Induction equivalence for equivariant $\mathcal{D}$-modules on rigid analytic spaces

Classical story

sketch of proof of the Kazhdan–Lusztig conjectures

a) Want to compute $[M_w : L_y]$.

M_w ← Verma module; L_y ← simple h.w. module.

$$\mathcal{O}_0 \subseteq \mathrm{coh}(U(\mathfrak{g})_0, B)$$

b) $\mathrm{coh}(U(\mathfrak{g})_0) \cong \mathrm{coh}(D_X)$ where $X = G/B$

$$\mathrm{coh}(U(\mathfrak{g})_0, B) \xrightarrow{\sim} \mathrm{coh}(D_X, B)$$

- B has only finitely many orbits on G/B!

$$|B \backslash G/B| < \infty.$$

c) $\mathrm{coh}(\mathcal{D}_X, B) = \mathrm{hol}(\mathcal{D}_X, B)$

d) $\left\{ \begin{array}{c} \text{irred. } B\text{-eq.} \\ \text{hol. } D_X\text{-mods} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (U, \mathcal{L}) : U \subseteq \overline{Y} \\ Y \in B \backslash X / B, \\ U \subset Y \text{ affine smooth}, \\ \mathcal{L} \in \mathrm{MIC}(U, G_U) \text{, irred} \end{array} \right\} \Big/ \sim$

$$\underbrace{i_{!*}\mathcal{L}}_{\text{minimal extension}} \longleftarrow\!\!\shortmid \ \mathcal{L}$$

$$\mathcal{M} \longmapsto (U, \mathcal{L})$$

$U \subseteq$ affine open dense smooth $\mathrm{Supp}(\mathcal{M})$

e) Apply the RH correspondence.

§ p-adic setting

Thm $G \subset \mathbb{G}(F)$ open, $\mathbb{G}$ split semisimple alg gp $/F$.

$X = (\mathbb{G}/\mathbb{B})^{an}$ (K-rigid flag var.)

Then $\mathcal{C}_{X/G} \xrightarrow[\sim]{\Gamma} \mathcal{C}_{D(G,K)_0}$.

Problem 1 Construct a 6-functor formalism for the categories $\mathcal{C}_{Y/H}$.

Problem 2 Try to construct/classify irreducible objects in $\mathcal{C}_{X/G}$.

By construction, every $\mathcal{M} \in \mathcal{C}_{X/G}$ is a sheaf on the rigid space X.

Recall: **Theorem** (Huber) Let Y be any rigid space. Then $\exists$ top. space $\tilde{Y}$ and an equivalence of ab. cats

$$\begin{array}{c} Y \\ \downarrow i \\ \tilde{Y} \end{array}$$

$$\mathrm{Ab}(Y) \underset{i^{-1}}{\overset{\mathcal{M} \mapsto \tilde{\mathcal{M}}}{\rightleftarrows}} \mathrm{Ab}(\tilde{Y}).$$

Also, if $\mathcal{M}(Y)$ is the Berkovich space associated with Y, then

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr] & \tilde{Y} = \mathcal{P}(Y) \supset S \arrow[d, "r"] \\ & \mathcal{M}(Y) \end{tikzcd}$$

s.t. $\mathcal{M}(Y)$ is the maximal Hausdorff quotient space of $\mathcal{P}(Y)$.

Defn The *support* of $\mathcal{M} \in \mathcal{C}_{Y/H}$ is $\mathrm{Supp}(\mathcal{M}) := \mathrm{Supp}(\tilde{\mathcal{M}})$
$= \{ p \in \mathcal{P}(Y) : \tilde{\mathcal{M}}_p \neq 0 \}$.

Note: if $\mathcal{M}$ is a G-eq. ab. sheaf on $\tilde{Y}$, then $\mathrm{Supp}(\mathcal{M})$ is G-stable.

Conjecture

1) $\mathcal{M} \in \mathcal{C}_{Y/H} \Rightarrow \mathrm{Supp}(\mathcal{M})$ is closed in $\mathcal{P}(Y)$.

2) $0 \neq \mathcal{M} \in \mathcal{C}_{X/G} \Rightarrow \mathrm{Supp}(\mathcal{M}) \cap X \neq \emptyset$.

