p-adic representations and arithmetic $\mathcal{D}$-modules

Kashiwara's equivalence, minimal extensions and Riemann-Hilbert correspondence

(most basic version)

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GOOD FILTRATIONS

The Weyl Algebra

$$A_n(\mathbb{C}) = \mathbb{C}[x's, \partial x's] / ([\partial_{x_i}, x_j] - \delta_{ij})$$

Bernstein Filtration: let $P = \sum_{(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^n}^{<\infty} \gamma_{(\alpha,\beta)} x^\alpha \partial^\beta$. We set

$$\deg(P) = \max\{|\alpha| + |\beta| \mid \gamma_{(\alpha,\beta)} x^\alpha \partial^\beta \text{ appearing in } P\}$$

$$B_k = \operatorname{Span}_{\mathbb{C}}(\{P \in A_n(\mathbb{C}) \mid \deg(P) \leq k\})$$

$\{B_k\}$ is a filtration for $A_n(\mathbb{C})$. Like in the order filt. We've

$$gr^B(A_n(\mathbb{C})) = \mathbb{C}[\underline{x}, \underline{\xi}] \qquad \sigma_1 : B_1 \longrightarrow gr^B_1(A_n(\mathbb{C}))$$

ξ_i principal symbol of ∂x_i $\qquad \partial x_i \longmapsto \xi_i$

DIMENSION THEORY

Hilbert polynomial

Thm: $M = \bigoplus_{i \in \mathbb{N}} M_i$ f.g graded module over $\mathbb{C}[x_1, \dots, x_n]$

$\exists\, \chi(t) \in \mathbb{Q}[t]$ and $N \in \mathbb{Z}_{>0}$ such that

$$\forall s \geq N \qquad \sum_{i=0}^{s} \dim_{\mathbb{C}} (M_i) = \chi(s)$$

- $\chi(t)$ is known as the Hilbert polynomial

- M is a f.g A_n-mod $\xrightarrow[\text{filt. } \Gamma]{}$ • $gr^{F}(M)$ f.g $gr^{B}(A_n(\mathbb{C}))$

  • $\chi(t, \Gamma, M)$ Hilbert polynomial of $gr^{\Gamma}(M)$.

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- $\chi(t, \Gamma, M)$ Hilb. polynomial of $gr^{\Gamma}(M)$

For $s >> 0$

$$\chi(s, \Gamma, M) = \sum_{i=0}^{s} \dim_{\mathbb{C}} \left(\Gamma_i M / \Gamma_{i-1} M \right) = \dim_{\mathbb{C}} (\Gamma_k M)$$

- $d(M) := \deg(\chi(s, \Gamma, M))$ is the dimension of M

<u>Remark:</u> χ does not depend of the filtration.

EXAMPLES

(1) Dimension of the Weyl algebra

$s >> 0 \quad \chi(s, B, A_n) = \dim_{\mathbb{C}}(B_s) = \#$ non-negative sol's $|\alpha| + |\beta| \leq s$

$$= \binom{s+2n}{2n}$$

$$\deg(\chi(t, B, A_n)) = 2n \implies d(A_n) = 2n$$

(2) Dimension of the ring of polynomials.

$\Gamma_k \mathbb{C}[\underline{x}] := \mathrm{Span}_{\mathbb{C}}(\{x^\alpha \mid |\alpha| \leq k\})$. $\dim_{\mathbb{C}} \Gamma_s = \binom{n+s}{s}$

$$\deg(\chi(t, \Gamma, \mathbb{C}[\underline{x}])) = n \implies \dim(\mathbb{C}[\underline{x}]) = n.$$

(3) Fourier transform

$$\mathcal{F} : A_n(\mathbb{C}) \longrightarrow A_n(\mathbb{C})$$
$$x_i \longmapsto \partial x_i$$
$$\partial x_j \longmapsto -x_j$$

- $\mathcal{F}(B_k) = B_k$ and defining $M_{\mathcal{F}} = \begin{cases} M_{\mathcal{F}} = M \\ p.m = \mathcal{F}(p)m \end{cases}$

We have $d(M) = d(M_{\mathcal{F}})$

Bernstein inequality:

<u>Thm</u>: If M is a f.g. non-zero left $A_n(\mathbb{C})$-module, then

$$d(M) \geqslant n$$

<u>Proof</u>: $M = \text{Span}_{A_n(\mathbb{C})}(u_1, \dots, u_s)$. $\Gamma_k = \sum_{i=1}^{s} B_k u_i$

$gr^{\mathcal{F}}(M) = \text{Span}(\{u_1, \dots, u_s\})$ and $B_i \longrightarrow Hom_{\mathbb{C}}(\Gamma_i, \Gamma_{2i})$

$gr^{B}(A_n(\mathbb{C}))$

$a \longmapsto \varphi_a(u) := au$.

