

p -adic representations and arithmetic \mathcal{D} -modules Grassmannians

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Definition

Plücker embedding

Totally
decomposable
vectors

Main theorem

Examples

Flag varieties

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Totally decomposable vectors

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Main idea.

To parametrize higher dimensional subspaces of a fixed (complex) vector space V .

Set-theoretic definition

Definition. Let us suppose that V is an n -dimensional vector space. We define the **Grassmannian** $G(k, V)$ as follows:

$$G(k, V) := \{U \leq V \mid \dim(U) = k\}.$$

Objective

To realize Grassmannians as projective varieties.

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Dimension of the Grassmannian

Key observation

\mathbb{G}_m acts on $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \mathbb{G}_m.$$

Geometric description

Fixing a basis $\{v_1, \dots, v_n\}$ of V we have $V \simeq \mathbb{C}^n$ and

$$G(k, n) = \{U \subseteq \mathbb{C}^n \mid \dim(U) = k\}.$$

The **Linear group** GL_k acts on $M_{k \times n}$ on the left and

$$G(k, n) = M_{k \times n}^{\mathrm{rank} \ k} / \mathrm{GL}_k$$

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If the first $k \times k$ minor is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & \cdots & b_{2,n-k} \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & \cdots & b_{k,n-k} \end{pmatrix}$$

\Updownarrow bijection

$$\mathbb{A}^{k(n-k)}$$

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We have fixed a basis $\mathcal{B} := \{v_1, \dots, v_n\}$ of V .

The exterior algebra

Expressing $w_1, \dots, w_k \in V$ in terms of the basis \mathcal{B} we have

$$\left(\sum_{i=1}^n a_{i,1} v_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^n a_{i,k} v_i \right) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det(A(ik)) v_{i_1} \wedge \cdots \wedge v_{i_k}$$

Example. If $\mathcal{A} := \{w_1, \dots, w_n\}$ is another basis then $\det(A(nn))$ is the determinant of the change of basis matrix $\mathcal{A} \rightarrow \mathcal{B}$.

The Plücker embedding

$$\begin{aligned} i : G(k, n) &\rightarrow \mathbb{P}(\wedge^k V) \\ W &\mapsto [w_1 \wedge \cdots \wedge w_k] \end{aligned}$$

$\{w_1, \dots, w_k\}$ is a basis for W . The map i is well defined by the preceding relation.

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Lemma. i is injective.

Remark. The **Plücker embedding** allows us to view $G(k, n)$ as subset of the projective space $\mathbb{P}^{\binom{n}{k}-1}$.

Plücker coordinates

$$I_{k,n} := \{\vec{i} := (i_1, \dots, i_k) \mid 1 \leq i_1 < \dots < i_k \leq n\}$$

the set of coordinates of $\mathbb{P}(\wedge^k V)$. We want to compute $\vec{i}(W)$.

We take $\mathcal{B}_W := \{w_1, \dots, w_k\}$ a basis of W . The basis \mathcal{B} of V gives rise to a matrix $M_W(a_{ij}) \in M_{n \times d}$. We have

$$w_1 \wedge \dots \wedge w_k = \sum_{\vec{i} \in I_{k,n}} \sum_{\sigma \in \mathcal{S}_k} \text{sig}(\sigma) a_{i_1, \sigma(1)} \dots a_{i_k, \sigma(k)} v_{\vec{i}}.$$

$\vec{i}(W) = \det(M_{\vec{i}})$, with $M_{\vec{i}}$ the $k \times k$ sub-matrix formed from the \vec{i} rows of M_W .

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Remark. The **Plücker embedding** allows us to view $G(k, n)$ as subset of the projective space $\mathbb{P}^{\binom{n}{k}-1}$.

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We will need the following notion.

Definition. Let $w \in \bigwedge^k V$. We say that w is **totally decomposable** if we can write $w = w_1 \wedge \cdots \wedge w_k$, with $\{w_1, \dots, w_k\} \subset V$ l.i.

Preparation

Lemma. $[w] \in \mathbb{P}(\bigwedge^k V)$ lies in the image of the Grassmannian under the Plücker embedding if and only if w is totally decomposable.

$$(w \in \bigwedge^k V) \quad L_w := \{v \in V \mid v \wedge w = 0 \text{ in } \bigwedge^{k+1} V\}$$

Lemma. The space L_w has dimension at most k . With equality occurring if and only if w is totally decomposable.

