

$\widehat{\mathcal{D}}$ -modules on rigid analytic spaces

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Outline

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K/\mathbb{Q}_p finite (for simplicity), $R \subset K$ the ring of integers.
Recall from last time:

$$\begin{aligned} \mathcal{O}(\mathbb{A}_K^{d,\text{an}}) &= \varprojlim K\langle \pi^n \xi_1, \dots, \pi^n \xi_d \rangle \\ &= \left\{ \sum_{\alpha \in \mathbb{N}^d} a_\alpha \xi^\alpha : |\pi^{-n|\alpha|} a_\alpha| \rightarrow 0 \forall n \right\} \end{aligned}$$

where $|\pi| < 1$ is non-zero, and $K\langle \pi^n \xi \rangle = \widehat{R[\pi^n \xi]} \otimes_R K$.
Note:

- This is a Fréchet–Stein algebra.
- Coadmissible $\mathcal{O}(\mathbb{A}^{d,\text{an}})$ -modules are the global sections of coherent \mathcal{O} -modules on $\mathbb{A}^{d,\text{an}}$.

There are natural non-commutative generalizations of this:

- Let \mathfrak{g}_K be a finite-dimensional Lie algebra over K , then we set

$$\widehat{U(\mathfrak{g}_K)} := \varprojlim \left(\widehat{U_R(\pi^n \mathfrak{g})} \otimes_R K \right)$$

for $\mathfrak{g} \subset \mathfrak{g}_K$ a Lie lattice over R .

- If $X = \mathrm{Sp}A$ is a smooth affinoid, $L = \mathcal{T}_X(X)$, then we set

$$\widehat{\mathcal{D}}_X(X) := \varprojlim \left(\widehat{U_{\mathcal{A}}(\pi^n \mathcal{L})} \otimes_R K \right)$$

for suitable ' R -versions' $\mathcal{A} \subset A$, $\mathcal{L} \subset L$ (see later for precise definitions!).

Example: The unit disk.

Let $A = K\langle x \rangle$. Then the tangent sheaf $\mathcal{T}_X \cong \mathcal{O}_X \partial$, where $\partial = \frac{d}{dx}$.

Let $\mathcal{A} = R\langle x \rangle$, $\mathcal{L} = \mathcal{A}\partial$. Note $\mathcal{L}(\mathcal{A}) \subset \mathcal{A}$, and \mathcal{L} is closed under the Lie bracket.

Write $D_n = \widehat{U_{\mathcal{A}}(\pi^n \mathcal{L})} \otimes_R K = K\langle x, \pi^n \partial \rangle$ (as always, with $\partial \cdot x - x\partial = 1$).

Then

$$\begin{aligned}\widehat{\mathcal{D}}_X(X) &= \varprojlim D_n = \varprojlim K\langle x, \pi^n \partial \rangle \\ &= \left\{ \sum f_i \partial^i : f_i \in A, |\pi^{-ni} f_i| \rightarrow 0 \forall n \right\},\end{aligned}$$

a Fréchet completion of the Weyl algebra.

In full generality:

Let S be a commutative base ring, A a commutative S -algebra.

Definition

An (S, A) -**Lie–Rinehart algebra** is a finitely generated A -module L , equipped with an S -bilinear Lie bracket and an A -linear map of Lie algebras

$$\rho : L \rightarrow \text{Der}_S(A)$$

such that $[\partial, a\partial'] = a[\partial, \partial'] + \rho(\partial)(a)\partial'$ for all $a \in A, \partial, \partial' \in L$.

Can define the universal enveloping algebra $U_A(L)$ with relations $a \cdot \partial = a\partial, \partial \cdot \partial' - \partial' \cdot \partial = [\partial, \partial'], \partial \cdot a = a\partial + \rho(\partial)(a)$.
 $U_A(L)$ is Noetherian if A is (PBW).

Now let A be an affinoid K -algebra and let L be a (K, A) -Lie–Rinehart algebra which is finitely generated projective as an A -module.

- An R -subalgebra $\mathcal{A} \subset A$ is an **affine formal model** if it is topologically of finite type and spans A .
- A f.g. \mathcal{A} -submodule $\mathcal{L} \subset L$ is an **\mathcal{A} -Lie lattice** if it is closed under the Lie bracket, spans L , and $\rho(\mathcal{L})(\mathcal{A}) \subset \mathcal{A}$.
In particular, \mathcal{L} is an (R, \mathcal{A}) -Lie–Rinehart algebra, and so is $\pi^n \mathcal{L}$ for all $n \geq 0$.

