \varepsilon\}$$

$$\text{(resp. } B_\varepsilon^-(a) = \{x \in K \mid |x - a| < \varepsilon\} \text{)} .$$

Due to the non-archimedean inequality, open and closed balls are both open and closed subsets of the topological field  $K$  and, if two balls intersect, one of them is contained in the other. In particular,  $K$  is totally disconnected.

Notice that,  $B_1(0) =: K^\circ$  is an open local subring of  $K$  and  $B_1^-(0) =: K^{\circ\circ}$  is its maximal ideal. In fact  $K^\circ$  is a  $(\pi)$ -adic linearly topologized ring, for any  $0 \neq \pi \in K^{\circ\circ}$ .

For any  $\varepsilon > 0$ ,  $B_\varepsilon(0)$  and  $B_\varepsilon^-(0)$  are additive subgroups of  $K$ , and that

$$B_\varepsilon(a) = a + B_\varepsilon(0) \quad , \quad B_\varepsilon^-(a) = a + B_\varepsilon^-(0) \quad ,$$

for any  $a \in K$ , are group cosets.

For example,  $(K = \mathbb{C}(\!(T)\!), |\cdot|)$ , where  $0 < |T| < 1$ , is such a field. Another is  $(\mathbb{Q}, |\cdot|_p)$  in which  $|p| = p^{-1}$ . In both cases the field  $K$  contains a bounded open subring  $K^\circ$  called *the ring of integers* whose topology is  $I$ -adic, where  $I$  is a finitely generated ideal. Here

$$(K^\circ, I) = (\mathbb{C}[[T]], T\mathbb{C}[[T]]) \quad \text{(resp. } = (\mathbb{Z}_{(p)}, p\mathbb{Z}_{(p)}) \text{)} .$$

**Convention 1.2.2.** Our non archimedean fields  $(K, |\cdot|)$  will always be assumed to be complete and non-trivially valued (i.e.  $|K| \neq \{0, 1\}$ ). All over this paper, “complete” stands for “Hausdorff and complete”.

The main character of our story is the completion  $(\mathbb{Q}_p, |\cdot|_p)$  of  $(\mathbb{Q}, |\cdot|_p)$ , with ring of integers  $\mathbb{Z}_p$  and ideal of definition  $p\mathbb{Z}_p$ . The absolute value of  $\mathbb{Q}_p$  uniquely extends to any finite extension  $L/\mathbb{Q}_p$ , and therefore to the algebraic closure  $\overline{\mathbb{Q}_p}$ . By completion we obtain  $\mathbb{C}_p$ , which happens to be algebraically closed by a theorem of Krasner. We have  $|\mathbb{C}_p^\times| = p^\mathbb{Q}$ .

The field of complex numbers is locally compact. This also holds true for any valued finite extension field  $L/\mathbb{Q}_p$ . But

**Proposition 1.2.3.**  $\mathbb{C}_p$  is not locally compact.

In fact,  $|\mathbb{C}_p| = p^\mathbb{Q}$ . If  $\mathbb{C}_p$  were locally compact there would exist a compact ball  $B_\varepsilon(0)$ , with  $\varepsilon = p^r$  and  $r \in \mathbb{Q}$ . The image in  $\mathbb{R}$  of such a ball via the continuous function  $|\cdot|$ , must be compact, hence closed, and should therefore coincide with  $[0, \varepsilon]$ . But it is contained in  $p^\mathbb{Q}$ . Absurd.

Fortunately, we only need a weak form of compactness.

**Definition 1.2.4.** A non archimedean field  $K$  is said to be spherically complete if for any decreasing sequence of balls  $B_1 \supset B_2 \supset \dots$  in  $K$ , the intersection  $\bigcap_{n \in \mathbb{N}} B_n$  is nonempty.

More generally,

**c-compact**

**Definition 1.2.5.** Let  $R$  be a linearly topologized ring and let  $M$  be a linearly topologized  $R$ -module. Then  $M$  is c-compact (as an  $R$ -module) if, for any filtered system  $\{P_\gamma\}_{\gamma \in \Gamma}$  of open submodules of  $M$ , the canonical map

$$M \longrightarrow \varprojlim_{\gamma} M/P_\gamma$$

is surjective. In particular, the ring  $R$  is c-compact iff it is so as an  $R$ -module.

The field  $K$  is c-compact as a  $K^\circ$ -module iff it is spherically complete.

**Proposition 1.2.6.**  $\mathbb{C}_p$  is not spherically complete.

*Proof.* The countable set  $\overline{\mathbb{Q}}$  is dense in  $\mathbb{C}_p$ . So, let's order it as a sequence  $a_1, a_2, \dots$ . Fix a sequence  $1 > \varepsilon_1 > \varepsilon_2 > \dots > \frac{1}{2}$ . Consider the equivalence relation  $a \sim_1 b$  if  $|b - a| \leq \varepsilon_1$ . The equivalence classes are balls of diameter  $\varepsilon_1$ . Fix one such  $B_1$  such that  $a_1 \notin B_1$ . In  $B_1$  let us consider the equivalence relation  $a \sim_2 b$  if  $|b - a| \leq \varepsilon_2$ . The equivalence classes are balls of diameter  $\varepsilon_2$ . We find a ball  $B_2 \subset B_1$  of diameter  $\varepsilon_2$  such that  $a_2 \notin B_2$ , and so on. Then  $C := \bigcap_{n \in \mathbb{N}} B_n$  is empty. Otherwise, let  $c \in C$ . Then,  $B_{1/2}(c) \subset C$ . But then no  $a_n$  is in  $B_{1/2}(c)$ . Absurd, since  $\{a_n\}_{n \in \mathbb{N}}$  is dense in  $\mathbb{C}_p$ .  $\square$

**Remark 1.2.7.** Obviously, any discretely valued complete field is spherically complete.

## 2 Locally convex spaces: lattices vs. seminorms

Let  $(K, |\cdot|)$  is a fixed n.a. field and let  $R = K^\circ$  be its ring of integers. Notice that  $K$  is a ring in both monoidal categories  $(\mathcal{LM}_R^c, \otimes_R^c)$  and  $(\mathcal{LM}_R^s, \otimes_R^s)$ . A locally convex (l.c. for short) (topological)  $K$ -vector space is a  $K$ -module  $V$  in  $(\mathcal{LM}_R^c, \otimes_R^c)$ , hence in  $(\mathcal{LM}_R^s, \otimes_R^s)$ . This is because  $K \otimes_R^s V = K \otimes_R^c V$ .

If  $V$  is a locally convex  $K$ -vector space, the topology of  $V$  has a basis of open  $K^\circ$ -submodules  $\{L_j\}_{j \in J}$  such that, for any  $v \in V$  and  $j \in J$ , there is  $a \in K^\times$  such that  $av \in L_j$ . So, any  $L_j$  is a lattice in  $V$ :  $L_j \otimes K \xrightarrow{\sim} V$ . Equivalently, any lattice  $L$  in  $V$  determines a seminorm  $q_L : V \rightarrow \mathbb{R}$ , namely

$$q_L(v) = \inf\{|a| \mid v \in aL\}.$$

This seminorm satisfies  $q_L(v + w) \leq \max(q_L(v), q_L(w))$ , so it is non archimedean, as well. It is called the *gauge* of  $L$ .

