# GAGA functor & Kiehl's thm

$K$ = complete nonarchimedean field (non-trivial)

Reference: "Lectures on formal and rigid geometry" Bosch

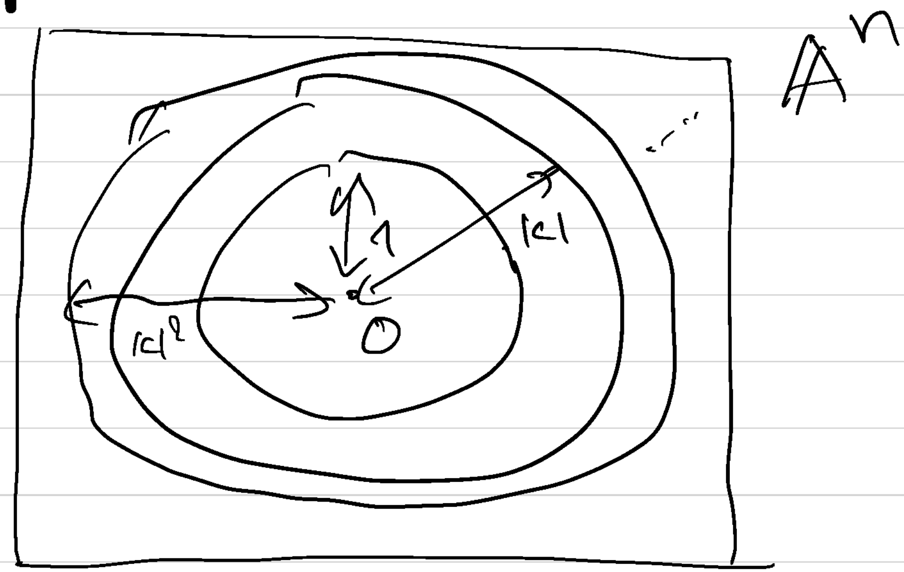
## § GAGA

Based on Serre's work over $\mathbb{C}$
"Géométrie algébrique et géométrie analytique".
now adapted to nonarchimedean setting.

Quⁿ: How to construct a rigid analytic $\mathbb{A}^n$ ?

We know how to define rigid analytic closed balls e.g. $\mathrm{Sp}\, K\langle \xi_1, \dots, \xi_n \rangle$ = unit ball $\subset \mathbb{A}^n$

Idea: Take larger and larger balls, and pass to the limit.




Pick $c \in K$ s.t. $|c| > 1$. For $i \geq 0$, let $\mathbb{B}_i$ = closed ball of radius $|c|^i$

i.e. $\mathbb{B}_i = \mathrm{Sp}\, K\langle c^{-i}\xi_1, \dots, c^{-i}\xi_n \rangle$

Have a tower of inclusions

$$K\langle \xi_1, \dots, \xi_n \rangle \hookleftarrow K\langle c^{-1}\xi_1, \dots, c^{-1}\xi_n \rangle \hookleftarrow K\langle c^{-2}\xi_1, \dots, c^{-2}\xi_n \rangle \dots$$

corresponding to $\mathbb{B}_0 \subseteq \mathbb{B}_1 \subseteq \mathbb{B}_2 \subseteq \dots$

We can glue these spaces to get

$$\mathbb{A}_K^{n,an} := \varinjlim_{i \geq 0} \mathbb{B}_i$$

This is a rigid analytic $K$-space with admissible affinoid covering $(\mathbb{B}_i)_{i\geq 0}$.

Moreover,

$$\mathcal{O}_{\mathbb{A}_K^{n,an}}(\mathbb{A}_K^{n,an}) = \varprojlim_{i\geq 0} K\langle c^{-i}\zeta_1, \dots, c^{-i}\zeta_n\rangle$$

$$= \left\{ \sum_{\alpha\in\mathbb{N}^n} a_\alpha \zeta^\alpha \;\middle|\; \begin{array}{l} a_\alpha \in K \text{ and } |c^{i|\alpha|} a_\alpha| \to 0 \\ \text{as } |\alpha| \to \infty, \text{ for all } i \geq 0 \end{array} \right\}$$

$(|\alpha| := \alpha_1 + \dots + \alpha_n)$

Can generalise previous construction to any affine $K$-scheme of finite type

$X = \operatorname{Spec}\left(K[\xi_1, \dots, \xi_n]/\mathfrak{a}\right)$

by replacing the $\mathbb{B}_i$ with

$$X_i := \operatorname{Sp}\left(K\langle c^{-i}\xi_1, \dots, c^{-i}\xi_n\rangle / (\mathfrak{a})\right)$$

