GEOMETRY OF FLAG VARIETY

mercoledì 25 novembre 2020 16:49

A (FULL/COMPLETE) FLAG is
$\underline{F} = (0 = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n)$ $\dim V_i = i$

The set of all flags is denoted $\mathcal{F}$

ASA: $\mathcal{F} \underset{\text{Zariski closed}}{\subset} Gr(1,n) \times \cdots \times Gr(1,n) \Rightarrow \mathcal{F}$ is a projective var.

Let $\{e_i\}_{i=1,\dots,n}$ be the canonical basis of $\mathbb{C}^n$

The COORDINATE FLAG is $\underline{F}_0$ $V_0 = \{0\}$
$V_i = \operatorname{span}\{e_j \mid j \leq i\}$ $i=1,\dots,n$

$G = GL_n(\mathbb{C}) \curvearrowright \mathbb{C}^n, \mathcal{F}$

$$G \times \mathcal{F} \longrightarrow \mathcal{F}$$
$$(g, \underline{F}) \longmapsto g\underline{F} = (gV_i)_{i=0,\dots,n}$$

This action is TRANSITIVE, $G \cdot \underline{F}_0 = \mathcal{F}$

$\operatorname{Stab}_G(\underline{F}_0) \underset{!}{=} T_n(\mathbb{C})$ sbgp of upper trg matrices in G
$= B$ a Borel sbgp of G

$\Rightarrow$ $G/B = \{gB \mid g \in G\} \overset{\sim}{\longleftrightarrow} \mathcal{F}$

① $\mathcal{F}$ is smooth (G/B is smooth, it is an homog. space)

② we can generalize the concept of flag variety.
So for any reductive gp G, one defines the flag variety of G as G/B, $B \underset{\text{Borel}}{\leq} G$

$$G/B \overset{\sim}{\longleftrightarrow} \mathcal{B} = \{B' \leq G \mid B' \text{ Borel sbgp of } G\}$$
$$gB \longmapsto gBg^{-1}$$

Proof. Use 1) $N_G(B) = B$ 2) all Borel sbgps of G are conj □

Idea: give a decomposition of G with some "nice" properties

and use the $\pi : G \longrightarrow G/B \simeq \mathcal{B}$ To deduce
$g \longmapsto gB$ info on the geom. of the flag variety

$T \leq B \leq G$
max'l torus, Borel group

$\frac{N_G(T)}{T} = W$ finite group

R root system ass. to G, T
finite set

fixing $B \leftrightarrow$ fixing $S \subset R$
$\leftrightarrow R^+$

$\forall\, \alpha \in R \Rightarrow$ the ROOT SBGP associated to α : $U_\alpha = \operatorname{im} u_\alpha$
$u_\alpha : (\mathbb{C}, +) \longrightarrow G$ alg gp morph

$t u_\alpha(z) t^{-1} = u_\alpha(\alpha(t) z)$ $z \in \mathbb{C}$, $t \in T$
$[\alpha : T \to \mathbb{C}^\times]$

$S \rightsquigarrow W(S)$ simple reflections in W

$W = \langle s_i \mid (s_i s_j)^{m(s_i s_j)} = 1 \rangle$, $m(s_i s_j) \in \mathbb{N}$

Example:

$\mathbb{D}_n(\mathbb{C}) \leq \mathbb{T}_n(\mathbb{C}) \leq Gl_n(\mathbb{C})$
diagonal invertible matrices; u.t.

$\frac{\{\text{monomial matrices}\}}{\mathbb{D}_n(\mathbb{C})} \simeq \mathfrak{S}_n$

$\varepsilon_i : \mathbb{D}_n(\mathbb{C}) \longrightarrow \mathbb{C}^\times$
$(t_1, \dots, t_n) \longmapsto t_i$

$R = \{\varepsilon_i \varepsilon_j^{-1} \mid 1 \leq i \neq j \leq n\}$
$S = \{\varepsilon_i \varepsilon_{i+1}^{-1} \mid i = 1, \dots, n-1\}$
$R^+ = \{\varepsilon_i \varepsilon_j^{-1} \mid i < j\}$

$u_{\varepsilon_i \varepsilon_j^{-1}} : (\mathbb{C}, +) \longrightarrow G$
$z \longmapsto id + z e_{ij}$

$$\mathfrak{S}_n = \langle \underset{s_i}{(i, i+1)} \mid s_i^2 = 1,\ (s_i s_{i+1})^3 = 1,\ s_i s_j = s_j s_i \ |i-j| \geq 2 \rangle$$

the length function
$\ell : W \longrightarrow \mathbb{N}$
$w \longmapsto \ell(w) = \min\{n \mid w = s_{i_1} \cdots s_{i_n} \quad s_{i_j} \in W(S)\}$
$\Rightarrow \exists!\, w_0 \in W$ s.t. $\ell(w_0)$ is maximal, the LONGEST ELEMENT