The datum of a lattice is equivalent to the one of its gauge. So, a locally convex  $K$ -vector space  $(V, \{L_j\}_{j \in J})$  may be seen as a  $K$ -vector space whose topology is induced by a family of seminorms  $(V, \{q_j\}_{j \in J})$ , where  $q_j = q_{L_j}$ . A seminorm  $q$  on  $V$  is a *norm* if  $q(v) = 0 \Rightarrow v = 0$ . A vector space  $V$  equipped with a single (semi-)norm  $q$  is a *(semi-)normed vector space*  $(V, q)$ . The distance function  $d(x, y) := q(x - y)$  makes  $V$  a metric space. The lattice  $L(q) = \{v \in V \mid q(v) \leq 1\}$  is the unit ball of  $(V, q)$  centered at 0. For  $R = K^\circ$ , the tensor product  $\otimes_R^s$  (resp.  $\otimes_R^c$ ) induces a tensor product on  $\mathcal{LC}_K$ . The category  $\mathcal{LC}_K$  is equipped with two structures of symmetric monoidal category :

$$\otimes_{K, \iota} = \otimes_{K^\circ}^s \quad \text{and} \quad \otimes_{K, \pi} = \otimes_{K^\circ}^c .$$

**Definition 2.0.1.** *A normed  $K$ -vector space is a  $K$ -Banach space if the corresponding metric space is complete. We denote by  $\text{Ban}_K$  the corresponding full subcategory of  $\mathcal{LC}_K$ .*

Morphisms of locally convex  $K$ -vector spaces are simply continuous  $K$ -linear maps. For normed spaces  $(V, p)$  and  $(W, q)$  there is a finer notion of morphism. Namely, a linear map  $f : V \rightarrow W$  is *norm-decreasing* if  $q(f(v)) \leq p(v)$ , for any  $v \in V$ . This is equivalent to  $f(L(p)) \subset L(q)$ . Isomorphisms in this latter sense are *isometries*. This finer theory of Banach spaces will not be discussed here.

Obviously, a locally convex  $K$ -vector space  $V$  is separated if and only if

$$\overline{(0)} (= \bigcap_j L_j) = (0) .$$

We let  $\mathcal{SLLC}_K$  denote the full subcategory of  $\mathcal{LC}_K$  consisting of separated objects. The inclusion of categories  $\mathcal{SLLC}_K \hookrightarrow \mathcal{LC}_K$  admits a left adjoint  $V \mapsto V^{\text{sep}} = V/\overline{(0)}$ . We also let  $\mathcal{CCLC}_K$  denote the full subcategory of  $\mathcal{SLLC}_K$  consisting of complete objects. Again the inclusion of categories  $\mathcal{CCLC}_K \hookrightarrow \mathcal{SLLC}_K$  admits a left adjoint  $V \mapsto \widehat{V}$ , called completion. The composition  $V \mapsto \widehat{V^{\text{sep}}}$  is separated completion.

**Definition 2.0.2.** *Let  $V$  be a l.c.  $K$ -vector space. A subset  $B \subset V$  is bounded if, for any neighborhood  $P$  of 0 in  $V$  there exists  $a \in K^\times$  such that  $aB \subset P$ .*

**Proposition 2.0.3.** *Normed spaces are characterized among Hausdorff locally convex vector spaces by the existence of a bounded open lattice  $L \subset V$ .*

**Definition 2.0.4.** *Let  $M$  be any  $K^\circ$ -module. The naive canonical topology of  $M$  is the  $K^\circ$ -linear topology for which a basis of open submodules is  $\{JM\}_J$  where  $J$  varies among open ideals of  $K^\circ$ .*

Let  $L$  be a bounded open lattice in the  $K$ -Banach space  $V$ . The subspace topology of  $L$  is its naive canonical topology, and  $L$  is complete in that topology. Conversely, let  $L$  be any torsion free  $K^\circ$ -module which is complete in its naive canonical topology. Then  $V := L \otimes_{K^\circ} K$  equipped with the gauge norm of  $L$  is a  $K$ -Banach space with bounded open lattice  $L$ .

The previous observation permits an abstract definition of  $K$ -Banach spaces without reference to a norm.

**Definition 2.0.5.** *A Fréchet space is an object of  $\mathcal{LC}_K$  which is C1 and complete. We let  $\text{Fr}_K$  denote the full subcategory of  $\mathcal{LC}_K$  consisting of Fréchet spaces.*

**Proposition 2.0.6.** *Let  $V$  be an object of  $\mathcal{LC}_K$ . TFAE :*

1.  $V$  is a  $K$ -Fréchet space;
2.  $V$  is a complete l.c.  $K$ -vector space whose topology is defined by a decreasing sequence of open lattices

$$L_1 \supset L_2 \supset \dots ;$$

(equivalently, by an increasing sequence of seminorms

$$q_1 \leq q_2 \leq \dots ).$$

3.  $V$  is metrizable and complete.

**Definition 2.0.7.** For  $V, W$  in  $\mathcal{B}an_K$  (resp.  $\mathcal{F}r_K$ ),

$$V \otimes_{K,\iota} W \cong V \otimes_{K,\pi} W$$

is an object of  $\mathcal{B}an_K$  (resp.  $\mathcal{F}r_K$ ). So,  $(\mathcal{B}an_K, \otimes_{K,\pi}) \subset (\mathcal{F}r_K, \otimes_{K,\pi})$  are tensor subcategories of both  $(\mathcal{L}C_K, \otimes_{K,\pi})$  and  $(\mathcal{L}C_K, \otimes_{K,\iota})$ . A  $K$ -Banach (resp. a  $K$ -Fréchet) algebra is a  $K$ -algebra in  $(\mathcal{B}an_K, \otimes_{K,\pi})$  (resp. in  $(\mathcal{F}r_K, \otimes_{K,\pi})$ ).

**exafcts**

**Example 2.0.8.**

1. For  $X$  any set,

$$\ell^\infty(X) := \text{all bounded functions } \phi : X \rightarrow K$$

with the obvious operations and with the supnorm on  $X$ ,  $\|\cdot\|_\infty$ , is a  $K$ -Banach space.

2. For  $X$  as before

$$c_o(X) := \{ \phi \in \ell^\infty(X) \mid \forall \varepsilon > 0, |\phi(x)| < \varepsilon, \forall x \in X \}$$

(“ $\forall\forall$ ” means “for all but a finite number”), is a Banach subspace of  $\ell^\infty(X)$ .

