

The Beilinson–Bernstein–Brylinski–Kashiwara localization theorem

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Outline

- 1 Statement of the theorem
- 2 The \mathfrak{sl}_2 case
- 3 Equivariant \mathcal{D} -modules

Recall from last time:

- G semisimple, $\mathfrak{g} = \text{Lie}G$.
- $X = G/B$.
- $U(\mathfrak{g})_\lambda = U(\mathfrak{g}) \otimes_{\mathbb{Z}} U(\mathfrak{h}) \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$.
- Tdo $\mathcal{D}_X^\lambda = \widetilde{\mathcal{D}} \otimes_{U(\mathfrak{h})} \mathbb{C}_\lambda$.
- $\mathcal{D}_X^\lambda(X) \cong U(\mathfrak{g})_\lambda$, higher cohomology vanishes.
- If λ is dominant integral, then $\mathcal{D}^\lambda = \mathcal{D}^{\mathcal{L}\lambda}$.

A weight $\lambda \in \mathfrak{h}^*$ is called

- **dominant** if $\lambda(h_i) \notin \{-1, -2, \dots\}$ for all simple coroots h_i .
- **regular** if $\lambda(h_i) \neq 0$ for all simple coroots h_i .

Useful fact: if λ is regular dominant and ν is integral dominant, then $\lambda + \nu$ is the only representative of its W -orbit contained in

$$\{\lambda + \mu : \mu \text{ a weight of } L_\nu\}.$$

If $\lambda + \rho$ is regular dominant, $\lambda + \nu$ is the only representative of its W -orbit under the shifted action!

Theorem (BB, BK, '81)

Let $\lambda + \rho$ be regular dominant, then $\Gamma(X, -)$ is exact and yields equivalences

$$\begin{aligned} \mathcal{D}_X^\lambda\text{-Mod} &\cong U(\mathfrak{g})_\lambda\text{-Mod} \\ \mathcal{D}_X^\lambda\text{-Mod}^{\text{coh}} &\cong U(\mathfrak{g})_\lambda\text{-Mod}^{\text{f.g.}} \end{aligned}$$

We say that X is ' \mathcal{D}_X^λ -affine'.

The quasi-inverse is the localisation functor

$$\text{Loc} : M \mapsto \mathcal{D}^\lambda \otimes_{\mathcal{D}^\lambda(X)} M.$$

It suffices to prove the following statements for $M \in \mathcal{D}^\lambda\text{-Mod}$:

- if $\lambda + \rho$ is dominant, then M has vanishing higher cohomology.
- if $\lambda + \rho$ is regular dominant, then M is generated by global sections.

The result then follows formally, as this makes \mathcal{D}^λ a projective generator, and $\Gamma(X, M) = \text{Hom}(\mathcal{D}^\lambda, M)$.

The two statements above should remind us of Serre vanishing!

Let ν be a dominant integral weight, and let \mathcal{L}_ν be the corresponding line bundle, with $\mathcal{L}_\nu(X) = L_\nu$ and the locally split surjection

$$\mathcal{O}_X \otimes_{\mathbb{C}} L_\nu \rightarrow \mathcal{L}_\nu.$$

This (and its dual) yields natural maps

$$M \otimes_{\mathbb{C}} L_\nu \rightarrow M \otimes_{\mathcal{O}} \mathcal{L}_\nu \text{ (surjective)}$$

and

$$i : M \rightarrow M \otimes_{\mathcal{O}} \mathcal{L}_\nu \otimes_{\mathbb{C}} L^{-}(-\nu) \text{ (injective)}.$$

All terms carry natural locally finite \mathfrak{sl}_2 -actions, and the decompositions into generalized eigenspaces show that the maps above split!

This allows to finish the proof using Serre vanishing.

E.g. if N is a coherent \mathcal{O} -submodule of M , then the injectivity of $H^j(i)$ in the diagram

$$\begin{array}{ccc}
 H^j(X, N) & \longrightarrow & H^j(X, M) \\
 \downarrow & & \downarrow \\
 H^j(X, N \otimes_{\mathcal{O}} \mathcal{L}_\nu \otimes_{\mathbb{C}} L^{-(-\nu)}) & \longrightarrow & H^j(X, M \otimes_{\mathcal{O}} \mathcal{L}_\nu \otimes_{\mathbb{C}} L^{-(-\nu)})
 \end{array}$$

yields that $H^j(X, N) \rightarrow H^j(X, M)$ is the zero map for all N , so $H^j(X, M) = 0$.

We now concentrate on the \mathfrak{sl}_2 case (and the trivial central character, i.e. $\lambda = 0$). The block \mathcal{O}_0 has exactly two irreducibles,

- $L(0) = \mathbb{C}$, and
- $L(-2) \cong M(-2) = \text{span}\{f^i v : i = 0, 1, \dots\}$ with $hv = -2v$.
- There is a short exact sequence

$$0 \rightarrow L(-2) \rightarrow M(0) \rightarrow L(0) \rightarrow 0.$$

What are the corresponding $\mathcal{D}_{\mathbb{P}^1}$ -modules?

Let $X = \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\}$. Let $\mathbb{A}^1 = \text{Spec}\mathbb{C}[z]$ and write $w = z^{-1}$.
Recall

- $e \mapsto -\partial_z = w^2\partial_w$
- $h \mapsto -2z\partial_z = 2w\partial_w$
- $f \mapsto z^2\partial_z = -\partial_w$.