Then glueing the $X_i$ one obtains

$$X^{an} := \varinjlim X_i .$$

Fact: Doesn't depend on choice of $c$

In gen$^{l}$, given a $K$-scheme $X$ locally of finite type, can cover $X$ by open affines $X_i$ which are of finite type over $K$, then can apply previous construction to get $X_i^{an}$

$\rightsquigarrow$ glue to get $X^{an}$

This way we get a functor

$$\left\{\begin{matrix}K\text{-schemes} \\ \text{locally of finite} \\ \text{type}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{rigid analytic} \\ K\text{-spaces}\end{matrix}\right\}$$

$$X \longmapsto X^{an}$$

called GAGA.

Properties: * Universal property: there is a morphism of locally ringed spaces $(i, i^*) : (X^{an}, \mathcal{O}_{X^{an}}) \longrightarrow (X, \mathcal{O}_X)$ s.t. for any rigid $K$-space $(Y, \mathcal{O}_Y)$, any morphism of loc. ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ factors uniquely through $(i, i^*)$.
The underlying map of sets $X^{an} \longrightarrow X$ identifies $X^{an}$ with the closed points of $X$.

* $\mathbb{A}^{n,an}_K$ satisfies the usual universal property of affine $n$-space i.e. given a rigid $K$-space $Y$, $\mathrm{Hom}(Y, \mathbb{A}^{n,an}_K) \cong \mathcal{O}_Y(Y)^n$

$(\cong \mathrm{Hom}_{K\text{-alg}}(K[\xi_1, \dots, \xi_n], \mathcal{O}_X(X)))$

via the morphism $\mathbb{A}^{n,an}_K \longrightarrow \mathbb{A}^n_K$.

* $X \mapsto X^{an}$ is faithful but not full

e.g. $\exists$ $\mathbb{A}^{1,an}_K \longrightarrow \mathbb{A}^{1,an}_K$ corresponding to $\xi \mapsto \sum_{n \geq 0} c^{-n^2} \xi^n$. This doesn't come from a morphism of schemes $\mathbb{A}^1_K \to \mathbb{A}^1_K$.

* Given an $\mathcal{O}_X$-module $\mathcal{F}$, can construct an $\mathcal{O}_{X^{an}}$-module $\mathcal{F}^{an} = i^{-1}(\mathcal{F}) \otimes_{i^{-1}(\mathcal{O}_X)} \mathcal{O}_{X^{an}}$.

Moreover:

$\mathcal{F}$ is coherent $\Longleftrightarrow$ $\mathcal{F}^{an}$ is coherent

& $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact.

Example: $\mathbb{P}^{n,an}_K$ is by def$^n$ covered by

$$U_i^{an} = \varinjlim_{j\geq 0} \operatorname{Sp} K\langle c^{-j}\frac{\xi_0}{\xi_i}, \dots, c^{-j}\frac{\xi_n}{\xi_i}\rangle$$
$$\cong \mathbb{A}^{n,an}_K \qquad (i=0,\dots,n)$$

In fact, there is an admissible affinoid covering of $\mathbb{P}^{n,an}_K$ by closed unit balls

$$V_i = \operatorname{Sp} K\langle \frac{\xi_0}{\xi_i}, \dots, \frac{\xi_n}{\xi_i}\rangle \subset U_i^{an}$$

Idea: for $x = (x_0 : \dots : x_n)$ $(x_i \in \bar{K})$, let $i$ be s.t. $|x_i| = \max\{|x_0|, \dots, |x_n|\}$. Then $x \in V_i$.

$\underline{n=1}$: Two hemispheres cover $\mathbb{P}^1$

## § Proper Mapping Theorem

First need notion of a proper morphism.

**Defⁿ:** A morphism $\varphi : X \longrightarrow Y$ of rigid $K$-spaces is a closed immersion if there is an admissible affinoid covering $(V_j = \mathrm{Sp}\, A_j)$ of $Y$ s.t. $\varphi^{-1}(V_j) = \mathrm{Sp}\, B_j$ is affinoid & the corresponding map $A_j \longrightarrow B_j$ is surjective.

**Remarks:** * This defⁿ doesn't depend on choice of covering $(V_j)$

* If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module then $\varphi_*(\mathcal{F})$ is coheren

Def$^n$: Let $\varphi : X \to Y$ be a morphism of rigid $K$-spaces.