This motivates looking at, for $y \in Y$

$$\mathcal{C}_{Y/H}^{(H\cdot y)} := \{ \mathcal{M} \in \mathcal{C}_{Y/H} : \mathrm{Supp}(\mathcal{M}) \subseteq H \cdot y \}.$$

s.t. $i(H\cdot y) \subseteq Y$ is <u>closed</u>

<u>Example</u> $Y = \mathbb{P}^{1,an}$, $y \in \mathbb{P}^1(F)$,

$H = SL_2(F) \circlearrowright Y$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}.$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot z = z+b \longrightarrow \infty$$

So $i(H\cdot y)$ is closed <u>only</u> if $\infty \in H\cdot y$
i.e. $y \in \mathbb{P}^1(F)$.

<u>Theorem 1</u> Let G be a p-adic Lie gp acting ctsly on smooth separated rigid space X, $x \in X$. Suppose $G\cdot x$ is closed in X. ($\iff G_x$ is co-compact in G)

Then $\exists$ equivalence of cats

$$\underline{\mathcal{C}^{G\cdot x}_{X/G}} \xrightarrow[\Gamma \circ i^{!} \circ \mathcal{H}^0_{\{x\}}]{\sim} \mathcal{C}_{D^\infty(G_x,K)}.$$

$U \leq G_x$ compact open.

Here $D^\infty(G_x, K) = D^\infty(U,K) \otimes_{K[U]} K[G_x]$

and $D^\infty(U,K) = \varprojlim_{N \triangleleft_o U} K[U/N]$.

$$\frac{D(G_x,K)}{\langle Lie(G_x) \rangle}$$

e.g. $U = \mathbb{Z}_p$ $\quad D^\infty(\mathbb{Z}_p, K) = \varprojlim K[\mathbb{Z}/p^n\mathbb{Z}]$

<u>Definition</u>
Let G be a group acting on a set X. Let $Y \subseteq X$. Then the G-orbit of Y is <u>regular</u> in X if:

$$gY \cap Y \neq \emptyset \Rightarrow gY = Y.$$

<u>e.g.</u>

Theorem 2 (Induction Equivalence)

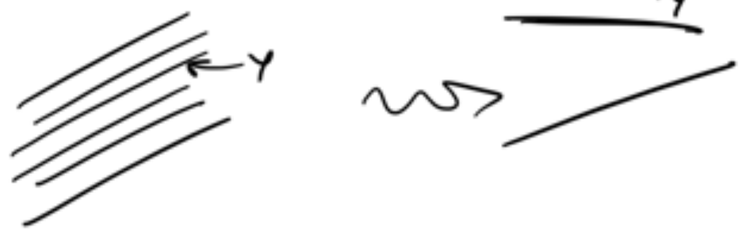
Let G be a p-adic Lie group acting ctsly on smooth rigid space X,
Let $Y \subseteq X$ be a Zariski closed subset, st.

a) X smooth & sep.
b) Y irred & quasi-compact
c) G-orbit of Y is regular in X
d) G_Y is co-compact in G.

Then $\exists$ equivalence of cats

$$\underline{\mathcal{H}^0_Y} : \boxed{\mathcal{C}^{G\cdot Y}_{X/G}} \xrightarrow{\sim} \mathcal{C}^{Y}_{X/G_Y}$$

with quasi-inverse $\underline{\mathrm{ind}^G_{G_Y}}$



Remarks: 1) If $Y \subset X$ is st. $X \setminus Y$ is admissible open, then

$$\mathcal{H}^0_Y(\mathcal{M})(U) = \ker\left(\mathcal{M}(U) \to \mathcal{M}(U \setminus Y)\right).$$

2) If $\mathcal{M} \in G\text{-}Ab_X$ then $\mathcal{H}^0_Y(\mathcal{M}) \in G_Y\text{-}Ab_X$.

Theorem 3 Let $\iota : Y \hookrightarrow X$ be smooth & Zariski closed. Let H be a p-adic Lie gp acting ctsly on X, and stabilising Y.
Then $\exists$ equivalence of cats

$$\iota_+ : \mathcal{C}_{Y/H} \xrightarrow{\sim} \mathcal{C}^{Y}_{X/H}$$

("equivariant Kashiwara").