$\forall\, w \in W$, fix a repr. $\dot{w} \in N_G(T)$ | $w \in \mathfrak{S}_n$

$\forall w \in W$, fix a repr. $\dot{w} \in N_G(T)$ | $w \in S_n$ $\dot{w} \in G \ni \left(e_i \mapsto e_{w(i)} \atop i=1,\dots,n \right)$

Def. The BRUHAT CELL relative to $w \in W$ is $C(w) = B\dot{w}B$

conn. alg gp

$$(B \times B) \times G \longrightarrow G$$

variety

$$((b_1, b_2), g) \longmapsto b_1 g b_2^{-1}$$

BgB is the orbit of $g \in G$ under this action

$C(w)$ are orbits: - locally closed irr subv. of G

- $\overline{C(w)} = C(w) \cup$ other BgB s.t. $\dim(BgB) < \dim C(w)$

Ex. $G = GL_2(\mathbb{C})$ $B = T_2(\mathbb{C})$

$W = S_2 = \{1, s\}$ $s = (12)$ $\dot{s} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in N_G(T)$

$C(1) = B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$

$x \in G \setminus B \Rightarrow x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $c \neq 0$, $ad - bc \neq 0$

you find that $x = u \dot{s} b'$ $\underset{B}{u} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $b' \in B$

$$x \in B\dot{s}B = C(\dot{s})$$

Act. $G = C(1) \sqcup C(\dot{s})$.

Properties of Bruhat cells. $w \in W$

1) $w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$ red. exp. $s_{i_j} \in W(S)$ $\ell = \ell(w)$

$$\varphi : \mathbb{A}^\ell(\mathbb{C}) \times B \longrightarrow G$$

$((x_1,\dots,x_\ell), b) \longmapsto u_{\alpha_{i_1}}(x_1)\dot{s}_{i_1} u_{\alpha_{i_2}}(x_2)\dot{s}_{i_2}\cdots\dot{s}_{i_\ell} b$

inj. morph. of varieties $\mathrm{im}\,\varphi \simeq C(w)$

$\Rightarrow$ $\dim C(w) = \dim B + \ell(w) \quad \forall w \in W$

BRUHAT'S LEMMA : $G = \bigsqcup_{w\in W} C(w)$ [BIBLIOGRAPHY, Springer, LAG ch. 6, 8]

1) $H = \bigcup_{w\in W} C(w) \subset G$

$G \subset H$, $GH \subset H$

This is true bec. $P_i = C(s_i) \cup C(1) \leq G$ closed conn

$G = \langle P_i \mid s_i \in W(S) \rangle$

and $P_i H \subset H \quad \forall i$ bec. $C(s)C(w) = \begin{cases} C(sw) \\ C(sw) \cup C(w) \end{cases}$

$GH \subset H$

2) Have to show that $C(w_1) \cap C(w_2) \neq \emptyset$

$\Rightarrow w_1 = w_2$

use the fact that they are orbits of $B\times B$ + + ind. on $\ell(w)$

We have our deco of G $\Rightarrow$ say something on $\mathcal{B}$

through $\pi : G \to G/B \simeq \mathcal{B}$

$g \mapsto gB$

<u>Def</u>. $\forall w \in W$, $X(w) = \pi(C(w))$ the SCHUBERT CELL rel. to w.

<u>Properties</u> : • $X(w)$ are B-orbits in G/B

$B \times G/B \to G/B$

$(b, gB) \mapsto bgB$

- $X(w)$ are disjoint, locally closed subv. of G/B
- $C(w) \simeq \mathbb{A}^{\ell(w)}(\mathbb{C}) \times B \Rightarrow X(w) \simeq \mathbb{A}^{\ell(w)}(\mathbb{C})$

<u>Def</u>. $\overline{X(w)} = S(w)$ SCHUBERT VARIETY corr to $w \in W$

<u>AIM</u>: describe Schubert varieties.

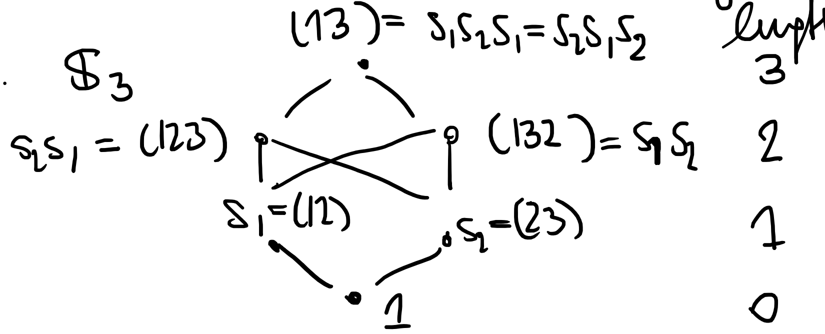
$\Rightarrow \overline{X(w)} = X(w) \sqcup$ (other B-orbits in G/B) other Schubert cells $X(w')$ s.t. $\dim X(w') < \dim X(w)$

Which ones?