3. A bounded  $K$ -analytic function on  $B_1^-(a)$  is the sum of a power series of the form

$$f(x) = \sum_{i=0}^{\infty} a_i (x-a)^i,$$

where the family  $\{|a_i|\}_{i \in \mathbb{N}}$  in  $\mathbb{R}_{\geq 0}$  is bounded. The ring

$$\mathcal{A}_K^{\text{bd}}(B_1^-(a)) = K \otimes_{K^\circ} K^\circ[[x-a]]$$

of bounded  $K$ -analytic functions on  $B_1^-(a)$ , equipped with the supnorm, is a  $K$ -Banach space isomorphic to  $\ell^\infty(\mathbb{N})$  via

$$\ell^\infty(\mathbb{N}) \xrightarrow{\sim} \mathcal{A}_K^{\text{bd}}(B_1^-(a)), \quad (a_i)_{i \in \mathbb{N}} \mapsto \sum_{i=0}^{\infty} a_i (x-a)^i$$

and it is a  $K$ -Banach algebra.

4. A  $K$ -analytic function on  $B_1(a)$  is the sum of a power series of the form

$$f(x) = \sum_{i=0}^{\infty} a_i (x-a)^i,$$

where  $|a_i| \rightarrow 0$  as  $i \rightarrow \infty$ . The ring

$$\mathcal{A}_K(B_1(a)) = K\{x-a\}$$

of  $K$ -analytic functions on  $B_1(a)$ , equipped with the supnorm, is a  $K$ -Banach space isomorphic to  $c_o(\mathbb{N})$  via

$$c_o(\mathbb{N}) \xrightarrow{\sim} \mathcal{A}_K(B_1(a)) \quad , \quad (a_i)_{i \in \mathbb{N}} \longmapsto \sum_{i=0}^{\infty} a_i(x-a)^i$$

and it is a  $K$ -Banach algebra. Its unit ball is

$$K^\circ\{x-a\} := \text{the } \pi\text{-adic completion of } K^\circ[x-a], \text{ for any } 0 \neq \pi \in K^{\circ\circ} .$$

5. A  $K$ -analytic function on  $B_1^-(a)$  is the sum of a power series of the form

$$f(x) = \sum_{i=0}^{\infty} a_i(x-a)^i ,$$

where, for any  $0 < r < 1$ ,  $r^i|a_i| \rightarrow 0$  as  $i \rightarrow \infty$ . For any such  $r$ , let  $\|\cdot\|_r$  denote the norm

$$\left\| \sum_{i=0}^{\infty} a_i(x-a)^i \right\|_r := \sup_{i \in \mathbb{N}} r^i |a_i| .$$

The ring  $\mathcal{A}_K(B_1^-(a))$  of  $K$ -analytic functions on  $B_1^-(a)$ , equipped with the increasing sequence of norms

$$\{\|\cdot\|_{(n)/(n+1)}\}_{n=1,2,\dots}$$

is a  $K$ -Fréchet algebra which may be seen as the limit of the projective system of  $K$ -Banach algebras

**FScomm**

(2.0.8.1)

$$\begin{aligned} \cdots &\longrightarrow (\mathcal{A}_K(B_{(n)/(n+1)}(a)), \|\cdot\|_{(n)/(n+1)}) \longrightarrow (\mathcal{A}_K(B_{(n-1)/n}(a)), \|\cdot\|_{(n-1)/n}) \cdots \\ &\cdots \longrightarrow \mathcal{A}_K(B_{1/2}(a)), \|\cdot\|_{1/2} . \end{aligned}$$

Notice that the Banach algebras appearing in (2.0.8.1) are Noetherian rings, while the transition morphisms there are flat since they are restrictions of functions from an affinoid space to an affinoid subdomain [1, Prop. 2.2.4 (ii)].

### 3 Limits and colimits of locally convex spaces

**Proposition 3.0.1.** *The category  $\mathcal{LC}_K$  admits all limits and colimits. Similarly for its full subcategory  $\mathcal{SLLC}_K$ .*

*Proof.* This follows from Corollary 1.1.6. We want to be more explicit. Limits are calculated as  $K$ -vector spaces and are endowed with the initial (*i.e.* weak) topology of the canonical morphisms. But if a  $K$ -vector space  $V$  is equipped with  $K$ -linear maps  $f_h : V \rightarrow V_h$  to locally convex vector spaces  $V_h$ , for  $h \in H$ , with fundamental systems of open lattices  $\{L_{h,j}\}_{j \in J_h}$  in  $V_h$ , then the initial topology of the maps  $f_h$  gives a structure of locally convex vector space to  $V$  with fundamental system of open lattices  $\{L_{h,j}\}_{h,j}$ . This settles the existence of limits.

For colimits we need to apply formula (1.1.6.1). In fact, quotients are even simpler: if  $V$  is locally convex and  $U \subset V$  is a vector subspace, the quotient topology on  $V/U$  is locally convex. On the other hand, given a family  $\{V_\gamma\}_{\gamma \in \Gamma}$  in  $\mathcal{LC}_k$ , the direct sum in  $\mathcal{LC}_k$  is the direct sum  $\bigoplus_\gamma V_\gamma$  of the  $V_\gamma$ 's as  $K$ -vector spaces, equipped with all  $K^\circ$ -submodules of the form  $\bigoplus_\gamma L_\gamma$ , where  $L_\gamma$  is an open lattice in  $V_\gamma$ ,  $\forall \gamma$ . Notice that these  $K^\circ$ -submodules are necessarily lattices in  $\bigoplus_\gamma V_\gamma$ . This topology is coarser than the strong topology of the family of maps  $V_\gamma \rightarrow \bigoplus_\gamma V_\gamma$ , for  $\gamma \in \Gamma$ .  $\square$

**Remark 3.0.2.** The category  $\mathcal{LC}_K$  is quasi-abelian in the sense of [6].

**Proposition 3.0.3.** *The category  $\mathcal{CLC}_K$  admits all limits and colimits.*

*Proof.* The category  $\mathcal{CLC}_K$  is stable in  $\mathcal{LC}_K$  under limits. For colimits we have, for any inductive system  $(V_\gamma)_{\gamma \in \Gamma}$

$$(3.0.3.1) \quad \mathcal{CLC}_K - \varinjlim_{\gamma \in \Gamma} V_\gamma = \text{the separated completion of } \mathcal{LC}_K - \varinjlim_{\gamma \in \Gamma} V_\gamma .$$

□

**Remark 3.0.4.** The category  $\mathcal{CLC}_K$  is not quasi-abelian. In fact a cokernel in  $\mathcal{CLC}_K$  is not necessarily surjective so that in general cokernels will not be stable under pull-back. However the full subcategories  $\mathcal{Ban}_K \subset \mathcal{Fr}_K$  of  $\mathcal{CLC}_K$  are quasi-abelian. They are however neither complete nor cocomplete.

amazing

**Remark 3.0.5.** An amazing fact is that if the inductive limit in (3.0.3.1) is simply the direct sum of a family of objects of  $\mathcal{CLC}_K$ , then the direct sum in  $\mathcal{LC}_K$  is already complete, so that it coincides with the direct sum in  $\mathcal{CLC}_K$  (cf. [4, Lemma 7.8]).

There is a third, more important, full subcategory of  $\mathcal{LC}_K$  containing  $\mathcal{CLC}_K$ .