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$$\begin{aligned} \varphi: \Lambda^k V &\rightarrow \text{Hom}(V, \Lambda^{k+1} V) \\ w &\mapsto \varphi(w) := w \wedge (\bullet) \end{aligned}$$

It is a linear map

Theorem. $i(G(k, n)) \subset \mathbb{P}(\Lambda^k V)$ is a projective variety.

Proof. First of all

$$[w] \in i(G(k, n)) \quad \Leftrightarrow \quad \text{rank}(\varphi(w)) = n - k$$

- The matrix $A(w)$ of $\varphi(w)$ has homogeneous entries of degree 1 in the coordinates
- $G(k, n)$ is the vanish of all $(n - k - 1) \times (n - k - 1)$ -minors of $A(w)$.

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- The Plücker embedding of $G(1, V)$ maps a linear subspace

$$W = \text{span}(\lambda_1 v_1 + \cdots + \lambda_n v_n) \quad (\lambda_1 : \cdots : \lambda_n) \in \mathbb{P}^{n-1}$$

As expected $G(1, V) = \mathbb{P}^{n-1}$.

- $\dim(V) = 3$ and $W = \text{span}(v_1 + v_2, v_1 + v_3) \in G(2, V)$. Since

$$(v_1 + v_2) \wedge (v_1 + v_3) = -v_1 \wedge v_2 + v_1 \wedge v_3 + v_2 \wedge v_3$$

the Plücker coordinates of W in \mathbb{P}^2 are given by the vector $(-1 : 1 : 1)$. Alternatively,

$$M_W = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $(-1:1:1)$ keeps the 2×2 -minors of M_W .

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Plücker quadratic

Let $\dim(V) = 4$.

The Plücker coordinates are the 2×2 -minors

$$p_{ij} := \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = x_i y_j - x_j y_i$$

of the 4×2 -matrix

$$M = (x_i \ y_i)_{0 \leq i \leq 3}$$

The Plücker embedding is defined by

$$\begin{aligned} i: G(2,4) &\rightarrow \mathbb{P}^5 \\ W &\mapsto (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) \end{aligned}$$

and the Plücker coordinates satisfy the **quadratic Plücker relation**

$$p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0.$$

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$$\begin{aligned} i: G(2,4) &\rightarrow \mathbb{P}^5 \\ W &\mapsto (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}) \end{aligned}$$

and the Plücker coordinates satisfy the **quadratic Plücker relation**

$$p_{01} p_{23} + p_{02} p_{31} + p_{03} p_{12} = 0.$$

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Let $\dim(V) = 4$.

The Plücker coordinates are the 2×2 -minors

$$p_{ij} := \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = x_i y_j - x_j y_i$$

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Plücker quadratic: bijection

Let $(q_{01} : q_{02} : q_{03} : q_{23} : q_{31} : q_{12}) \in \mathbb{P}^5$. If $q_{01} \neq 0$

$$M_W = \begin{pmatrix} q_{01} & 0 \\ 0 & q_{01} \\ -q_{12} & q_{02} \\ q_{31} & q_{03} \end{pmatrix}$$

has rank 2 (defining a line W in \mathbb{P}^3).

Key point:

If the coordinates q_{ij} satisfy the quadratic relation, then they are the coordinates of W .

We have proved

$$G(2, 4) = V(p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}) \subset \mathbb{P}^5$$

is the **Plücker quadratic**.

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$$\dim_{\mathbb{C}}(V) = n.$$

Definition. A **flag** in V is a strictly increasing sequence of subspaces

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l \subsetneq V.$$

The **signature** of the flag is $(\dim(V_1), \dots, \dim(V)_l)$.

For every sequence of integers

$$\underline{a} := 0 < a_1 < \cdots < a_l < n$$

We let

$$\mathbb{F}(\underline{a}; n) := \{0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l \subsetneq V \mid \dim(V_i) = a_i\}$$

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Remark. $\mathbb{F}(a_1; n) = G(a_1, n)$ is a projective variety.

Proposition. $\mathbb{F}(\underline{a}; n)$ is a Zariski closed subset of $\prod_{i=1}^l G(a_i, n)$.