$\cdot\ B_i \hookrightarrow \mathrm{Hom}_{\mathbb{C}}(\Gamma_i, \Gamma_{2i})$. For $s \gg 0$

$$\binom{s+2n}{n} = \dim_{\mathbb{C}}(B_s) \leq \dim_{\mathbb{C}} \mathrm{Hom}(\Gamma_s, \Gamma_{2s}) = \dim_{\mathbb{C}}(\Gamma_s) \cdot \dim(\Gamma_{2s}) = \chi(s) \cdot \chi(2s)$$

We have $\quad 2n \leq \deg\left(\chi(t)\chi(2t)\right) = 2d(M)$

So, $\quad n \leq d(M)$ ▪

$$\dim(A_n) = 2n$$

Remarks : (i) M. f.g A_n-mod : $A_n^r \overset{\varphi}{\twoheadrightarrow} M \Rightarrow \overset{\shortparallel}{\dim}(A_n^r) = \max\{d(M), d(\ker(\varphi))\}$

(ii) The bounds are atteined by $\quad d(A_n) = 2n \quad$ and $\quad d(\mathbb{C}[\underline{x}]) = n$.

HOLONOMIC A_n-MODULES

Definition: A f.g A_n-mod M is holonomic if $d(M) = n$.

Examples: (i) $\mathbb{C}[\underline{x}]$ is holon.

(ii) $A_n(\mathbb{C})$ is not holon.

(iii) If $\chi(t, \Gamma, M) = a_d t^d + \dots$ $m(M) = d!\, a_d$ the multiplicity (>0).

Claim: $(0) \neq I \subsetneq A_1$ then A_1/I is holonomic.

In fact, if $(0) \neq I \subsetneq A_n$ then $\dim(A_n/I) \leq 2n-1$.

Reason: It is enough to consider $I = A_n \cdot P$

The exact sequence $0 \to A_n \xrightarrow{(\cdot)P} A_n \longrightarrow A_n/I \to 0$

$$\left\{ \begin{array}{l} \text{If } \dim(A_n/I) = \dim(A_n) = 2n \\ \Downarrow \\ \bullet\ m(A_n) = m(A_n) + m(A_n/I) \end{array} \right.$$

IMPOSSIBLE!

CHARACTERISTIC VARIETIES

<u>Definition</u>: M f.g A_n-module:

$$Ch(M) = V\left(\sqrt{\mathrm{Ann}_{\mathbb{C}[\underline{x},\underline{\xi}]}(gr(M))}\right) \subseteq \mathbb{C}^{2n} = T^*\mathbb{A}^n_{\mathbb{C}}.$$

<u>Connecting the definitions</u> : X smooth alg. variety and $U=(x_i, dx_i) \subseteq X$

$$Ch(\mathcal{M}) \cap T^*U = \left\{ p \in T^*U \mid f(p) = 0 \quad \forall f \in \sqrt{\mathrm{Ann}(gr(\mathcal{M}|_U))} \right\}$$

We also find

[Hartshorne]

$$\dim(M) = \dim\left(V\left(\sqrt{\mathrm{Ann}(gr(M))}\right)\right) = \dim(\mathrm{supp}(gr(M))) \overset{\downarrow}{=} d(M)$$

$$\| $$

$$\mathrm{Krulldim}\left(gr(A_n(\mathbb{C})) \big/ \sqrt{\mathrm{Ann}(gr(M))} \right)$$

$$= 2n - j(M)$$

PRODUCING HOLONOMIC D-MODULES

Remarks: (i) : $A_n \otimes A_m \xrightarrow{\simeq} A_{n+m}$ $\quad ([A_n, A_m] = 0)$.

$$P \otimes Q \longmapsto P.Q$$

(ii) M left A_n-mod & N a left A_m-module $\begin{cases} M \otimes N \text{ is a left } A_{n+m}\text{-mod} \\ PQ.(m \otimes n) = Pm \otimes Qn \end{cases}$

