Lie algebras

Francesco Zerman

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Detailed summary

- Definition of Lie algebra. Every (associative) algebra A can be endowed with a structure of a Lie algebra by defining

  $$[a,b] := ab - ba$$

  for every $a, b \in A$.

- Let G be an (affine) algebraic group over the algebraically closed field K of characteristic 0. Three ways of seeing the tangent space to G at 1.

  1. Let $D := K[t]/(t^2) \cong K \oplus K\varepsilon$, with $\varepsilon = t + (t^2)$. Let $G \subseteq \mathbb{A}^n$. Then $v \in \mathbb{A}^n$ is tangent to G at $x \in G$ if the map

     $$\begin{aligned} \phi_v : K[G] &\longrightarrow D \\ f &\longmapsto f(x + \varepsilon v) \end{aligned}$$

     is an algebra map.
  2. $T_1(G) := \{\delta \in \mathrm{Hom}(K[G], K) : \delta(fg) = (\delta f)g(1) + f(1)(\delta g) \text{ for every } f, g \in K[G]\}$.
  3. Let $\mathrm{Der}(K[G])$ be the set of derivations of $K[G]$. Let $g \in G$ and $f \in K[G]$ we define $\lambda_g : K[G] \longrightarrow K[G]$ as

     $$(\lambda_g(f))(h) = f(g^{-1}h)$$

     for every $h \in G$, and

     $$\mathrm{Lie}(G) = \mathrm{Der}_G(K[G]) := \{\delta \in \mathrm{Der}(K[G]) : \delta\lambda_g = \lambda_g\delta \quad \text{for every } g \in G\}.$$

     $\mathrm{Lie}(G)$ is a Lie algebra.

  Sketch of the proof of the 1-to-1 correspondence between these three objects and examples of the tangent spaces to $\mathrm{GL}_n(K)$ and $\mathrm{SL}_n(K)$ in 1, namely $\mathfrak{gl}_n(K)$ and $\mathfrak{sl}_n(K)$.

- There is a correspondence between some structure of the algebraic group G and the structure of $T_1(G)$, which is a Lie algebra. Moreover, to every rational representation of an algebraic group G, we can associate a representation of the Lie algebra $T_1(G)$. These are some motivations that lead us to study in detail Lie algebras and its representations.

- Let L be a Lie algebra. Definitions of subalgebras and ideals of L. Definition of the adjoint representation

$$\begin{aligned} \mathrm{ad}_L : L &\longrightarrow \mathrm{Der}(L) \\ x &\longmapsto [x, -]. \end{aligned}$$

  Simple Lie algebras, and $\mathfrak{sl}_2(K)$ as example.

- **Nilpotent Lie algebras**: definitions, first properties. L is nilpotent if and only if every element of L is ad-nilpotent. If $L \subseteq \mathfrak{gl}(V)$ for some fin. dim. K-vector space V, then there esists a basis for V such that L is a subalgebra of the algebra of strictly upper triangular matrices.

- **Solvable Lie algebras**: definitions, first properties. Definition of $\mathrm{Rad}(L)$ as the maximal solvable ideal of every Lie algebra L. If $L \subseteq \mathfrak{gl}(V)$ for some fin. dim. K-vector space V, then there esists a basis for V such that L is a subalgebra of the algebra of upper triangular matrices. L solvable if and only if $[L, L]$ is nilpotent. Cartan solvability criterion.

- **Semisimple Lie algebras**: definitions, first properties. The killing form k. L is semisimple if and only if k is nondegenerate on L. Decomposition of every semisimple Lie algebra into a sum of simple ideals. Every fin. dim. representation of a semisimple Lie algebra is completely reducible (Weyl's theorem).

- Abstract and usual Jordan decomposition: if L is semisimple, any element x of L can be written uniquely as a sum of a nilpotent and a semisimple element. If $L \subseteq \mathfrak{gl}(V)$ then usual=abstract.

- Toral subalgebras of semisimple Lie algebras: any toral subalgebra is abelian, any toral subalgebra is maximal if and only if it is self-centralizing.

- Let L be a semisimple Lie algebra, $H \subseteq L$ a maximal toral subalgebra. Decomposition of L with respect to the action of $\mathrm{ad}_L(H)$ on L. Example: decomposition of $\mathfrak{sl}_2(K)$ and (maybe) characterization of the irreducible representations of $\mathfrak{sl}_2(K)$.