\Rightarrow can define $\widehat{U_{\mathcal{A}}(L)} := \varprojlim (\widehat{U_{\mathcal{A}}(\pi^n \mathcal{L})} \otimes_R K)$.

Theorem

Let X be a smooth affinoid K -variety.

- $\widehat{\mathcal{D}}_X$ defines a sheaf on X with vanishing higher Čech cohomology for all finite affinoid coverings.
- $\widehat{\mathcal{D}}_X(X)$ is a Fréchet–Stein algebra.

We'll prove this for $X = \mathrm{Sp}K\langle x \rangle$. Note that for any affinoid subdomain U ,

$$\widehat{\mathcal{D}}_X(U) \cong \varprojlim (\mathcal{O}(U) \widehat{\otimes}_K K\langle \pi^n \xi \rangle)$$

as left $\mathcal{O}(U)$ -modules, so the first bullet point follows from the corresponding result for \mathcal{O}_X and a Mittag-Leffler result on the exactness of \varprojlim .

We now show that $\widehat{\mathcal{D}}(X)$ is Fréchet–Stein. The only subtle part is flatness of the maps $D_{n+1} \rightarrow D_n$.

Consider $n = 0$, let $U_1 = R\langle x \rangle[\pi\partial]$, $U_0 = R\langle x \rangle[\partial] \subset K\langle x \rangle[\partial]$.

We want to show that $\widehat{U}_1 \otimes K \rightarrow \widehat{U}_0 \otimes K$ is flat. Set $V = \widehat{U}_1 \cdot U_0 \subset \widehat{U}_1 \otimes K$. Then

$$V \otimes K = \widehat{U}_1 \otimes K = D_1, \text{ and } \widehat{V} \otimes K \cong \widehat{U}_0 \otimes K = D_0.$$

Filtering V by $F_i V = \widehat{U}_1 F_i U_0$ shows that V is Noetherian, as $\text{gr} V$ is generated by $\text{gr} \partial$ over \widehat{U}_1 . Hence \widehat{V} is flat over V by Artin–Rees, and $\widehat{V} \otimes K$ is flat over $V \otimes K$.

There is a slight issue when we pass to a smaller affinoid subdomain:

Let $X = \mathrm{Sp}A$ is affinoid, \mathcal{A} an affine formal model, $\mathcal{L} \subset \mathcal{T}(X)$ an \mathcal{A} -Lie lattice.

Let $Y = \mathrm{Sp}B$ be an affinoid subdomain, \mathcal{B} an affine formal model containing the image of \mathcal{A} . Then $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{L}$ need NOT be a \mathcal{B} -Lie lattice in $\mathcal{T}(Y) = B \otimes_{\mathcal{A}} \mathcal{T}(X)$.

E.g. $\mathbb{R}\langle\pi^{-1}x\rangle\partial$ does not preserve $\mathbb{R}\langle\pi^{-1}x\rangle$.

\Rightarrow A ‘Banach version’ only makes sense on a suitable subsite of ‘ \mathcal{L} -admissible’ subdomains.

In taking the limit, this doesn’t matter: $\mathcal{B} \otimes_{\mathcal{A}} \pi^n \mathcal{L}$ works for sufficiently large n .

Theorem

Let $Y = \mathrm{Sp}B$ be an affinoid subdomain of a smooth affinoid $X = \mathrm{Sp}A$. Let \mathcal{A}, \mathcal{L} be as before. Then there exists B, n_0 such that $B \otimes_{\mathcal{A}} \pi^n \mathcal{L}$ is a Lie lattice for all $n \geq n_0$, and

$$\mathcal{D}_n(X) := \widehat{U_{\mathcal{A}}(\pi^n \mathcal{L})} \otimes K \rightarrow \mathcal{D}_n(Y) := \widehat{U_B(B \otimes \pi^n \mathcal{L})} \otimes K$$

is flat on both sides for all $n \geq n_0$.

Proof: Reduce to Y of a special form (' $\pi^n \mathcal{L}$ -accessible' rational subdomains), so that B admits a specific presentation.