(iii) $B_s(A_{m+n}) = \sum'_{p+q=s} B_p(A_m)\, B_q(A_n)$

(iv) $\Gamma_k(M \otimes N) = \sum_{i+j=k} \Gamma_i(M) \otimes \Gamma_j(N)$ & $gr(M \otimes N) = \bigoplus_{i+j=k} gr_i(M) \otimes gr_j(N)$

$$\Gamma_k(M\otimes N)=\sum_{i+j=k}{}' \Gamma_i(M)\otimes\Gamma_j(N) \quad \& \quad gr_k(M\otimes N)=\bigoplus_{i+j=k} gr_i(M)\otimes gr_j(N)$$

$$\dim_{\mathbb{C}}(\Gamma_k(M\otimes N))=\sum_{r=0}^{k}\sum_{r_i+r_j=r}{}' \dim_{\mathbb{C}} gr_{r_i}(M)\cdot\dim_{\mathbb{C}} gr_{r_j}(N)$$

$$\leq\sum_{i=0}^{k}\dim_{\mathbb{C}}(gr_{r_i}(M))\cdot\sum_{j=0}^{k}\dim gr_{r_j}(N)$$

$$=\dim_{\mathbb{C}}\Gamma_k(M)\cdot\dim_{\mathbb{C}}\Gamma_k(N)\leq\dim\Gamma_{2k}(M\otimes N)$$

For $k\gg 0$ $\quad \chi(k,M\otimes N)\leq\chi(k,M)\cdot\chi(k,N)\leq\chi(2k,M\otimes N)$

$\Rightarrow \dim(M\otimes N)=\dim(M)+\dim(N)$

$\left\{\begin{array}{l} M \text{ holon. } A_n \ \& \ N \text{ holon } A_m \\ \Rightarrow \\ M\otimes N \text{ holon } A_{n+m}\text{-mod.}\end{array}\right.$

Inverse images

Notation: $K[x] := K[x_1, \dots, x_n]$ and $K[y] := K[y_1, \dots, y_m]$ $M \in \mathrm{Mod}(A_m)$

$F : (F_1, \dots, F_m) : X \to Y$: $F^{q} : K[Y] \longrightarrow K[x] \rightsquigarrow F^* M := K[x] \underset{K[y]}{\otimes} M$

$$(*) \dots \quad \partial_{x_i} (q \otimes u) = \partial_i (q) \otimes u + \sum_{k=1}^{m} q \, \partial_{x_i} (F_k) \otimes \partial_{y_k} u .$$

Projection: $\pi : X \times Y \longrightarrow Y$ | $\pi^* M = K[X, Y] \underset{K[Y]}{\otimes} M = K[x] \underset{K}{\otimes} M$

$x^\alpha q_\alpha \otimes m \longmapsto x_\alpha \otimes q_\alpha m$

. $\pi^* M$ is an A_{m+n} - module.

Compatibility with $(*)$
$$\begin{cases} \cdot \ \partial y_j (x^\alpha q_\alpha \otimes u) = x^\alpha \otimes \partial y_j (q_\alpha u) \\ \\ \cdot \ \partial x_i (x^\alpha q_\alpha \otimes u) = \partial_{x_i} (x^\alpha) \otimes q_\alpha u . \end{cases}$$

HOLONOMICITY UNDER INVERSE IMAGES (Projections)

<u>Theorem</u>: let M be a f.g A_m-module and

$\pi : X \times Y \longrightarrow Y$ the projection.

$$d(\pi^* M) = n + d(M)$$

<u>Proof</u>: $d(\pi^* M) = d(k[x] \otimes_k M) = d(k[x]) + d(M) = n + d(M)$

<u>Corollary</u>: If $M \in \underset{hol}{\mathrm{Mod}}(A_m) \Rightarrow \pi^* M \in \underset{hol}{\mathrm{Mod}}(A_{m+n})$.