(1) Say $\varphi$ is <u>quasi-compact</u> if for each quasi-compact $Y' \subset Y$ (i.e. $Y'$ has a finite adm. affinoid covering), $\varphi^{-1}(Y')$ is also quasi-compact.

(2) $\varphi$ is <u>separated</u> (resp. <u>quasi-separated</u>) if the diagonal $\Delta : X \to X \times_Y X$ is a closed immersion (resp. quasi-compact).

(3) Say $X$ is <u>separated</u> if $X \to \mathrm{Sp}\, K$ is separated.

In alg. geom, have $X \to Y$ is separated iff $\mathrm{Im}(\Delta) \subset X \times_Y X$ is closed. Here:

Prop$^n$: $\varphi : X \longrightarrow Y$ is separated iff:

(i) $\varphi$ is quasi-separated; and

(ii) the image of $\Delta$ is a closed analytic subset in $X \times_Y X$ i.e. locally on open affinoids it is Zariski closed.

To define proper, need a notion of relative compactness over a base

**Def$^n$:** $\varphi : X \to Y$ morphism with $Y$ affinoid, $U \subset U' \subset X$ open affinoids. Say $U$ is <u>relatively compact</u> in $U'$, denoted $U \Subset_Y U'$, if $\exists\, f_1, \dots, f_r \in \mathcal{O}_X(U')$ s.t.

$$\mathcal{O}_Y(Y)\langle \xi_1, \dots, \xi_r \rangle \twoheadrightarrow \mathcal{O}_X(U')$$
$$\xi_i \longmapsto f_i$$

and s.t. $U \subset \{ x \in U' \mid |f_i(x)| < 1 \}$.

This property behaves well under intersections and fibre products.

**Def$^n$:** $\varphi : X \longrightarrow Y$ is <u>proper</u> if

(i) $\varphi$ is separated; and

(ii) $\exists$ admissible affinoid covering $(Y_i)_{i \in I}$ of $Y$ and, for each $i \in I$, two finite adm. affinoid coverings $(X_{ij})_{j=1,\dots,n_i}$ and $(X'_{ij})_{j=1,\dots,n_i}$ of $\varphi^{-1}(Y_i)$ s.t. $X_{ij} \Subset_{Y_i} X'_{ij}$ $\forall i, j$.

[This def$^n$ is inspired by the theory of compact Riemann surfaces.]

**Ex:**
- Finite/projective morphisms are proper.
- $X$ proper $K$-scheme $\Longrightarrow$ $X^{an}$ is proper.

**Thm:** (Kiehl) Let $\varphi: X \to Y$ be a proper morphism of rigid analytic $K$-spaces, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $R^i\varphi_*(\mathcal{F})$ is a coherent $\mathcal{O}_Y$-module for all $i \geq 0$.

**Application:** Stein factorisation

$\varphi: X \to Y$ proper. Then $\varphi_*(\mathcal{O}_X)$ is a coherent $\mathcal{O}_Y$-module.

$\rightsquigarrow$ rigid $K$-space $Y'$ that is finite over $Y$.

$\rightsquigarrow$ factorise $\varphi = X \xrightarrow[\substack{\text{proper} \\ \text{w/ connected} \\ \text{fibres}}]{} Y' \xrightarrow[\text{finite}]{} Y$

Some GAGA result:

<u>Theorem</u>: Let $X$ be a proper $K$-scheme, $\mathcal{F}, \mathcal{G}$ coherent $\mathcal{O}_X$-modules. Then:

(i) $H^i(X, \mathcal{F}) \xrightarrow{\cong} H^i(X^{an}, \mathcal{F}^{an}) \quad \forall i \geq 0$

(ii) $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an})$.

Moreover if $\mathcal{F}'$ is a coherent $\mathcal{O}_{X^{an}}$-module then there exists a unique coherent $\mathcal{O}_X$-module $\mathcal{F}$, up to isomorphism, s.t $\mathcal{F}' = \mathcal{F}^{an}$. In particular

$$\left\{\begin{array}{c}\text{coherent}\\ \mathcal{O}_X\text{-mod}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{coherent}\\ \mathcal{O}_{X^{an}}\text{-mod}\end{array}\right\}$$

$$\mathcal{F} \longmapsto \mathcal{F}^{an}$$

is an equivalence.

e.g. $X = \mathbb{P}^n_K$, $\mathcal{F}' =$ coherent ideal $\mathcal{I}' \subset \mathcal{O}_{X^{an}}$

then $\mathcal{F}' =$ analytification of ideal $\mathcal{I} \subset \mathcal{O}_X$

$\rightsquigarrow$ nonarchimedean Chow's thm: closed analytic subsets of $\mathbb{P}^{n,an}_K$ are algebraic.