**Definition 3.0.6.** *An object  $V$  of  $\mathcal{LC}_K$  is quasi-complete if any bounded closed  $K^\circ$ -submodule of  $V$  is complete in the subspace topology. We denote by  $\mathcal{QLC}_K$  (resp.  $\mathcal{QSLC}_K$ ) the full subcategory of  $\mathcal{LC}_K$  (resp.  $\mathcal{SLLC}_K$ ) whose objects are the quasi-complete objects of  $\mathcal{LC}_K$  (resp.  $\mathcal{SLLC}_K$ ).*

**Remark 3.0.7.** The category  $\mathcal{QSLC}_K$  is stable by limits taken in  $\mathcal{LC}_K$ , so it is complete. It also stable by direct sums taken in  $\mathcal{LC}_K$ , but not for cokernels. So,  $\mathcal{QSLC}_K$  is not quasi-abelian. Notice however that the amazing fact mentioned in Remark 3.0.5 also happens in this case (cf. [4, Prop. 7.12]).

### 3.1 The strict inductive limit

Suppose we have a countable inductive system in  $\mathcal{LC}_K$

$$V_1 \subset V_2 \subset \dots$$

where for any  $n$ ,  $V_n$  is a subspace of  $V_{n+1}$ , and equip  $V = \bigcup_{n \in \mathbb{N}} V_n$  with the locally convex final topology of the inclusions  $V_n \subset V$ . This  $V$  will be called the *strict inductive limit* of the inductive system  $\{V_n\}_{n \in \mathbb{N}}$ .

**Proposition 3.1.1.**

1.  $V_n$  is a subspace of  $V$ ;
2. if all  $V_n$  are Hausdorff, then  $V$  is Hausdorff;
3. if  $V_n$  is closed in  $V_{n+1}$  for any  $n$ , then  $V_n$  is closed in  $V$ ;
4. if  $V_n$  is closed in  $V_{n+1}$  for any  $n$ , then a bounded subset  $B$  of  $V$  is contained in some  $V_m$  and is bounded in  $V_m$ .

**Example 3.1.2.** Let  $X$  be a locally compact topological space, and Let  $C_c(X)$  be the  $K$ -vector space of continuous  $K$ -valued functions  $\phi : X \rightarrow K$  with compact support. For any compact subset  $A \subset X$ , the vector space

$$C_A(X) := \{\phi \in C_c(X) \mid \phi|_{X \setminus A} = 0\}$$

equipped with the supnorm  $\| \cdot \|_\infty$  on  $X$  is a  $K$ -Banach space. Clearly  $C_c(X)$  is the union of all its subspaces  $C_A(X)$ , for  $A \subset X$  compact. We endow  $C_c(X)$  with the locally convex final topology of all inclusions  $C_A(X) \subset C_c(X)$ . For any  $x \in X$  there is a continuous linear form  $\delta_x : \phi \mapsto \phi(x)$ , the *Dirac measure at  $x$* . Since  $C_A(X) = \bigcap_{x \notin A} \ker(\delta_x)$ , every  $C_A(X)$  is closed in  $C_c(X)$ . Since  $\{0\} = \bigcap_{x \in X} \ker(\delta_x)$ ,  $C_c(X)$  is Hausdorff.

Let us assume that  $X$  is moreover  $\sigma$ -compact. So, there is an increasing sequence  $A_1 \subset A_2 \subset \dots$  of compact subsets of  $X$  such that, for any  $n$ ,  $A_n$  is contained in the interior of  $A_{n+1}$ . Since the inclusions  $C_{A_n}(X) \subset C_{A_{n+1}}(X)$  are isometries,  $C_c(X)$  is the strict inductive limit of the sequence  $C_{A_n}(X)$ .

The locally convex space  $C_c(X)$  is the starting point for measure theory. Continuous linear forms on  $C_c(X)$  are called *Radon measures* on  $X$ .

## 4 Boundedness and topologies on $\mathcal{L}(V, W)$

For two objects  $V$  and  $W$  of  $\mathcal{LC}_K$ , we set

$$\mathcal{L}(V, W) = \{f \in \text{Hom}_K(V, W) \mid f \text{ is continuous}\}$$

**Definition 4.0.1.** A subset  $H$  of  $\mathcal{L}(V, W)$  is equicontinuous if, for any open lattice  $M$  in  $W$ , there exists an open lattice  $L$  in  $V$  such that  $h(L) \subset M$ , for any  $h \in H$ .

Let  $V$  and  $W$  be locally convex spaces and let  $B \subset V$  be a bounded subset. For any open lattice  $M \subset W$  the subset

$$\mathcal{L}(B, M) := \{f \in \mathcal{L}(V, W) \mid f(B) \subset M\}$$

is a lattice in  $\mathcal{L}(V, W)$ .

**Definition 4.0.2.** Let  $\mathcal{B}$  be a family of bounded subsets of  $V$  closed under finite unions. Then the family of lattices  $\{\mathcal{L}(B, M)\}_{B \in \mathcal{B}}$  defines a l.c. topology called the  $\mathcal{B}$ -topology on  $\mathcal{L}(V, W)$ . We denote by  $\mathcal{L}_{\mathcal{B}}(V, W)$  the corresponding object of  $\mathcal{LC}_K$ .

If  $W$  is Hausdorff and  $\bigcup_{B \in \mathcal{B}} B$  generates a dense vector subspace of  $V$ , then  $\mathcal{L}_{\mathcal{B}}(V, W)$  is Hausdorff.

**Example 4.0.3.**

1. Let  $\mathcal{B}$  be the family of finite subsets of  $V$ . We write  $\mathcal{L}_s(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$  and the corresponding topology is the *weak topology* or topology of simple convergence of  $\mathcal{L}(V, W)$ .
2. Let  $\mathcal{B}$  be the family of all bounded subsets of  $V$ . Then the  $\mathcal{B}$ -topology is the *strong topology* or the topology of bounded convergence of  $\mathcal{L}(V, W)$ . We write  $\mathcal{L}_b(V, W) := \mathcal{L}_{\mathcal{B}}(V, W)$ .

**Proposition 4.0.4.** For any l.c.  $K$ -vector spaces  $V$  and  $W$ , the linear map

$$\mathcal{L}(V, W) \longrightarrow \mathcal{L}(W'_b, V'_b) \quad , \quad f \longmapsto f'$$

where  $f'(\ell) = \ell \circ f$ , is continuous.

**Proposition 4.0.5.** Let  $\{V_h\}_{h \in H}$  be a family in  $\mathcal{LC}_K$ . Then we have natural isomorphisms

$$\left( \bigoplus_{h \in H} V_h \right)'_b \xrightarrow{\sim} \prod_{h \in H} (V_h)'_b$$

$$\begin{aligned} \left(\bigoplus_{h \in H} V_h\right)'_s &\xrightarrow{\sim} \prod_{h \in H} (V_h)'_s \\ \left(\prod_{h \in H} V_h\right)'_b &\xrightarrow{\sim} \bigoplus_{h \in H} (V_h)'_b \end{aligned}$$

**Definition 4.0.6.** Let  $V$  be an object of  $\mathcal{LC}_K$ . A  $K^\circ$ -submodule  $L$  of  $V$  is bornivorous if, for any bounded subset  $B$  of  $V$ , there is  $a \in K^\times$  such that  $aB \subset L$ . We say that  $V$  is bornological if any bornivorous  $K^\circ$ -submodule of  $V$  is open.