Corollary

Let $M = \varprojlim M_n$ be a coadmissible $\widehat{\mathcal{D}}_X(X)$ -module. Then

$$\widehat{\mathcal{D}}_X(Y) \widehat{\otimes}_{\widehat{\mathcal{D}}(X)} M := \varprojlim (\mathcal{D}_n(Y) \otimes_{\mathcal{D}_n(X)} M_n)$$

is a coadmissible $\widehat{\mathcal{D}}(Y)$ -module.

The assignment $Y \mapsto \widehat{\mathcal{D}}(Y) \widehat{\otimes} M$ defines a sheaf $\text{Loc}M$ of $\widehat{\mathcal{D}}$ -modules with vanishing higher Čech cohomology on finite affinoid coverings.

The first part is clear from the definition of coadmissibility. For any finite affinoid covering \mathfrak{U} of X , choose n large enough so that all intersections of elements of \mathfrak{U} are as in the theorem. Let $0 \rightarrow N \rightarrow \mathcal{D}_n(X)^r \rightarrow M_n \rightarrow 0$ be short exact, giving rise to

$$0 \rightarrow C^i(\mathfrak{U}, \mathcal{D}_n \otimes_{\mathcal{D}_n(X)} N) \rightarrow C^i(\mathfrak{U}, \mathcal{D}_n)^r \rightarrow C^i(\mathfrak{U}, \mathcal{D}_n \otimes M_n) \rightarrow 0,$$

by flatness.

Hence we have a long exact sequence of Čech cohomology, and the result follows by induction on i and the acyclicity of \mathcal{D}_n (together with Mittag-Leffler).

Corollary

The functor $\text{Loc} : \mathcal{C}_{\widehat{\mathcal{D}}(X)} \rightarrow \widehat{\mathcal{D}}\text{-mod}$ is exact and fully faithful.

Definition

Let X be a smooth rigid analytic K -variety. A $\widehat{\mathcal{D}}_X$ -module \mathcal{M} is called **coadmissible** if there exists an admissible covering (X_i) by affinoids such that $\mathcal{M}|_{X_i}$ is a localization of a coadmissible $\widehat{\mathcal{D}}(X_i)$ -module for each i .

We denote the category of coadmissible $\widehat{\mathcal{D}}_X$ -modules by \mathcal{C}_X .

Theorem

If X is a smooth affinoid, then $\text{Loc} : \mathcal{C}_{\widehat{\mathcal{D}}(X)} \rightarrow \mathcal{C}_X$ is an equivalence of categories, with quasi-inverse $\Gamma(X, -)$.

Proposition

Let X be a smooth rigid analytic K -variety. Then \mathcal{C}_X is an abelian category.

Example 1: Integrable connections.

Let \mathcal{M} be a coherent \mathcal{O}_X -module, endowed with an integrable connection $\mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}$. If $Y \subset X$ is affinoid, then the action of the tangent sheaf on $\mathcal{M}(Y)$ extends to a $\mathcal{D}_n(Y)$ -module structure for all sufficiently large n , and

$$\mathcal{D}_n(Y) \otimes_{\mathcal{D}_{n+1}(Y)} \mathcal{M}(Y) \cong \mathcal{M}(Y).$$

Moreover,

$\mathcal{M}(Y') \cong \mathcal{O}(Y') \otimes_{\mathcal{O}(Y)} \mathcal{M}(Y) \cong \mathcal{D}_n(Y') \otimes_{\mathcal{D}_n(Y)} \mathcal{M}(Y)$, so \mathcal{M} is a coadmissible $\widehat{\mathcal{D}}_X$ -module.

Example 2: Completing algebraic \mathcal{D} -modules.

Let X be a smooth rigid analytic K -variety. Write \mathcal{D}_X for the sheaf of finite-order differential operators, i.e.

$\mathcal{D}_X(Y) = U_{\mathcal{O}(Y)}(\mathcal{T}(Y))$ (Noetherian).

If M is a finitely generated $\mathcal{D}(Y)$ -module, then $\widehat{\mathcal{D}}(Y) \otimes_{\mathcal{D}(Y)} M$ is finitely presented, hence coadmissible, inducing an exact and faithful functor

$$\{\text{coh. } \mathcal{D}\text{-modules}\} \rightarrow \mathcal{C}_X.$$