An overview of Emerton's Jacquet functor

Notations $L/\mathbb{Q}_p$ finite $G = L$-pts of a conn reductive L-gr, $P \subset G$ parab, $\bar{P}$ opposite parab

$$N = R_u(P) \rightsquigarrow M = P/N \xleftarrow{\sim} P \cap \bar{P}$$

(see M inside P) $P = MN$

K/L complete, discretely valued coeff field

Convention: all vect spaces are over K

all top vect sp are of compact type

H p-adic Lie gr $\quad \mathrm{Rep}\, H$ = loc. analytic reps of H

$$\mathrm{Rep}^{sm} H \subset \mathrm{Rep}\, H \supset \mathrm{Rep}^{ad} H$$

$\mathrm{Rep}^{sm} H$ = smooth reps, $\mathrm{Rep}^{ad} H$ = adm loc an reps

The smooth Jacquet functor $\mathrm{Rep}^{sm} M \to \mathrm{Rep}^{sm} G$

$$V \to (\mathrm{Ind}_P^G V)^{sm}$$

has a left adjoint $\underbrace{(-)_N}_{\text{Jacquet functor}}$ (N-coinvariants)

$$\mathrm{Hom}_G(V, (\mathrm{Ind}_P^G W)^{sm}) \overset{\text{Frob recipr}}{=} \mathrm{Hom}_P(V, W) = \mathrm{Hom}_M(V_N, W) \qquad (1)$$

Th 1) (Jacquet) $(\ )_N$ is exact & preserves admissibility

2) (Casselman, Bernstein 2nd adjointness th)

(2) $\mathrm{Hom}_G\left((\mathrm{Ind}_{\bar P}^G U)^{sm}, V \right) \xrightarrow{\sim} \mathrm{Hom}_M(U(\delta), V_N)$ (δ = modulus character of P)

The locally analytic Jacquet functor

One still has

$$\mathrm{Hom}_G(V, \mathrm{Ind}_P^G W) \xrightarrow{\sim} \mathrm{Hom}_M(\underbrace{(V_N)^{sep}}_{V_N/\overline{\{0\}}}, W)$$

V, W loc an reps

Pb: $V \to (V_N)^{sep}$ is very hard to control

"Easy" adjunction There will be a full subcat $\mathrm{Rep}^z M \subset \mathrm{Rep}\, M$ (if $Z(M)$ is compact this is $=$) & a functor $J_P : \mathrm{Rep}\, P \longrightarrow \mathrm{Rep}^z M$ with good finiteness properties, extending $(-)_N$ on adm smooth reps of G.

$U \in \mathrm{Rep}^z M$ ⇝ fct with support in $\bar P N$ $\subset \mathrm{Ind}_{\bar P}^G U$ ← P-stable, closed

$\simeq C_c^{la}(N, U)$

$(N \underset{open}{\hookrightarrow} \bar P \backslash G)$

iii) $P \circlearrowleft C_c^{la}(N, U) \supset C_c^{sm}(N, U)$ ← P-stable

$$(m f)(n) = m . f(m^{-1} n m)$$

Th (Emerton) $\exists$ functor

$$J_P : \mathrm{Rep}\, P \longrightarrow \mathrm{Rep}^{z} M$$

st we have $J_P(C_c^{sm}(N, U)) \xrightarrow{\sim} U(\delta)$

& passage to J_P gives a bijection

$$\mathrm{Hom}_P(C_c^{sm}(N, U), V) \xrightarrow{\sim} \mathrm{Hom}_M(U(\delta), J_P(V))$$

Moreover J_P is indep of choice of $\overline{P}$ & $\underline{J_P(V)} \xrightarrow{\sim} V_N$ for V adm sm rep of G.