**Proposition 4.0.7.** TFAE for a locally convex vector space  $V$ :

1.  $V$  is bornological;
2. a  $K$ -linear map  $f : V \rightarrow W$  into any other locally convex vector space  $W$  is continuous if and only if it is bounded.

**Example 4.0.8.**

1. If  $V$  contains a bounded open lattice then  $V$  is bornological.
2. If the topology on  $V$  is the locally convex final topology of a family of linear maps  $f_h : V_h \rightarrow V$  and all the  $V_h$  are bornological, then  $V$  is bornological.
3. In particular, quotient spaces, locally convex direct sums, and strict inductive limits of bornological vector spaces are bornological.
4. A metrizable vector space is bornological.

**Definition 4.0.9.** A locally convex vector space  $V$  is barrelled if any closed lattice in  $V$  is open.

It follows from Baire's category theory that if  $V$  is completely metrizable (i.e. if  $V$  is a Fréchet space) then  $V$  is barrelled. Moreover, quotient spaces, locally convex direct sums, and strict inductive limits of barrelled vector spaces are barrelled. Direct products of barrelled vector spaces are barrelled.

**Proposition 4.0.10.** (Banach-Steinhaus) If  $V$  is barrelled then any bounded subset  $H \subset \mathcal{L}_s(V, W)$  is equicontinuous

**Proposition 4.0.11.** If  $V$  is barrelled, the linear map

$$V \longrightarrow (V'_b)'_b, \quad v \longmapsto \delta_v$$

where  $\delta_v(\ell) = \ell(v)$ , for any  $v \in V$ , is continuous.

If both  $V$  and  $W$  are normed, then  $\mathcal{L}_b(V, W)$  is also normed via the operator norm

$$\|f\| = \sup\left\{\frac{\|f(v)\|}{\|v\|} \mid v \neq 0\right\},$$

where we have denoted by  $\|\cdot\|$  the norm in every space. If both  $V$  and  $W$  are Banach, so is  $\mathcal{L}_b(V, W)$ .

**Definition 4.0.12.** For a  $K$ -Banach space  $V$ ,  $V'_b := \mathcal{L}(V, K)$  endowed with the operator norm is the dual  $K$ -Banach space of  $V$ .

**Example 4.0.13.** For any set  $X$ , and any  $x \in X$ , let  $1_x$  denote the  $K$ -valued characteristic function of  $\{x\}$ . The map

$$c_o(X)' \xrightarrow{\sim} \ell^\infty(X), \quad \ell \longmapsto \phi_\ell \text{ where } \phi_\ell(x) = \ell(1_x), \forall x \in X,$$

is an isometry.

## 5 Various notions of compactness

We assume here that the n.a. field  $(K, | \cdot |)$  is spherically complete. We already said in Definition 1.2.5 when a linearly topologized  $K^\circ$ -module  $A$  is c-compact. Now, if  $f : V \rightarrow W$  is a morphism of  $\mathcal{LC}_K$  and  $A$  is a c-compact  $K^\circ$ -submodule of  $V$ , then  $f(A)$  is c-compact, as well. Moreover,  $A$  is closed in  $V$ . If  $B$  is closed in  $A$ , then  $B$  is c-compact, as well. Let  $\{A_h\}_{h \in H}$  be a family of c-compact  $K^\circ$ -modules. Then  $\prod_{h \in H} A_h$  is c-compact.

compactoid

**Definition 5.0.1.** Let  $V$  be an object of  $\mathcal{LC}_K$ . Then a subset  $A \subset V$  is compactoid if for any open lattice  $L \subset V$ , there exist  $v_1, \dots, v_m \in V$  such that

$$A \subset L + K^\circ v_1 + \dots + K^\circ v_m .$$

A compactoid subset  $A \subset V$  is bounded in  $V$ . For any morphism  $f : V \rightarrow W$  in  $\mathcal{LC}_K$ , if  $A$  is compactoid, so is  $f(A)$ . Moreover, if  $A$  is compactoid in  $V$ , so is  $\overline{A}$ .

We mention the following definition only for a better understanding of the previous notions.

**Definition 5.0.2.** A  $K^\circ$ -module is linearly compact if it is c-compact in its discrete topology.

**Proposition 5.0.3.** Let  $V$  be an object of  $\mathcal{LC}_K$ . TFAE for a  $K^\circ$ -submodule  $A \subset V$

1.  $A$  is c-compact and bounded;
2.  $A$  is compactoid and complete.

**Remark 5.0.4.** If  $K$  is locally compact and  $V$  is a l.c.  $K$ -vector space, then a  $K^\circ$ -submodule  $A \subset V$  is c-compact and bounded iff it is compact.

**Example 5.0.5.** Let  $a \in K$  be such that  $0 < |a| < 1$ , and let

$$A = \{ \phi \in c_o(\mathbb{N}) \mid |\phi(n)| \leq |a|^n \forall n \in \mathbb{N} \} .$$

We want to show that  $A$  is closed and c-compact in  $c_o(\mathbb{N})$ .

Clearly  $A$  is bounded in the  $K$ -Banach space  $(c_o(\mathbb{N}), \| \cdot \|_\infty)$ . The map

$$\left( \prod_{n \in \mathbb{N}} K^\circ, \text{product topology} \right) \xrightarrow{\sim} (A, \| \cdot \|_\infty)$$

sending  $(a_n)_n \mapsto [n \mapsto a^n a_n]$ , is an isomorphism of topological  $K^\circ$ -modules. Then,  $K^\circ$  is c-compact because  $K$  was assumed to be spherically complete. So,  $\prod_{n \in \mathbb{N}} K^\circ$  is also c-compact, and therefore so is  $A$ . But  $A$  is also bounded in  $c_o(\mathbb{N})$ . So  $A$  is (compactoid and) complete in the subspace topology, hence closed, in  $c_o(\mathbb{N})$ .

## 6 Spaces of compact type

**Definition 6.0.1.** Let  $f : V \rightarrow W$  be a morphism in  $\mathcal{LC}_K$ . We say that  $f$  is compact if there is an open lattice  $L \subset V$  such that  $f(L) \subset W$  is bounded and c-compact.

**Lemma 6.0.2.** ([4, Lemma 16.4].) Let  $f : V \rightarrow W$  be a morphism in  $\mathcal{Ban}_K$ . Then  $f$  is compact iff so is the dual morphism  $f' : W'_b \rightarrow V'_b$ .

**Proposition 6.0.3.** ([4, Cor. 16.6].) For any projective system in  $\mathcal{SLC}_K$

$$\dots V_{n+1} \rightarrow V_n \rightarrow \dots \rightarrow V_1$$

with compact transition maps, the limit  $\varprojlim_n V_n$  is a reflexive Fréchet space.