Proof.

- The case $\underline{a} \in \mathbb{Z}_{>0}$ is the preceding remark.

Reduction to $\underline{a} \in \mathbb{Z}_{>0}^2$.

- Let

$$\pi_{ij} : G(a_1, n) \times \cdots \times G(a_l, n) \rightarrow G(a_i, n) \times G(a_j, n) \times G(a_j, n)$$

be the projection.

$$\mathbb{F}(\underline{a}; n) = \bigcap_{1 \leq i < j \leq l < n} \pi_{ij}^{-1}(\mathbb{F}(a_i, a_j; n))$$

It is enough to consider $l = 2$.

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Let $r < s$ and

$$\underbrace{(\text{span}(u_1, \dots, u_r))}_U, \underbrace{(\text{span}(w_1, \dots, w_s))}_W \in G(r, n) \times G(s, n).$$

If $u := u_1 \wedge \dots \wedge u_r$, $w := w_1 \wedge \dots \wedge w_s$, and

$$\begin{aligned} \varphi \oplus \varphi: \underbrace{\bigwedge^r V \oplus \bigwedge^s W}_{\underline{v}_1 \oplus \underline{v}_2} &\rightarrow \text{Hom}(V, \bigwedge^{r+1} V \oplus \bigwedge^{s+1} V) \\ &\mapsto \underbrace{\underline{v}_1 \wedge (\bullet)}_{\varphi_{\underline{v}_1}} \oplus \underbrace{\underline{v}_2 \wedge (\bullet)}_{\varphi_{\underline{v}_2}}, \end{aligned}$$

$\text{then } \ker(\varphi_u \oplus \varphi_w) = U \cap W$

$$U \subsetneq W \Leftrightarrow \text{rank}(\varphi_u \oplus \varphi_w) = n - r$$

- The $(n - r + 1) \times (n - r + 1)$ -minors give polynomial conditions for $\mathbb{F}(r, s; n)$.

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Example

$$\dim(V = \text{span}(v_1, \dots, v_4)) = 4.$$

Let

$$(\text{span}(u), \text{span}(w_1, w_2)) \in \mathbb{F}(1, 2; 4)$$

with

$$u = \sum_{i=1}^4 a_i v_i \text{ and } w_1 \wedge w_2 = \sum_{i < j} b_{ij} v_i \wedge v_j.$$

By definition

$$0 = u \wedge w_1 \wedge w_2 = \sum_{i < j < k} (a_i b_{jk} - a_j b_{ik} + a_k b_{ij}) v_i \wedge v_j \wedge v_k$$

and

$$\mathbb{F}(1, 2; 4) = V(X_1 X_{2,3} - X_2 X_{13} + X_3 X_{12}, \dots)$$

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$$\mathbb{F}(1, 2; 4) = V(X_1 X_{2,3} - X_2 X_{13} + X_3 X_{12}, \dots)$$

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Example

$$\dim(V = \text{span}(v_1, \dots, v_4)) = 4.$$

Let

$$(\text{span}(u), \text{span}(w_1, w_2)) \in \mathbb{F}(1, 2; 4)$$

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$$u = \sum_{i=1}^4 a_i v_i \text{ and } w_1 \wedge w_2 = \sum_{i < j} b_{ij} v_i \wedge v_j.$$

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Complete flag varieties

Definition. A flag variety $\mathbb{F}(1, \dots, n-1; n)$ is called **complete**.

Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{C}^n . we have a **full flag**

$$\mathcal{F} := 0 \subsetneq \mathbb{C} \cdot e_1 \subsetneq \dots \subsetneq \bigoplus_{i=1}^{n-1} \mathbb{C} \cdot e_i \subsetneq \mathbb{C}^n$$

Facts.

- GL_n acts transitively on complete flag varieties.
- The stabilizer $\mathbb{B} := \mathrm{stab}_{\mathrm{GL}_n}(\mathcal{F})$ is the (**Borel**) subgroup of upper triangular matrices in GL_n , and

$$\mathrm{GL}_n/\mathbb{B} = \mathbb{F}(1, \dots, n-1; n).$$

In particular, $\mathbb{P}^1 = \mathbb{F}(1, 2) = G(1, 2) = \mathrm{GL}_2 / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$.

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