BGG Category $\mathcal{O}$

$\mathcal{O}$ full subcategory of $U(\mathfrak{g})$-mod

contains
- fd$^{\ell}$ modules
- hw modules
- simple modules

$\mathfrak{g}$ ss Lie alg / $\mathbb{C}$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$$

$$\mathfrak{sl}(n+1) \quad \begin{pmatrix} 0 & & 0 \\ & \ddots & \\ * & & 0 \end{pmatrix} \oplus \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \oplus \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

$$\mathfrak{g}\text{-mod} \rightsquigarrow U(\mathfrak{g})\text{-mod}$$

$$U(\mathfrak{g}) = T(\mathfrak{g}) \Big/ \left(x\otimes y - y\otimes x - [x,y]_{\mathfrak{g}} \quad x,y\in\mathfrak{g} \right)$$

$\cong$ v.space

$$U(\mathfrak{n}^-) \otimes \underbrace{U(\mathfrak{h})}_{\text{commutative}} \otimes U(\mathfrak{n})$$

$$f_{\alpha_N}^{a_N} \cdots f_{\alpha_1}^{a_1} \qquad h_1^{b_1} \cdots h_n^{b_n} \quad e_{\alpha_1}^{c_1} \cdots e_{\alpha_N}^{c_N}$$

$\underline{sl(n+1)}$

$f_\alpha \rightsquigarrow e_{ij} \quad i>j$

$h_i \rightsquigarrow e_{ii} - e_{i+1,i+1}$

$e_\alpha \rightsquigarrow e_{ij} \quad i<j$

Recall

① V $U(\mathfrak{g})$-mod , $\lambda \in \mathfrak{h}^*$

$$V_\lambda = \{ v \in V \mid h.v = \lambda(h) v \ \forall h \in \mathfrak{h} \}$$

weight spaces

$$e_\alpha \cdot V_\lambda \subseteq V_{\lambda+\alpha}$$

$$f_\alpha \cdot V_\lambda \subseteq V_{\lambda-\alpha}$$

② highest weight modules :

$$V = U(\mathfrak{g}) v \quad , \quad \begin{cases} v \in V_\lambda \\ \mathfrak{n}.v = 0 \end{cases}$$

maximal vector

$$U(\mathfrak{g})v = U(\mathfrak{n}^-) \cancel{U(\mathfrak{g})} U(\cancel{\mathfrak{n}}) v$$

$$= \underline{U(\mathfrak{n}^-) v}$$

$\dim V_\lambda = 1$, $V \subseteq \bigoplus_{\beta \in \mathbb{Z}_+ \Phi^+} V_{\lambda - \beta}$

$\dim V_\mu < \infty \quad \forall \mu \in \mathfrak{h}^*$

Rk Simple modules are hwm

$\forall \lambda \in \mathfrak{h}^*$ $\exists!$ universal

hw module $\boxed{M(\lambda) \twoheadrightarrow V}$

projecting on any other hwm

of hw λ.

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{n}+\mathfrak{h})} \mathbb{C}_\lambda$$

$$\mathfrak{n}.1 = 0 \;, \quad h.1 = \lambda(h)$$

$$M(\lambda) = U(\mathfrak{g}) \Big/ \left(\underset{h \in \mathfrak{h}}{h - \lambda(h)1}, \; e_\alpha, \alpha \in \Phi^+ \right)$$

CATEGORY $\mathcal{O}$

Full subcategory of $U(\mathfrak{g})$-mod whose objects M satisfy

① M f.g.

② $M = \bigoplus_{\lambda \in \mathfrak{h}} M_\lambda$

I can find generators that are weight vectors (f. many)

③ $\forall v \in M$, $U(\mathfrak{n}).v$ is finite dimensional

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- Highest weight modules are objects in $\mathcal{O}$

- M is an object in $\mathcal{O}$ then $\dim M_\lambda < \infty$

pf similar to the pf that in hw modules $\dim V_\mu < \infty$ (use PBW).

$\rightsquigarrow$ M in $\mathcal{O}$ then

$$\operatorname{ch} M = \sum_\lambda \dim M_\lambda \, e^\lambda$$

$$e^\lambda(\mu) = \delta_{\lambda\mu} \qquad e^\lambda : \mathfrak{h}^* \longrightarrow \mathbb{C}$$

THM ⓐ $\mathcal{O}$ is noetherian
(because $U(\mathfrak{g})$ is so)

ⓑ $\mathcal{O}$ closed to taking submodules, quotients, finite $\oplus$

ⓒ $\mathcal{O}$ is abelian
($U(\mathfrak{g})$-mod is abelian)

ⓓ L fd^{e} $U(\mathfrak{g})$-mod

$$\mathcal{O} \longrightarrow \mathcal{O}$$
$$M \longrightarrow L \otimes M$$

$$\left(\underset{x \in \mathfrak{g}}{x.} (\ell \otimes m) = x.\ell \otimes m + \ell \otimes x.m \right)$$

is an exact functor

(e) $Z(\mathfrak{g}) = Z(U(\mathfrak{g}))$

preserves M_λ weight spaces

M in $\mathcal{O}$, $v \in M$

$Z(\mathfrak{g})v$ is finite dim^e

$v \in \bigoplus_{\lambda \text{ finite}} M_\lambda \Rightarrow \forall z \in Z(\mathfrak{g})$

$zv \in \bigoplus_{\lambda \text{ finite}} M_\lambda$ finite dimensional

(f) $M \in \mathcal{O}$ is a f.g. $U(\mathfrak{n}^-)$-module

(g) Duality:

M in $\mathcal{O}$ $\quad M^\vee = \bigoplus_\lambda M_\lambda^*$

$x \in \mathfrak{g}$

$(x \cdot f)(w) = f(\tau(x) w)$

$\tau(e_\alpha) = f_\alpha$, $\tau(f_\alpha) = e_\alpha$,

$\tau(h_i) = h_i$ $\qquad \tau : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{op}$

(9) Every object in $\mathcal{O}$ has a finite filtration whose quotients are hw modules.