**Definition 6.0.4.** A l.c.  $K$ -vector space  $V$  is of compact type if it is the colimit in  $\mathcal{LC}_K$  of a sequence

$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_n \xrightarrow{i_n} V_{n+1} \longrightarrow \cdots$$

of  $K$ -Banach spaces and injective compact maps.

**Example 6.0.5.** Let  $L$  be a locally compact n.a. field and let  $K/L$  be a spherically complete extension. Let  $a \in L$ ,  $r \in |L^\times|$ , and let  $B = B_r(a) \subset L^n$  be a closed ball. We fix a decreasing sequence  $\{r_m\}_{m \in \mathbb{N}}$  in  $|L^\times| \cap (0, r]$ , with  $r_m \rightarrow 0$ . A function  $f : B \rightarrow K$  is *locally  $L$ -analytic* if  $\forall b \in B$ ,  $\exists m \in \mathbb{N}$  such that

$$f|_{B_{r_m}(b)} \in \mathcal{A}_K(B_{r_m}(b)) = \left\{ \sum_{i=0}^{\infty} a_i (x-b)^i \mid a_i \in K, \lim_{i \rightarrow \infty} |a_i| r_m^i = 0 \right\}.$$

We set

$$C^{\text{an}}(B, K) := \{K\text{-valued locally analytic functions on } B\}.$$

For any  $m \in \mathbb{N}$ , we set

$$V_m := \{f \in C^{\text{an}}(B, K) \mid f|_{B_{r_m}(b)} \in \mathcal{A}_K(B_{r_m}(b)) \forall b \in B\}.$$

Then

$$V_1 \subset V_2 \subset \cdots \subset V = C^{\text{an}}(B, K) = \bigcup_{m \in \mathbb{N}} V_m.$$

By compactness of  $B$ , for any  $m \in \mathbb{N}$ ,  $\exists b_1, \dots, b_{n_m} \in B$  such that

$$V_m \xrightarrow{\sim} \bigoplus_{i=1}^{n_m} \mathcal{A}_K(B_{r_m}(b_i)), \quad f \longmapsto \sum_{i=1}^{n_m} f|_{B_{r_m}(b_i)}.$$

So,  $V_m$  with the norm

$$\|f\|_m := \max_{i=1, \dots, n_m} \|f|_{B_{r_m}(b_i)}\|$$

is a  $K$ -Banach space. Obviously the inclusions  $V_m \hookrightarrow V_{m+1}$  are continuous. We equip  $C^{\text{an}}(B, K)$  with the  $\mathcal{LC}_K$ -colimit topology. It is an l.c.  $K$ -vector space of compact type since  $V_m \hookrightarrow V_{m+1}$  is compact for any  $m \in \mathbb{N}$ . This follows from the fact, already seen, that the  $\mathcal{LC}_K$ -morphism

$$\mathcal{A}_K(B_{r_m}(b)) \cong c_o(\mathbb{N}) \longrightarrow c_o(\mathbb{N}) \cong \mathcal{A}_K(B_{r_{m+1}}(b))$$

where  $c_o(\mathbb{N}) \longrightarrow c_o(\mathbb{N})$  takes  $\phi \mapsto [n \mapsto (r_{m+1}/r_m)^n \phi(n)]$ , is compact. In fact the image of the open lattice  $\{\phi \in c_o(\mathbb{N}) \mid \|\phi\|_\infty \leq 1\}$  is dense in

$$A = \{\phi \in c_o(\mathbb{N}) \mid |\phi(n)| \leq (r_{m+1}/r_m)^n \forall n \in \mathbb{N}\}$$

which is bounded and c-compact in  $c_o(\mathbb{N})$ . We conclude that  $C^{\text{an}}(B, K)$  is a l.c.  $K$ -vector space of compact type.

**Example 6.0.6.** We point out that the set-theoretic image of an injective morphism of Banach spaces is not necessarily closed. Consider for simplicity the case of  $L = K = \mathbb{Q}_p$  and the compact inclusion

$$\mathcal{A}_{\mathbb{Q}_p}(B_1(0)) = \mathbb{Q}_p\{T\} \hookrightarrow \mathbb{Q}_p\{T/p\} = \mathcal{A}_{\mathbb{Q}_p}(B_{1/p}(0)).$$

Then  $T^n \rightarrow 0$  in  $\mathbb{Q}_p\{T/p\}$ , since  $T^n = p^n(T/p)^n$ , but  $T$  is not topologically nilpotent in  $\mathbb{Q}_p\{T\}$ . In fact,  $\mathbb{Q}_p\{T\}$  is dense in  $\mathbb{Q}_p\{T/p\}$ .

## 7 Duality

Important theorems on duality are the following.

**Theorem 7.0.1. Closed Graph.** *Let  $K$  be any n.a. field. Let  $f : V \rightarrow W$  be a  $K$ -linear map from a barrelled space  $V$  to a Fréchet space  $W$ . Then  $f$  is continuous iff  $\Gamma(f)$  is closed.*

**Theorem 7.0.2. Open Mapping.** *Let  $K$  be any n.a. field. Let  $f : V \rightarrow W$  be a continuous  $K$ -linear map from a Fréchet space  $V$  to a Hausdorff barrelled space  $W$ . Then if  $f$  is surjective then it is open.*

**Theorem 7.0.3. Hahn-Banach.** *Let  $K$  be a n.a. spherically complete field. Let  $V$  be a  $K$ -vector space,  $V_0$  be a vector subspace,  $L$  be a lattice in  $V$ , and  $L_0 := L \cap V_0$ . Let  $\ell_0 : V_0 \rightarrow K$  be a  $K$ -linear form such that  $\ell_0(L_0) \subset K^\circ$ . Then there exists a  $K$ -linear form  $\ell : V \rightarrow K$  such that  $\ell|_{V_0} = \ell_0$  and  $\ell(L) \subset K^\circ$ .*

**Definition 7.0.4.** *A Hausdorff l.c. space is reflexive if the duality map*

$$\delta : V \longrightarrow (V'_b)'_b$$

*is a topological isomorphism.*

**Proposition 7.0.5.** *A Hausdorff l.c. space  $V$  is reflexive iff it is barrelled and any closed and bounded  $K^\circ$ -submodule of  $V$  is  $c$ -compact.*

Let  $V$  be a l.c.  $K$ -vector space. For any open lattice  $L$  in  $V$ , we let  $V_L$  be the vector space  $V$  equipped with the l.c. topology with fundamental system of open lattices  $\{aL\}_{a \in K^\times}$ . We also denote by  $\widehat{V}_L$  the separated completion of  $V_L$ , which is a  $K$ -Banach space. If  $M \subset L$  are both open lattices in  $V$ , then the identity of  $V$  induces a  $\mathcal{LC}_K$ -morphism  $V_M \rightarrow V_L$ , which therefore extends to the completions  $\widehat{V}_M \rightarrow \widehat{V}_L$ .