TRANSFERT MODULES

- $F^* M = K[X] \otimes_{K[Y]} M = \underbrace{K[X] \otimes_{K[Y]} A_m}_{} \otimes_{A_m} M = D_{X \to Y} \otimes_{A_m} M$

- If $N \in \mathrm{Mod}(A_n)^{op} \Rightarrow F_*(N) = N \otimes_{A_n} D_{X \to Y}$

- (Right $\longleftrightarrow$ Left) : $\tau : A_n \to A_n : p(x)\partial^\alpha \longmapsto (-1)^{|\alpha|} \partial^\alpha h$

$$D_{Y \leftarrow X} := (D_{X \to Y})^\tau = \underset{\substack{\uparrow \\ \mathrm{Mod}(A_m, A_n)}}{A_m \otimes_{K[Y]} K[X]}$$

- If $N \in \mathrm{Mod}(A_n)$: $F_*(N) := D_{Y \leftarrow X} \otimes_{A_n} N$

Holonomicity under direct images (embeddings)

$$i : X \hookrightarrow X \times Y \; : \; x \longmapsto (x, 0)$$

- $\mathcal{D}_{X\times Y \leftarrow X} = \left(\mathcal{D}_{X \to X \times Y} \right)^{\tau} = \left(k[x] \otimes_{k[x,y]} A_{m+n} \right)^{\tau}$ $\qquad y_j \partial^{\alpha} = -\alpha_j \partial^{\alpha - e_j}.$

$$= \left(k[x, y, \partial_y\text{'s}] \otimes_k k[\partial_x\text{'s}] \right) \otimes_{k[x,y]} k[x] = k[\partial_y\text{'s}] \otimes_k A_n$$

- If $N \in \mathrm{Mod}(A_n)$: $i_*(N) = \mathcal{D}_{X\times Y \leftarrow X} \otimes_{A_n} N = k[\partial_y\text{'s}] \otimes_k N.$

Theorem: If $N \in \underset{hol}{\mathrm{Mod}}(A_n) \Rightarrow i_*(N) \in \underset{hol}{\mathrm{Mod}}(A_{m+n})$

Proof: $d(i_* N) = d(K[\partial_y's] \otimes_K N) = d(K[\partial_y's]) + d(N)$

$= d(K[y]_F) + d(N) = m + n$ ∎

Kashiwara's equivalence (I)

$$i : X \hookrightarrow X \times \mathbb{C}$$

- $M \in \mathrm{Mod}(A_{n+1})$ | $\Gamma_H(M) = \{u \in M \mid y^N u = 0\} = \mathrm{Ker}(M \to M[y^{-1}]) \in \mathrm{Mod}(A_{n+1})$

From the Weyl algebra relations

$$y^{k+1}(\partial_y u) = \partial_y(y^{k+1} u) - (k+1) y^k u = 0 \quad (y^k u = 0)$$

- On the other hand $M \in \mathrm{Mod}(A_{n+1})$

$$\mathrm{Ker}(y) = \{u \in M \mid yu = 0\} \quad \in \mathrm{Mod}(A_n)$$

Properties

$$M_0 := \mathrm{Ker}(y) = \{u \in M \mid yu = 0\}$$

(1) $\mathrm{Ker}\left(\partial_y^k M_0 \xrightarrow{\partial_y} \partial_y^k M_0\right) = 0$ | By def. $y^k \partial_y^k u = (-1)^k k!\, u$

(2) $y(A_{n+1} M_0) = A_{n+1} M_0$ | By def. $A_{n+1} \cdot M_0 = M_0 + \partial_y M_0 + \partial_y^2 M_0 + \cdots$

$$y \partial_y^{k+1} u = (k+1) \partial_y^k u$$

(3) $A_{n+1} M_0 = M_0 \oplus \partial_y M_0 \oplus \cdots$ | $y^k \sum_{i=0}^{k} \partial_y^i u_i = 0 \Rightarrow u_k = 0$ + Ind. $\blacksquare$

<u>Theorem</u> $i_*(\mathrm{Ker}(y)) = \Gamma_H(M)$ $\left(\simeq \text{ in } \mathrm{Mod}(A_{n+1})\right)$.

<u>Proof</u>: $0 \longrightarrow i_*(M_0) = K[\partial_y] \underset{K}{\otimes} M_0 \longrightarrow A_{n+1} M_0 \overset{\supseteq ?}{\subseteq} \Gamma_H(M)$; $f \otimes u \mapsto f.u$

$$\cdot\, y^k u = 0 \quad \begin{cases} k = 0 \,;\; yu = 0 \text{ and } u \in M_0 \\ \cdot\, k > 1 \,; \quad \text{(i) } \partial_y (y^k u) = 0 \\ \qquad\qquad \text{(ii) } [\partial_y, y^k] = k y^{k-1} \end{cases} \quad \begin{cases} \text{(iii) } y^{k-1}\left(ku + y\partial_y u\right) = 0 \\ \text{iv) } y^k u = y^{k-1}(yu) = 0 \end{cases}$$

$$\underbrace{\overbrace{ku + y\partial_y u}^{\in A_{n+1} M_0 \text{ (iii)}} - \overbrace{\partial_y y u}^{A_{n+1} M_0 \text{ (iv)}}}_{[\partial_y, y] = 1} = (k-1)u \in A_{n+1} M_0 .$$