DECOMPOSITION OF $\mathcal{O}$

$Z(\mathfrak{g}) \circlearrowright M$ $\qquad$ M in $\mathcal{O}$

on hw modules : $M = U(\mathfrak{g})v$

$v \in M_\lambda$

$e_\alpha v = 0$

$z \in Z(\mathfrak{g})$

$zv \in M_\lambda \leftarrow \underline{\dim 1}$

$zv = \chi_\lambda(z)\, v$, $z \in Z(\mathfrak{g})$

$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$

$\rightsquigarrow$ z acts by $\chi_\lambda(z)$ on any element of M.

χ_λ depends on λ, not on the choice of the hw module.

$M(\lambda)$ Verma module
$\parallel$
$U(\mathfrak{g})v$

$$z = \sum_{\underline{a},\underline{b},\underline{c}} \ell \, f_{\alpha_N}^{a_N} \cdots f_{\alpha_1}^{a_1} h_1^{b_1} \cdots h_n^{b_n} \underbrace{e_{\alpha_1}^{c_1} \cdots e_{\alpha_N}^{c_N}}_{\text{DO NOT contr.}}$$

$$\underset{\substack{\cap \\ M(\lambda)_\lambda}}{zv} = \sum_{\underline{a},\underline{b}} \ell_{\underline{a},\underline{b},\underline{0}} \, \overbrace{f \cdots f}^{} \, \lambda(h_1^{b_1} \cdots h_n^{b_n}) v$$

$$zv = \lambda(\mathrm{pr}_{\mathfrak{h}} z) v$$

$$\mathrm{pr}_{\mathfrak{h}} : U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}) \longrightarrow 1 \otimes U(\mathfrak{h}) \otimes 1$$

$$\chi_\lambda(z) = \lambda(\mathrm{pr}_{\mathfrak{h}}(z)) \quad \forall \lambda \in \mathfrak{h}^*$$

$$\mathrm{HC} : Z(\mathfrak{g}) \longrightarrow U(\mathfrak{h})$$

$$\mathrm{HC} = \mathrm{pr}_{\mathfrak{h}}\big|_{Z(\mathfrak{g})}$$

alg. morphism

Recall dot action of W on $\mathfrak{h}^*$

$\rho \in \mathfrak{h}^*$ $\quad \rho(h_i) = 1 \quad i = 1, \dots, n$

$$w \bullet \lambda = w(\lambda + \rho) - \rho$$

<u>Thm</u>

1) $\lambda, \mu \in \mathfrak{h}^*$ Then

$$\chi_\lambda = \chi_\mu \iff \lambda \in W \bullet \mu$$

2)
$$\begin{tikzcd} \widetilde{HC} : Z(\mathfrak{g}) \arrow[r, "HC"] & S(\mathfrak{h}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & S(\mathfrak{h}) \\ & \mathbb{C}[\mathfrak{h}^*] \arrow[ur] & \end{tikzcd}$$

$$p(\lambda) \longmapsto p(\lambda - \rho)$$

is inj. its image is $S(\mathfrak{h})^W$

3) All central characters

are of the form χ_λ for some λ.

The subcategories $\mathcal{O}_{\chi_\lambda}$

M in $\mathcal{O}$ $\quad$ χ central character

$$M^\chi = \{ v \in M \mid (z - \chi(z))^{n_z} v = 0 \}$$
$\forall z \in Z(\mathfrak{g}),\ n_z \in \mathbb{N}$

is a submodule $\Rightarrow$ M^χ in $\mathcal{O}$

$$M_\mu = \bigoplus_\chi (M^\chi \cap M_\mu)$$

$$\rightsquigarrow \quad M = \bigoplus_{\chi \text{ char}}^{\text{finite}} M^\chi$$

$\mathcal{O}_\chi$: full subcategory on which $M = M^\chi$.

PROP $\mathcal{O} = \bigoplus_{\lambda \in \mathfrak{h}^*/W} \mathcal{O}_{\chi_\lambda}$

$\mathcal{O}_{\chi_\lambda}$ contains only f.m. simples, fm Verma modules

All indecomposables lie in some $\mathcal{O}_{\chi_\lambda}$.

RK When λ is an integral weight $\Rightarrow$

$\mathcal{O}_{\chi_\lambda}$ is a block in the category

$\lambda, \mu \in \mathfrak{h}^*$, $\nu = \lambda - \mu \in \mathbb{Z}\Phi$ $\quad L(\bar{\nu})$

$$T_\lambda^\mu : \mathcal{O}_{\chi_\lambda} \longrightarrow \mathcal{O}_{\chi_\mu}$$

$$M \longrightarrow pr_\mu(L(\bar{\nu}) \otimes M)$$

<u>PROP</u> 1) $\lambda, \mu \in \mathfrak{h}^*$

T_λ^μ is exact, commutes with $^\vee$, preserves projectives

2) $\mathrm{Hom}_{\mathcal{O}}(T_\lambda^\mu M, N) \cong \mathrm{Hom}_{\mathcal{O}}(M, T_\mu^\lambda N)$

3) conditions on λ, μ to guarantee that they are equivalences