**Definition 7.0.6.** *A l.c. space  $V$  is nuclear if for any open lattice  $L \subset V$  there exists another open lattice  $M \subset L$  such that the canonical map  $\widehat{V}_M \rightarrow \widehat{V}_L$  is compact.*

The main result is here

**Theorem 7.0.7.** *The functor  $V \mapsto V'_b$  induces an antiequivalence between the category of l.c.  $K$ -vector spaces of compact type and the category of nuclear Fréchet spaces.*

## 8 Distributions on locally analytic groups

Here  $L$  is a locally compact n.a. field and  $K/L$  is a spherically complete extension. There is a rather obvious notion of *locally  $L$ -analytic manifold* which we do not repeat. We mainly think of a honest (rigid [2] or Berkovich [1] or Huber [3])  $L$ -analytic space  $X$ , and to an open (locally compact) subspace  $M$  of the space of its  $L$ -points  $X(L)$ . Similarly, a *locally  $L$ -analytic group* will be an open subgroup of the group  $G(L)$  of  $L$ -valued points of an  $L$ -analytic linear group  $G$ .

For any topological space  $X$  and any complete l.c.  $K$ -vector space  $V$  we define the  $K$ -vector space

$$C(X, V) := \{ \text{continuous functions } X \longrightarrow V \}$$

equipped with the topology of uniform convergence on compact subsets of  $X$ . This is a complete l.c.  $K$ -vector space. If  $X$  is compact  $C(X, K)$  is a Banach space and

$$C(X, V) = C(X, K) \widehat{\otimes}_{K, \pi} V.$$

For a locally  $L$ -analytic manifold  $M$  and any  $K$ -vector space of compact type  $V$ , we have  $C^{\text{an}}(M, V)$ , defined as we did before in the case of  $M =$  the closed unit ball in  $L^n$ . We denote by  $C^\infty(M, V)$  the subspace of  $C^{\text{an}}(M, V)$  of locally constant functions. Its elements are called *smooth functions*.

If  $M$  is compact and  $V$  is of compact type, then  $C^{\text{an}}(M, V)$  is also of compact type, and

$$C^{\text{an}}(M, V) = C^{\text{an}}(M, K) \widehat{\otimes}_{K, \pi} V .$$

Moreover, if  $M$  and  $N$  are compact

$$C^{\text{an}}(M \times N, K) = C^{\text{an}}(M, K) \widehat{\otimes}_{K, \pi} C^{\text{an}}(N, K) .$$

If  $M$  is not compact, we still have that  $M$  is a disjoint union of compact balls  $M = \bigcup_{i \in I} M_i$ . Then

$$C^{\text{an}}(M, V) = \prod_{i \in I} C^{\text{an}}(M_i, V) .$$

This implies

**Corollary 8.0.1.** *If  $V$  is of compact type, then  $C^{\text{an}}(M, V)$  is complete, barrelled and reflexive.*

**Definition 8.0.2.** *If  $M$  is a locally  $L$ -analytic manifold and  $V$  is a l.c. complete  $K$ -vector space, we set*

$$D(M, V) := C^{\text{an}}(M, V)'_b .$$

*Its elements are called locally analytic  $V$ -valued distributions on  $M$ .*

When  $V$  is of compact type,  $C^{\text{an}}(M, V)$  is reflexive hence

$$C^{\text{an}}(M, V) = D(M, V)'_b .$$

The duality is usually denoted as

$$C^{\text{an}}(M, V) \times D(M, V) \longrightarrow K \quad , \quad (f, \lambda) \longmapsto \int_M f(x) d\lambda(x) .$$

We now assume, for simplicity, that our locally  $L$ -analytic spaces and groups are compact. The  $K$ -algebra operations of  $C^{\text{an}}(M, K)$ , namely

$$\mu : C^{\text{an}}(M, K) \widehat{\otimes}_{K, \pi} C^{\text{an}}(M, K) \longrightarrow C^{\text{an}}(M, K) \quad , \quad (f, g) \longmapsto fg$$

$$\iota : K \longrightarrow C^{\text{an}}(M, K) \quad , \quad c \longmapsto \text{constant function } c \text{ on } M$$

transpose to induce a  $K$ -coalgebra structure on  $D(M, K)$ . We observe that for  $M$  is compact,  $D(M, K)$  is a Fréchet space and

$$D(M \times M, K) \cong D(M, K) \widehat{\otimes}_{K, \pi} D(M, K) = D(M, K) \widehat{\otimes}_{K, \iota} D(M, K) .$$

The coproduct of  $D(M, K)$  is then

$$\mathbb{P} : D(M, K) \longrightarrow D(M \times M, K) \cong D(M, K) \widehat{\otimes}_{K, \pi} D(M, K)$$

where

$$\int_{M \times M} (f \otimes g) d(\mathbb{P}\lambda) = \int_M f(x)g(x) d\lambda(x)$$

and counit

$$\varepsilon : D(M, K) \longrightarrow K \quad , \quad \lambda \longmapsto \int_M d\lambda(x) .$$

Important  $K$ -valued distributions on  $M$  are the *Dirac measures*  $\delta_x$ , for  $x \in M$ . They satisfy

$$\mathbb{P}(\delta_x) = \delta_x \otimes \delta_x \quad , \quad \varepsilon(\delta_x) = 1 \quad .$$

When  $M = G$  a locally  $L$ -analytic group (compact, for simplicity),  $C^{\text{an}}(G, K)$  is a  $K$ -Hopf algebra with coproduct

$$\mathbb{P} : C^{\text{an}}(G, K) \longrightarrow C^{\text{an}}(G \times G, K) \cong C^{\text{an}}(G, K) \widehat{\otimes}_{K, \pi} C^{\text{an}}(G, K) \quad ,$$

where  $(\mathbb{P}f)(x, y) = f(xy)$ , for all  $x, y \in G$ . The counit of  $C^{\text{an}}(G, K)$  is

$$\varepsilon : C^{\text{an}}(G, K) \longrightarrow K \quad , \quad f \longmapsto f(1_G) \quad ,$$

and the antipodism

$$\rho : C^{\text{an}}(G, K) \longrightarrow C^{\text{an}}(G, K)$$

is given by  $\rho(f)(x) = f(x^{-1})$ , for any  $x \in G$ . Then,  $D(G, K)$  is a  $K$ -algebra in  $(\mathcal{C}\mathcal{L}\mathcal{C}_K, \widehat{\otimes}_{K, \pi})$ . We have, for distributions  $\lambda, \nu$  on  $G$ , and any  $f \in C^{\text{an}}(G, K)$

$$\int_G f(z) d(\lambda * \nu)(z) = \int_{G \times G} f(x \cdot y) d\lambda(x) d\nu(y)$$

so that  $\delta_g * \delta_h = \delta_{gh}$ , for  $g, h \in G$  and the unit is  $\delta_{1_G}$ .

