Equivariant $\mathcal{D}$-modules on rigid analytic spaces

Padova, 4/5/21

0. Motivation

X — smooth, rigid space$/K$.

$\mathcal{C}_X = \{$coadmissible $\widehat{\mathcal{D}}_X$-modules$\}$.

$\mathbb{Q}_p \underset{f}{\subsetneq} F \subset K$; $\mathbb{B} \subset \mathbb{G} \leftarrow$ conn. split. semisimple alg gp.

$\mathfrak{g} := (\mathrm{Lie}\, \mathbb{G}) \otimes K$.

$(\mathbb{G}/\mathbb{B} \otimes K)^{an} =: X$, the rigid analytic flag variety

Thm (analogue of Beilinson–Bernstein)

1) $\Gamma : \mathcal{C}_{(\mathbb{G}/\mathbb{B}\otimes K)^{an}} \longrightarrow \mathcal{C}_{\widehat{\mathcal{D}}(X)}$ is an equivalence

2) $\widehat{\mathcal{D}}(X) \cong \widehat{U(\mathfrak{g})} \underset{Z(\mathfrak{g})}{\otimes} K$.

Would like to do something similar

for $D(G) := D^{F\text{-}an}(G, K)$

where $G := \mathbb{G}(F)$.

<u>Good news</u> : $\widehat{U(\mathfrak{g})} \hookrightarrow D(G)$

<u>Bad news</u> : $D(G)$ is much larger than $\widehat{U(\mathfrak{g})}$ e.g.

$K[G] \hookrightarrow D(G)$, indeed

$$\bigoplus_{g \in G} \widehat{U(\mathfrak{g})} \cdot g \subseteq D(G)$$

$$\cong$$

$\boxed{\widehat{U(\mathfrak{g})} \rtimes G}$, a skew-group ring.

Q. How do we "localize" this?

1. Equivariant sheaves (Grothendieck)

Let X be a top. space.
Let $G \xrightarrow[\rho]{} \mathrm{Homeo}(X)$ be an action.
Let R be a comm. base ring.

Defn 1 ① An R-linear G-equivariant sheaf on X
is a pair $(\mathcal{F}, \{g^{\mathcal{F}} : g \in G\})$, where

$$\boxed{g^{\mathcal{F}} : \mathcal{F} \longrightarrow g^* \mathcal{F}} \qquad (g : \mathcal{F}(U) \to \mathcal{F}(gU))$$

is a map of sheaves of R-modules on X
s.t. $(gh)^{\mathcal{F}} = h^*(g^{\mathcal{F}}) \circ h^{\mathcal{F}} \quad \forall g, h \in G$.

② A morphism $\varphi : \mathcal{F} \to \mathcal{F}'$ is an R-linear map of sheaves s.t.

$$g^*(\varphi) \circ g^{\mathcal{F}} = g^{\mathcal{F}'} \circ \varphi \quad \forall g \in G.$$

— these form an abelian category

— defn. extends to any site.

Defn 2 ① A sheaf $\mathcal{A}$ of R algebras is G-equivariant if $\mathcal{A}$ carries a G-eq. structure, s.t.

$$g^{\mathcal{A}} : \mathcal{A} \longrightarrow g^* \mathcal{A} \text{ is a map of}$$

sheaves of R-algebras on X.

② A G-$\mathcal{A}$-module is an R-linear G-eq. sheaf $(\mathcal{M}, \{g^{\mathcal{M}}\}_{g \in G})$ s.t.

$$g^{\mathcal{M}}(a \cdot m) = g^{\mathcal{A}}(a) \cdot g^{\mathcal{M}}(m)$$
$$\forall g \in G \quad \forall a \in \mathcal{A} \quad \forall m \in \mathcal{M}$$

These form an abelian category

G-$\mathcal{A}$-mod.

Example $X = \{*\}$ $\mathcal{A} = R$ const. sheaf

Then $G-\mathcal{A}-\text{mod} = R[G]-\text{mod}$.

Defn 3 Let A be an R-algebra. Let $\rho: G \to \mathrm{Aut}_{R\text{-alg}}(A)$, write

$$g \cdot a := \rho(g)(a) \quad \text{so that}$$
$$g \cdot (ab) = (g \cdot a)(g \cdot b) \quad \forall a, b \in A \ \forall g \in G.$$

The <u>skew-group ring</u> $A \rtimes G$ is

$\bigoplus_{g \in G} A g$ as a left A-module; its multiplication is given by

$$(a g) \cdot (a' g') = (a (g \cdot a')) (g g').$$
$$\forall a, a' \in A \ \forall g, g' \in G.$$

e.g.

- $g a g^{-1} = g \cdot a. \quad \forall g \in G \ \forall a \in A.$

Prop. Let $\mathcal{A}$ be a G-equivariant sheaf of R-algebras on $X = \{*\}$.

Then $\Gamma(X, -) : G-\mathcal{A}-\text{mod} \longrightarrow$

$$\longrightarrow \mathcal{A}(X) \rtimes G-\text{mod}$$

is an equivalence of categories.

2. Naive Localisation Functor

Defn 4 Let M be a $D(G)$-module.

Define $\mathrm{Loc}_{\text{naive}}(M) := \mathcal{D}_{(G/B)^{an}} \otimes_{U(\mathfrak{g})} M$.

It is a G-$\mathcal{D}_X$-module! $\forall g \in G$,

$$\mathcal{D}(U) \otimes_{U(\mathfrak{g})} M \xrightarrow{g^{\mathcal{D}}(U) \otimes g_M} \mathcal{D}(g \cdot U) \otimes_{U(\mathfrak{g})} M$$

$\forall$ adm. open $U \subseteq X$.

Main Theorem 1) Let G be a p-adic Lie group acting continuously on smooth rigid variety X.

Then $\exists$ an abelian category

$$\boxed{\mathcal{C}_{X/G}}$$

of coadmissible G-equivariant $\mathcal{D}$-modules on X, s.t.

- when $G = 1$, $\mathcal{C}_{X/G} = \mathcal{C}_X$
- when $X = \{*\}$, $\mathcal{C}_{X/G} = \mathcal{C}_{D^\infty(G)}$

Here $D^\infty(G) := D(G)/\langle \mathfrak{g} \rangle$.

2) When $G = \mathbb{G}(F)$ as before, and $X = (\mathbb{G}/\mathbb{B})^{an}$

$$\Gamma : \mathcal{C}_{X/G} \longrightarrow \mathcal{C}_{D(G) \underset{Z(\mathfrak{g})}{\otimes} K}$$

is an equivalence, with quasi-inverse

$$\mathrm{Loc} : \mathcal{C}_{D(G) \underset{Z(\mathfrak{g})}{\otimes} K} \longrightarrow \mathcal{C}_{X/G}.$$

Rk every $\mathcal{M} \in \mathcal{C}_{X/G}$ <u>is</u> a $\widehat{\mathcal{D}}_X$-module

But because G is infinite, it's <u>not</u> coadmissible as such!

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- $\mathcal{C}_X \cap \mathrm{coh}(\mathcal{O}_X) = \mathrm{MIC}(X)$