$\blacksquare$

General Kashiwara's equivalence

$X \hookrightarrow Y$ closed embedding

$$\begin{array}{ccc} \mathrm{Mod}_{qc}(\mathcal{D}_X) & \overset{\int_i^0 \;\simeq}{\longrightarrow} & \mathrm{Mod}_{qc}^{X}(\mathcal{D}_Y) \\ \cup\!| & \overset{H^0 i^+}{\longleftarrow} & \cup\!| \\ \mathrm{Mod}_{c}(\mathcal{D}_X) & \overset{\simeq}{\longrightarrow} & \mathrm{Mod}_{c}^{X}(\mathcal{D}_Y) \end{array}$$

Local + Induction

$Y = (x_1, \dots, x_n, y, \partial's)$, $X = \{y = 0\}$

$\int_i : D_{qc}(\mathcal{D}_X) \longrightarrow D_{qc}(\mathcal{D}_Y)$; $M^\bullet \longmapsto \mathbb{R} i_* (\underbrace{\mathcal{D}_{Y \leftarrow X}} \otimes^{\mathbb{L}}_{\mathcal{D}_X} M^\bullet)$

$$\int_i^k M = H^k\left(\int_i M\right) = \begin{cases} \int_i^0 M \\ k \neq 0, \; \int_i^k M = 0 \end{cases}$$

In fact

$$\mathcal{D}_{Y \leftarrow X} = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} \mathcal{D}_X$$

$$\left(i_* M \text{ “=” }\right) \int_i^0 M = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} M$$

- $i : X \hookrightarrow X \times Y$. $x \longmapsto (x, 0)$ and $M \in \mathrm{Mod}_{hol}(A_{n+m})$

$$i^* M \simeq i_1^* i_2^* M \qquad \in \mathrm{Mod}_{hol}(A_n)$$

$$X \xhookrightarrow{i_1} X \times K^{m-1} \xrightarrow{i_2} X \times Y$$

<u>Theorem</u>: $F : (F_1, \ldots, F_m) : X \longrightarrow Y$ and $M \in \mathrm{Mod}_{hol}(A_m) \Rightarrow F^* M \in \mathrm{Mod}_{hol}(A_n)$

<u>Reason</u>:

$$X \xhookrightarrow{i} X \times Y \xrightarrow{G} X \times Y \xrightarrow{\pi} Y$$

$$(x, y) \longmapsto (x, y + F(x))$$

$$F^* M = i^* \underbrace{G^* \pi^* M} \qquad \blacksquare$$

- $\pi : X \times Y \longrightarrow Y \quad | \quad M \in \mathrm{Mod}_{hol}(A_{m+n}) \implies \pi_*(M) \in \mathrm{Mod}_{hol}(A_m)$

Reason: $\pi_*(M) = D_{Y \leftarrow Y \times X} \otimes_{A_{m+n}} M = \left(D_{X \times Y \to Y} \right)^{\tau} \otimes_{A_{m+n}} M$

$$= \left({}^{A_{m+n}}/_{A_{m+n}(\partial_x\text{'s})} \right)^{\tau} \otimes_{A_{m+n}} M \simeq {}^{M}/_{(\partial_x\text{'s})M}$$

$$\simeq \left({}^{M}/_{M(x\text{'s})} \right)_{\mathcal{F}} \simeq \left(j^*(M) \right)_{\mathcal{F}} \quad | \quad Y \xhookrightarrow{j} X \times Y$$

- $F_*(M) \in \mathrm{Mod}_{hol}(A_m)$.