<u>THM</u> [Chevalley] $S(w) = \overline{X(w)} = \bigsqcup\limits_{w' \leq w} X(w')$
in the Bruhat order on W

<u>Recall</u> $w' \underset{B.O.}{\leq} w \iff$ a reduced expression of w' can be obtained from a red. exp. of w by deleting some simple reflections

Ex. S_3



<u>Pf</u>. Equivalently, we show $\overline{C(w)} = \bigsqcup\limits_{\substack{w' \leq w \\ \text{Bruhat}}} C(w')$

$w = s_{i_1} \cdots s_{i_\ell}$ $\quad \ell = \ell(w)$ and $s_{i_j} \in W(S)$

$P_{i_j} = C(s_{i_j}) \cup C(e) \underset{\text{conn. closed}}{\leq} G$

$C(w) \subset P := P_{i_1} \cdots P_{i_\ell} = \bigcup\limits_{w' \leq w} C(w')$ $\Big|$ $C(s)C(w) = \begin{cases} C(sw) \\ \ldots \end{cases}$

$$\underset{\text{irr}}{C(w)} \subset P := \underbrace{P_{i_1} \cdots P_{i_\ell}}_{\text{irr. closed}} = \bigcup_{w' \leq w} C(w') \quad \Bigg| \; C(s)C(w) = \begin{cases} C(sw) \\ \cup \\ C(w) \end{cases}$$

$$\Rightarrow \underset{\substack{\text{irr} \\ \dim = \ell(w)}}{\overline{C(w)}} \subset P = \bigcup_{w' \leq w} C(w')$$

$\dim = \ell(w) \Rightarrow$ equality $\blacksquare$

$\Rightarrow$ $(X(w))_{w \in W}$ give a stratification of $\mathcal{B} \cong G/B$ into finitely many loc. closed smooth affine strata

Application cohomology of $\mathcal{B}$

Regard $\mathcal{B}$ as a compact differentiable manifold of dim $2\ell(w_0)$

$\Rightarrow$ The SCHUBERT CLASSES are $[S(w)]$ $w \in W$

- $C(w_0)$ open dense G
- $\Rightarrow X(w_0)$ open dense in G/B
- Ex. compute dimensions.

they form an additive basis of $H^*(\mathcal{B})$

$\dim S(w) = \ell(w) \Rightarrow [S(w)] \in H^{2(\ell(w_0) - \ell(w))}(\mathcal{B})$

Poincaré duality

Eg. $G = GL_2(\mathbb{C}) \geq B = T_2(\mathbb{C})$

$$G/B \simeq \mathcal{F} = \{(0 \subsetneq \langle v \rangle \subsetneq \mathbb{C}^2) \mid 0 \neq v \in \mathbb{C}\} \simeq \mathbb{P}^1(\mathbb{C})$$

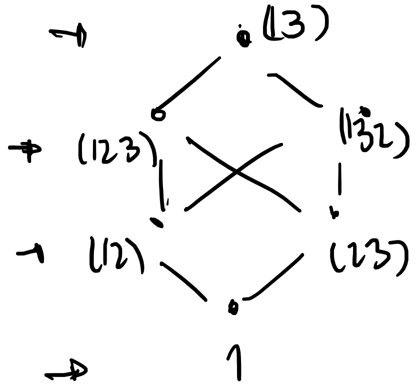
$$\begin{array}{cccc} & C(1) & & C(s) \\ G = & B & \sqcup & B\dot{s}B \\ \pi \downarrow & & & \\ G/B = & \{pt\} & \sqcup & X(s) \simeq \mathbb{A}^1(\mathbb{C}) \end{array}$$

$$H^*(\mathcal{B}, \mathbb{C}) \underset{\text{Vr}}{=} H_0(\mathcal{B}, \mathbb{C}) \oplus H_2(\mathcal{B}, \mathbb{C})$$

P. poly $1 + t^2$

$H^*(\mathbb{B}, \mathbb{C}) \underset{VS}{\cong} H_0(\mathbb{B}, \mathbb{C}) \oplus H_2(\mathbb{B}, \mathbb{C})$ P. poly $1 + t^2$

eg. $n = 3$



length	degree
3	0
2	2
1	4
0	6

Poincaré Poly : $1 + 2t^2 + 2t^4 + t^6$.