Remark: Analogous to Serre's original results on complex projective varieties in GAGA

Few words on proof: (proof ~ 10-15 pages)

Work locally so WLOG $Y$ is affinoid.

$\Gamma(Y, R^i\varphi_*(\mathcal{F})) = H^i(X, \mathcal{F})$ so need to show two things:

(1) $H^i(X, \mathcal{F})$ is a finite $\mathcal{O}_Y(Y)$-module

(2) $R^i\varphi_*(\mathcal{F})$ = sheaf of $\mathcal{O}_Y$-modules associated to $H^i(X, \mathcal{F})$

i.e. if $Y' = \mathrm{Sp}\, B' \subset Y = \mathrm{Sp}\, B$, need to show $H^i(X, \mathcal{F}) \otimes_B B' \xrightarrow{\cong} H^i(X \times_Y Y', \mathcal{F})$

(1) $B := \mathcal{O}_Y(Y)$. Assume $\exists$ two finite admissible coverings $\mathcal{U} = (U_i)$ & $\mathcal{V} = (V_i)$ of $X$ s.t. $V_i \Subset_Y U_i \ \forall i$ (replace $Y$ by $Y_i$, $X$ by $\varphi^{-1}(Y_i)$)

Čech cohomology $\rightsquigarrow$ $H^i(\mathcal{U}, \mathcal{F}) \cong H^i(\mathcal{V}, \mathcal{F})$

$\cong H^i(X, \mathcal{F})$.

Write $C^\bullet(\mathcal{V}, \mathcal{F}) =$ Čech complex w.r.t $\mathcal{V}$

$Z^\bullet(\mathcal{V}, \mathcal{F}) =$ cocycles of Čech complex

Need to show that cokernel of

$$f^i : C^{i-1}(\mathcal{V}, \mathcal{F}) \xrightarrow{d} Z^i(\mathcal{V}, \mathcal{F})$$

is a finite $B$-module.

Using $V_i \subseteq U_i$ $\forall i$, have restriction maps $C^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{F})$ & so a morphism

$$r^i : Z^i(\mathcal{U}, \mathcal{F}) \longrightarrow Z^i(\mathcal{V}, \mathcal{F}) \quad (i \geq 0)$$

Then as $H^i(\mathcal{U}, \mathcal{F}) \cong H^i(\mathcal{V}, \mathcal{F})$, see that

$$f^i + r^i : C^{i-1}(\mathcal{V}, \mathcal{F}) + Z^i(\mathcal{U}, \mathcal{F}) \longrightarrow Z^i(\mathcal{V}, \mathcal{F})$$

is surjective. So can "disturb" $f^i$ by some map $r^i$ to get something surjective.

<u>V. roughly:</u> Need a subtle approximation argument. Say a cts B-linear morphism $g : M \to N$ is <u>completely cts</u> if $g = \lim g_i$, where $g_i$ is a cts B-linear hom with $\mathrm{Im}(g_i)$ f.g over B.

Main tool: When $f, g : M \to N$ are cts homs of complete normed B-modules with $f$ surjective & $g$ completely continuous, there's a thm of L. Schwarz that says $\mathrm{Coker}(f+g)$ is f.g. / B.

Want to apply this thm to $f = f^i + r^i$ &
$g = -r^i$.
Problem: $r^i$ may not be completely continuous.
But one can show $r^i$ is "part of" a map
that is even strictly completely continuous.
Then one can reprove L. Schwarz' thm by
using this new cond$^n$ on $g$.
(2) Work by induction on $d$ = Krull dim of $B$.
Let $X' = X \times_Y Y'$. To show $H^i(X, F) \otimes_B B' \to H^i(X', F)$
is an iso, enough to show the localisation
at each max$^l$ ideal $m'$ of $B'$ is an iso.

As $B'_{m'} \longrightarrow \widehat{B'_{m'}}$ is faithfully flat, suffices to show $H^i(X, \mathcal{F}) \otimes_B \widehat{B'_{m'}} \longrightarrow H^i(X', \mathcal{F}) \otimes_{B'} \widehat{B'_{m'}}$ are all isomorphisms.

Idea: Work instead with $b$-adic completions for a well-chosen $b \in m' \cap B$.

Can use induction hypothesis on $B/(b^i)$ and pass to the limit to get the result.

There again, the details involve subtle arguments.