General Kashiwara's equivalence

$X \hookrightarrow Y$ closed embedding

$$\begin{array}{ccc} \mathrm{Mod}_{qc}(\mathcal{D}_X) & \xrightarrow[\simeq]{\int_i^0} & \mathrm{Mod}_{qc}^X(\mathcal{D}_Y) \\ \cup\!| & \xleftarrow{H^0 i^+} & \cup\!| \\ \mathrm{Mod}_c(\mathcal{D}_X) & \xrightarrow{\simeq} & \mathrm{Mod}_c^X(\mathcal{D}_Y) \end{array}$$

Local + Induction

$Y = (x_1, \dots, x_n, y, \partial's)$, $X = \{y = 0\}$

$$\int_i : D_{qc}(\mathcal{D}_X) \longrightarrow D_{qc}(\mathcal{D}_Y) \ ; \quad M^\bullet \longmapsto \mathbb{R} i_* \big(\underbrace{\mathcal{D}_{Y\leftarrow X}} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} M^\bullet \big)$$

$$\int_i^k M = H^k\Big(\int_i M\Big) = \begin{cases} \int_i^0 M \\ k \neq 0, \ \int_i^k M = 0 \end{cases}$$

In fact

$$\mathcal{D}_{Y\leftarrow X} = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} \mathcal{D}_X$$

$$\left(i_* M \text{ "="}\right) \int_i^0 M = \mathbb{C}[\partial_y] \otimes_{\mathbb{C}} M$$

$i^+ : D_{qc}(D_X) \longrightarrow D_{qc}(D_Y)$; $\quad M^\bullet \longmapsto \mathbb{R}i^*(M^\bullet)[d_X - d_Y]$

$$f\,dy \longmapsto yf$$

(codimension 1) $\quad 0 \longrightarrow i^{-1}\mathcal{O}_Y\, dy \longrightarrow i^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X \longrightarrow 0$

$i^+M = \mathbb{R}i^*M[-1]$: $\quad 0 \longrightarrow i^{-1}M \xrightarrow{y} i^{-1}M \longrightarrow 0$

$H^0 i^+ M = \ker(y : M \to M)$. $\quad H^j i^+ M = 0 \quad j \neq 1, 0$

$H^1 i^+ M = 0 \quad$ (Thm)

We have shown that

$$\int_i^0 H^0 i^+ M \simeq M \qquad M = \Gamma_H(M) \simeq i_*(\mathrm{Ker}(y)).$$

General Bernstein inequality

$$M \in \mathrm{Mod}_c(\mathcal{D}_X) \implies \dim(\mathrm{Ch}(M)) \geq \dim(X)$$

<u>sketch</u>: We may assume X affine $\quad i: X \hookrightarrow \mathbb{A}^m_{\mathbb{C}}$

(Technical result) $\quad \dim\left(\mathrm{ch}\left(\int_i^0 M\right)\right) = \dim(\mathrm{ch}(M)) + \dim(\mathbb{A}^m_{\mathbb{C}}) - \dim(X)$

$$\dim(\mathrm{ch}(M)) - \dim(X) = \dim\left(\mathrm{ch}\left(\int_i^0 M\right)\right) - \dim(\mathbb{A}^m_{\mathbb{C}})$$

≥ 0 (By Bernstein $\geq$ for Weyl alg)

$\blacksquare$

Minimal extensions

X smooth alg. variety, $Y \subseteq X$ locally closed s.t:

- $i : Y \hookrightarrow X$ is affine.
- M simple, holonomic $\mathcal{D}_Y$-module

1) $\begin{cases} \int_i^0 M \neq 0 \\ \int_i^k M = 0 \quad k \neq 0 \end{cases}$ | $\int_i M$ has a unique, simple submod, this is denoted by $L(Y, M)$.

2) Any simple, holonomic $\mathcal{D}_X$-module is of the form $L(Y, M)$.

3) $L(Y, M) \simeq L(Y', M')$ iff. $M|_U \simeq M'|_U$.

$\bar{Y} = \bar{Y}'$, $\exists\, U \subseteq Y \cap Y'$ open and dense

Irreducible representations of the Weyl algebra $A_1(\mathbb{C})$.

$X = \mathbb{A}^1_{\mathbb{C}}$, $Y \subseteq X$ locally closed and connected, and $Y \subseteq U$ open.

i) $\dim Y < \dim X \Rightarrow Y = \{p\}$, and $L(\{p\}, \mathcal{O}_{\{p\}}) = \mathcal{D}_X \otimes_{\mathcal{O}_{X,p}} \mathbb{C}$.

ii) $Y = U \Rightarrow \eta \in Y$ (the generic point) : $L(U, M) \simeq L(U', M')$ iff

$M|_\eta \simeq M'|_\eta$.

iii) The simple, holonomic $\mathcal{D}_{\mathbb{A}^1_{\mathbb{C}}}$ are, up to isomorphism,

$L(\{p\}, \mathcal{O}_{\{p\}})$ or $L(\eta, M)$ where M simple + holonomic $\mathcal{D}_\eta$-mod.