Rk 1) J_P is left exact, but not right exact (see below)

(so $J_P \neq (-)_N^{sep}$)

2) $U = J_P(V)(\delta^{-1}) \rightsquigarrow \mathrm{id}_{J_P(V)}$ induces

a map

$$\begin{array}{ccc} C_c^{sm}(N, U) & \longrightarrow & V \\ \| & & \\ C_c^{sm}(N, J_P(V))(\delta^{-1}) & & \\ \cup & & \\ J_P(V) & & \end{array}$$

$N_0 \leq N$ open $\rightsquigarrow$ $\left(n \mapsto \begin{cases} v & \text{if } n \in N_0 \\ 0 & \text{if not} \end{cases}\right) \longleftarrow v$

"canonical lifting" is $M_0 N_0$-equiv if we start with $P_0 \leq P$ cpt open & $M_0 = P_0 \cap M$, $N_0 = P_0 \cap N$

Construction of $J_p(V)$ Let $Z = Z(M)$

Th (Schneider, Teitelbaum) $\exists$ rig analytic space $\hat{Z}$ over L st $\forall\, E/L$ complete

$$\hat{Z}(E) = \{\chi : Z \to E^\times \text{ loc } L\text{-analytic characters}\}$$

$\big(\mathcal{O}_p^\times \rightsquigarrow \underbrace{\mathcal{B}_m^{an}}_{\chi(p)} \times \coprod_{\mu \in \mu_{p-1}} \underset{\text{unit}}{\text{open disc}} \big)$

$\exists$ dense image map ↙ K-valued rig an fct on $\hat{Z}$

$$\underbrace{D(Z)}_{K\text{-valued dist on } Z} \longrightarrow \mathcal{O}(\hat{Z}) \qquad \mu \mapsto \left(\chi \mapsto \int_Z \chi(x)\, d\mu(x)\right)$$

$\simeq$ iff Z is compact

Def $\mathrm{Rep}^{z} M := \{V \in \mathrm{Rep}\, M \mid Z \circlearrowright V_b'$ extends to a structure of topological $\mathcal{O}(\hat{Z})$-module$\}$

Ex $\underset{\text{cp open}}{M_0 \leq M} \rightsquigarrow A = \mathcal{O}(\hat{Z}) \hat{\otimes}_K D(M_0)$ FS-alg

$\forall\, V \in \mathrm{Rep}^z M \quad A \circlearrowright V_b'$

$\mathrm{Rep}^{es} M := \{V \in \mathrm{Rep}^z M \mid V_b'$ is coadm over $A\}$

essentially adm reps of M

$\to$ abelian cat $\supset \mathrm{Rep}^{ad} M \quad$ [$=$ if $Z = Z(M)$ is compact)

(indep of M_0)

Rk if $M = Z$ then

1) $\mathrm{Rep}^{es} M \xrightarrow{\sim} \mathrm{Coh}\,\hat{Z}$

$V \longrightarrow V'_b$

2) $V \in \mathrm{Rep}^{fs} M \Rightarrow \forall \chi : Z \to K^\times$ loc L-an $(\Leftrightarrow \chi \in \hat{Z}(K))$

then $\underbrace{V^\chi}$ $\in \mathrm{Rep}^{ed} M$

χ-eigenspace for Z

Fix $P_0 \underset{co}{\leq} P \rightsquigarrow M_0 = M \cap P_0$

$N_0 = N \cap P_0$

$M^+ = \{ m \in M \mid m N_0 m^{-1} \} \subset N_0$

$\cup$

$Z^+ := M^+ \cap Z \leftarrow$ generating submonoid of Z & $M = Z M^+$

$\forall V \in \mathrm{Rep}\, P$ $\quad V^{N_0}$ has an action of M^+

by C^0 Hecke operators

$$m.v = \int_{N_0} n m v \, dn = \delta(m) \sum_{n \in N_0 / m N_0 m^{-1}} n m v$$