Now the central element $1/4h(h-2) + ef$ acts as

$$z\partial_z(z\partial_z + 1) - (2z\partial_z^2 - z^2\partial_z^2) = 0,$$

as required, and $\mathcal{D}(X) \cong U(\mathfrak{sl}_2)_0$.

- $\text{Loc}L(0) = \mathcal{O}_X = L(\mathbb{A}^1, \mathcal{O})$ (minimal extension).
- $L(\infty, \mathcal{O}) = i_+ \mathbb{C}$ has global sections $\mathbb{C}[\partial_w] = A_1/A_1 w$ (Kashiwara). An easy calculation shows this is indeed $L(-2)$.
 E.g. $h \cdot 1 = 2w\partial_w = 2\partial_w \cdot w - 2 = -2$.
- What is $j_* \mathcal{O}_{\mathbb{A}^1}$? $(j_* \mathcal{O})(X) = \mathbb{C}[z]$. Note that z^i has weight $-2i$, but this is NOT $M(0)$ (e.g. $f \cdot 1 = 0$).
 Indeed $M(0) \cong \Gamma(X, \mathbb{D}j_* \mathcal{O})$, and the short exact sequence

$$0 \rightarrow i_+ \mathbb{C} \rightarrow \mathbb{D}j_* \mathcal{O} \rightarrow \mathcal{O}_X \rightarrow 0$$

is the dual of the usual ‘localization triangle’.

We now understand Lie algebra representations. What about group representations?

Definition

Let $K \leq G$ be a closed subgroup of G , and let $\mathfrak{k} = \text{Lie}(K)$. A K -equivariant \mathfrak{g} -module is a \mathfrak{g} -representation M with a K -action which is compatible in the sense that

- *differentiating the K -action yields the action of $\mathfrak{k} \leq \mathfrak{g}$.*
- *$k \cdot (x \cdot m) = (\text{Ad}(k)(x))(km)$ for all $k \in K, x \in \mathfrak{g}, m \in M$.*

Definition

Let K be an algebraic group acting on a smooth variety Y with action map $\sigma : K \times Y \rightarrow Y$.

A **(strongly) K -equivariant \mathcal{D}_Y -module** is a pair (M, θ) where M is a \mathcal{D}_Y -module and

$$\theta : p_2^! M \rightarrow \sigma^! M$$

is an isomorphism of $\mathcal{D}_{K \times Y}$ -modules such that $(1 \times \sigma)^*(\theta)p_{23}^*(\theta) = (m \times 1)^*(\theta)$ on $K \times K \times Y$.

Theorem (Equivariant localization)

Let G, X etc. be as before and let $K \leq G$ be a closed subgroup. The global sections functor induces an equivalence of categories

$$K - \mathcal{D}_X\text{-Mod} \cong K - U(\mathfrak{g})_0\text{-Mod}$$

Can also prove the twisted case, but needs some care with the K -action.

Idea of proof:

As K is affine, $K \times X$ is \mathcal{D} -affine. Moreover,

$$\mathcal{D}(K \times X) \cong (\mathcal{O}(K) \otimes U(\mathfrak{k})) \otimes \mathcal{D}(X).$$

Unpacking the definitions, we see that giving a K -equivariant structure for a \mathcal{D}_X -module M is the same as an isomorphism

$$\Gamma(K \times X, p_2^! M) \rightarrow \Gamma(K \times X, \sigma^! M)$$

satisfying a cocycle condition. Both sides are isomorphic to $\mathcal{O}(K) \otimes \mathcal{M}(X)$ as vector spaces. By comparing the actions, we find that giving such an isomorphism is the same as an action map

$$K \times \mathcal{M}(X) \rightarrow \mathcal{M}(X)$$

compatible with the \mathfrak{g} -action.

Theorem

Let K be an algebraic group acting on a smooth variety Y with finitely many orbits. Then any coherent K -equivariant \mathcal{D}_Y -module is holonomic.

Proof sketch: Induction on the number of orbits.

If $Y = K/H$, the map $p : K \rightarrow Y$ factors as $p = p_2 \circ i$,

$$\begin{aligned} K &\rightarrow K \times Y \rightarrow Y \\ k &\mapsto (k^{-1}, kH) \mapsto kH. \end{aligned}$$

Since $p_2^! M \cong \sigma^! M$ and σ factors through a point, coherence of $p^! M$ implies $p_2^! M$ is holonomic, and then so is M by smoothness of p .

Recall that all irreducible holonomics are classified as minimal extensions.

The same thing happens here:

Theorem

Let K, Y be as before. Then every irreducible K -equivariant coherent \mathcal{D}_Y -module is of the form $L(O, M)$, where O is a (locally closed) K -orbit and M is an irreducible K -equivariant integrable connection.

Remark:

The previous theorem actually shows that all K -equivariant coherent \mathcal{D} -modules are regular holonomic. Riemann–Hilbert then implies that M corresponds to an irreducible K -equivariant local system on O^{an} .

Examples:

- The previous \mathfrak{sl}_2 -example discussed all irreducible B -equivariant $U(\mathfrak{sl}_2)_0$ -representations.
- Suppose G is real and K is a maximal compact subgroup, e.g. $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}_2$ is isomorphic to diagonal matrices in $\mathrm{SL}_2(\mathbb{C})$. We can now classify admissible irreducibles in terms of local systems on K -orbits (here: 0 , ∞ and $\mathbb{P}^1 \setminus \{0, \infty\}$).