Conclusion

- $\Gamma(\{\eta\}, \mathcal{D}_\eta) = S^{-1}A_1(\mathbb{C})$, where $S = K[x] - \{0\}$

$\Rightarrow M$ simple $\mathcal{D}_\eta$-module $\Rightarrow \Gamma(\{\eta\}, M)$ simple $S^{-1}A_1(\mathbb{C})$

- M simple $S^{-1}A'(\mathbb{C}) \Rightarrow \mathrm{Socl}_{A'(\mathbb{C})}(M) = \sum_{\substack{N \subseteq M \\ N \text{ simple}}} N$ is simple $A_1(\mathbb{C})$ [Block]

In other words (by unicity in the thm)

$$\Gamma(\mathbb{A}^1_{\mathbb{C}}, L(\eta, M)) \simeq \mathrm{Socl}_{A_1(\mathbb{C})}(\Gamma(\{\eta\}, M)).$$

$\blacksquare$

Riemann - Hilbert correspondence

- $\mathcal{F}$ locally free sheaf of rank n on a Riemann surface X

$$\nabla : \mathcal{F} \longrightarrow \mathcal{F} \otimes \Omega_X \quad (\Leftrightarrow \mathcal{F} \in \mathrm{Mod}(\mathcal{D}_X))$$

$(U, z) \subseteq X$

$$\nabla : \mathcal{O}(U)^n \longrightarrow \mathcal{O}^n(U)\, dz = \bigoplus_{i=1}^{n} \mathcal{O}(U) e_i \, dz$$

$$\begin{cases} \cdot\ \nabla(e_i) = -\sum_{j=1}^{n} a_{ij} e_j \, dz \\ \cdot\ w = \sum_{i=1}^{n} w_i e_i \Rightarrow dw = \sum_{i=1}^{n} \left(\frac{dw_i}{dz} - \sum_{j=1}^{n} a_{ij} dw_j \right) e_i dz \end{cases}$$

- $\nabla(w) = 0 \Longleftrightarrow$ Sol. of a syst. of eqn's $\frac{dw_i}{dz} = \sum_{j=1}^{n} a_{ij} dw_j$

Conversely, $dw = Aw$ on a Riemann surface X

$$\nabla : \mathcal{O}^n \longrightarrow \mathcal{O}^n \otimes_{\mathcal{O}_X} \Omega$$

$$w \longmapsto \sum_{i=1}^{n} \left(dw_i - \sum_{j=1}^{i} a_{ij} w_j \right)$$

It satisfies $\nabla(w) = 0 \iff dw = Aw$.

<u>Definition</u>: $(\underline{\mathcal{F}}, \nabla) \mid \underline{\mathcal{F}}^{\nabla} = \{ f \in \mathcal{F} \mid \nabla(f) = 0 \}$

- $\{\text{holomorphic connexions}\} \xleftrightarrow{\sim} \{\text{Local systems}\} : (\mathcal{F}, \nabla) \mapsto \mathcal{F}^{\nabla}$.

$$\{\text{holomorphic connexions}\} \longleftrightarrow \{\text{Local systems}\}$$

$$\begin{tikzcd} (\mathcal{F}, \nabla) \arrow[r, mapsto] & \mathcal{F}^{\nabla} \\ {} & \mathcal{L} \arrow[l, bend left, "?"] \end{tikzcd}$$

<u>Proof</u> : $\mathcal{F}_{\mathcal{L}}(U) := \mathcal{L}(U) \otimes_{\mathbb{C}} \mathcal{O}(U)$

$$\begin{cases} \mathcal{L}_{|U} = \bigoplus_{i=1}^{n} \mathbb{C} \cdot s_i \\ \nabla_{\mathcal{L}}(U)\left(\sum_{i=1}^{n} s_i \otimes f_i\right) = \sum_{i=1}^{n} s_i \otimes df_i \end{cases}$$

$\blacksquare$

<u>Remark</u> : $\mathrm{Rep}_{f.d}(\pi_1(X, x)) \longleftrightarrow \{\text{Local syst's}\} \longleftrightarrow \{\text{holomorphic connexions}\}$

$$\mathcal{F}_x \longleftarrow \mathcal{F}$$

Feliz Navidad