Def $J_P(V) = \mathrm{Hom}_{Z^+}(\mathcal{O}(\hat{Z}), V^{N_0})$

$Z \circlearrowleft$ $\quad Z \circlearrowleft$

$M^+ \rightsquigarrow$

$M^+ \circlearrowleft J_P(V)$

Th (Emerton) 1) the 2 actions turn $J_P(V)$ into an object of $\mathrm{Rep}^z M$ st

$$J_P(V)'_b \simeq \mathcal{O}(\hat{Z}) \hat{\otimes}_{K[Z^+]} (V^{N_0})'_b$$

2) $J_P(V)$ is indep of P_0 &

$J_P(C_c^{sm}(N,V)) = U(\delta)$ &

$$\mathrm{Hom}_P(C_c^{sm}(N,U),V) = \mathrm{Hom}_M(U(\delta), J_P(V))$$

The deep results — Emerton

Th 1 $J_P(\mathrm{Rep}^{es} G) \subset \mathrm{Rep}^{es} M$

Z false with Rep^{ad} instead of Rep^{es}

$U \in \mathrm{Rep}^z M$ $\quad \mathrm{Ind} := \mathrm{Ind}_{\bar{P}}^G U \xhookleftarrow[P]{} C_c^{sm}(N,U)$

$I(U) := \overline{G \cdot C_c^{sm}(N,U)} \subset \mathrm{Ind}$ stable

ex 1) U adm sm $\Rightarrow I(U) = \mathrm{Ind}^{sm}$

2) $\chi : \mathbb{Q}_p^\times \to K^\times$ C^0 character $k = \chi'(1)$ $\quad G = GL_2(\mathbb{Q}_p)$

$P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$U = K$ $\quad M \circlearrowright U$ $\quad \begin{pmatrix} a & \\ & d \end{pmatrix} \cdot x = \chi(a)\, x$

$M = \begin{pmatrix} * & \\ & * \end{pmatrix}$

- if $k \notin \mathbb{Z}_{\geq 0} \Rightarrow I(U) = \mathrm{Ind}$
- if $k \in \mathbb{Z}_{\geq 0} \Rightarrow I(U) = (\mathrm{Ind})^{\text{lalg}}$

$= \pi \otimes \mathrm{Sym}^k K^2$ $\quad \pi$ certain explicit sm rep

Th (Emerton) G q-split, P Borel ($\Rightarrow M$ max torus) then $\forall\, U \in \mathrm{Rep}^t M$

$$J_P(I(U)) = J_P(\mathrm{Ind}_{\bar P}^G U)$$

If $\dim U < \infty \Rightarrow$ is $\hookrightarrow$ & $\forall\, I(U) \twoheadrightarrow_G W$

the map $J_P(I(U)) \to J_P(W)$ is surjective

ex (J_P not exact) back to prev ex $k \in \mathbb{Z}_{\geq 0}$

$$\Rightarrow \quad 0 \to I(U) \to \mathrm{Ind}_{\bar P}^G U \to \mathrm{Ind}_{\bar P}^G \delta \to 0$$

for some character δ

$\forall U$ $J_P(\mathrm{Ind}_{\bar P}^G U) \hookleftarrow U(\delta)$ $\left(C_c^{sm}(N, U) \right.$

$\hookrightarrow \mathrm{Ind}_{\bar P}^G U$ & J_P is left exact$\left.\right)$

In this ex $J_P(\mathrm{Ind}_{\bar P}^G U)$ is 2-dim & explicit

$\underline{\text{Thm}}$ (Emerton) $U \in \mathrm{Rep}^{es} M$, Π adm Ban rep of G, $\forall$ V closed subrep of Π^{la}

$$\mathrm{Hom}_G(I(U), V) \simeq \mathrm{Hom}_{\mathfrak{g}, P}(U(\mathfrak{g}) \cdot C_c^{sm}(N, U), V)$$

$\mathfrak{g} \leftarrow \mathrm{Lie}\, G$

(both embed naturally in $\mathrm{Hom}_M(U(\sigma), J_P(V))$)

$C_c^{sm}(N, U) \hookrightarrow \mathrm{Ind}_{\bar{P}}^G U$

$U(\mathfrak{g}) \cdot C_c^{sm}(N, U) = U(\mathfrak{g})$-submodule gener by $C_c^{sm}(N, U)$

$$\uparrow$$

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} C_c^{sm}(N, U)$$

$\mathfrak{p} \hookrightarrow \mathrm{Lie}\, P$