**Proposition 8.0.3.** *For any locally  $L$ -analytic locally compact group  $G$ , the l.c. complete  $K$ -algebra  $D(G, K)$  has a separately continuous multiplication with  $\delta_1$  as identity. When  $G$  is compact  $D(G, K)$  is a  $K$ -Fréchet algebra.*

**Definition 8.0.4.** *A locally analytic representation of  $G$  is a l.c. barrelled  $K$ -vector space  $V$  such that, for each  $v \in V$ , the orbit map  $g \mapsto gv$  belongs to  $C^{\text{an}}(G, V)$ . A locally analytic representation of  $G$  is smooth if the orbit maps are locally constant.*

**Proposition 8.0.5.** *Let  $G$  be a compact locally  $L$ -analytic group, and let  $V$  be a locally analytic representation of  $G$ . Then the  $G$ -action extends to a separately continuous  $D(G, K)$ -module structure on  $V$ . Moreover,  $G$ -equivariant continuous linear maps extend to module homomorphisms.*

**Proposition 8.0.6.** *Let  $G$  be compact, as before. The function  $V \longmapsto V'_b$  is an anti-equivalence of categories between the category of locally analytic  $G$ -representations on  $K$ -vector spaces of compact type, with continuous linear  $G$ -maps, and the category of continuous  $D(G, K)$ -modules on nuclear Fréchet spaces, with continuous  $D(G, K)$ -module homomorphisms.*

## 9 The case of $G = \mathbb{Z}_p$

A theorem of Mahler says that the  $K$ -Banach space of continuous functions  $\mathbb{Z}_p \rightarrow K$  is isometric to  $c_o(\mathbb{N})$  via the map

$$c_o(\mathbb{N}) \xrightarrow{\sim} C(\mathbb{Z}_p, K) \quad , \quad (a_n)_n \longmapsto \sum_{n \in \mathbb{N}} a_n \binom{x}{n} \quad .$$

For any  $f \in C(\mathbb{Z}_p, K)$ , we have

$$f(x) = \sum_{n \in \mathbb{N}} (\delta_1 - \delta_0)^n (f) \binom{x}{n}$$

that is

$$\begin{aligned} a_0 &= f(0), \quad a_1 = (\delta_1 - \delta_0)(f) = f(1) - f(0), \\ a_2 &= (\delta_1 - \delta_0)^2(f) = (\delta_2 - 2\delta_1 + \delta_0)(f) = f(2) - 2f(1) + f(0), \dots \end{aligned}$$

So,  $D^c(\mathbb{Z}_p, K)$  is isomorphic to  $\ell^\infty(\mathbb{N})$  as a  $K$ -Banach space. As Banach  $K$ -coalgebra  $C(\mathbb{Z}_p, K)$  is given by

$$\mathbb{P}\left(\binom{x}{n}\right) = \sum_{i+j=n} \binom{x}{i} \otimes \binom{x}{j}$$

$\varepsilon\left(\binom{x}{0}\right) = \varepsilon(1) = 1$ ,  $\varepsilon\left(\binom{x}{n}\right) = 0 \quad \forall n > 0$ . Therefore the algebra structure of  $D^c(\mathbb{Z}_p, K)$  is simply the power-series algebra  $K \otimes_{K^\circ} K^\circ[[\delta_1 - \delta_0]]$ . Notice in fact that

$$(\delta_1 - \delta_0)^n \left(\binom{x}{m}\right) = \delta_{m,n}.$$

This shows that  $D^c(\mathbb{Z}_p, K)$  is the  $K$ -Banach algebra of bounded  $K$ -analytic functions on  $B_1^-(0)$  as described before.

An improvement of the Mahler theorem, due to Yvette Amice, says that a function

$$f(x) = \sum_{n \in \mathbb{N}} a_n \binom{x}{n} \in C(\mathbb{Z}_p, K)$$

is locally analytic of radius  $r \in K^\times$ ,  $0 < r < 1$ , iff  $|a_n|r^{-n} \rightarrow 0$ . So the dual space  $D(\mathbb{Z}_p, K)$  consists of all power series  $\sum_{n \in \mathbb{N}} b_n (\delta_1 - \delta_0)^n$  such that, for any  $0 < r < 1$ ,  $|b_n|r^n \rightarrow 0$ . This can obviously be identified with the ring  $\mathcal{A}_K(B_1^-(0))$  of holomorphic functions on the open unit disc. The Fréchet topology of  $D(\mathbb{Z}_p, K)$ , that is on  $\mathcal{A}_K(B_1^-(0))$ , which was described before, makes  $D(\mathbb{Z}_p, K) \cong \mathcal{A}_K(B_1^-(0))$  a  $K$ -Fréchet algebra. We have seen in 5 of Examples 2.0.8 that  $\mathcal{A}_K(B_1^-(0))$  is the limit of the countable projective system (2.0.8.1) of Noetherian  $K$ -Banach commutative algebras via flat transition morphisms. In general, the algebra  $D(G, K)$  will have similar properties, but the group  $G$  will not be commutative. This state of facts pressed Schneider and Teitelbaum to introduce a new class of non-commutative  $K$ -Fréchet algebras, the Fréchet-Stein algebras, and to view them as an analog in non-commutative geometry of rings of analytic functions on classical open polydiscs.

## 10 Fréchet-Stein algebras

**Definition 10.0.1.** A  $K$ -Fréchet-Stein algebra  $A$  is a  $K$ -Fréchet algebra such that, for a decreasing fundamental system

$$L_1 \supset L_2 \supset \dots$$

of open lattices such that, for any  $n \in \mathbb{N}$ ,

1. the multiplication of  $A$  extends by continuity to a multiplication on  $A_n := A_{L_n}$  which makes it a left noetherian  $K$ -Banach algebra;
2. the identity map of  $A$ , extends to a  $K$ -Banach algebra morphism  $A_{n+1} \rightarrow A_n$  that makes  $A_n$  a flat right  $A_{n+1}$ -module;
- 3.

$$(A, (A \rightarrow A_n)_{n \in \mathbb{N}}) \xrightarrow{\sim} \varprojlim_n A_n.$$

**Definition 10.0.2.** A coherent sheaf for the Fréchet-Stein algebra  $(A, (L_n)_n)$  is a family  $(M_n)_{n \in \mathbb{N}}$  of finitely generated left  $A_n$ -modules  $M_n$  together with  $A_n$ -module isomorphism

$$A_n \otimes_{A_{n+1}} M_{n+1} \xrightarrow{\sim} M_n ,$$

for any  $n$ . We let  $\text{Coh}((A, (L_n)_n))$  be the category of coherent sheaves for  $(A, (L_n)_n)$

**Definition 10.0.3.** Let  $(A, (L_n)_n)$  be a Fréchet-Stein algebra. A left  $A$ -module  $M$  is called coadmissible if it is of the form

$$M \cong \Gamma((M_n)_n) := \varprojlim_n M_n .$$

We let  $\mathcal{C}_A$  denote the category of coadmissible  $A$ -modules.

**Theorem 10.0.4.** Let  $(A, (L_n)_n)$  be a Fréchet-Stein algebra. The category  $\mathcal{C}_A$  is abelian, and the functor

$$\Gamma : \text{Coh}((A, (L_n)_n)) \longrightarrow \mathcal{C}_A , \quad (M_n)_n \longmapsto \Gamma((M_n)_n)$$

is an equivalence of categories.

**Theorem 10.0.5.** Let  $G$  be a compact locally  $L$ -analytic group, and suppose that  $K$  is discretely valued. Then  $D(G, K)$  is a Fréchet-Stein algebra.

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