# LINEAR ALGEBRAIC GROUPS

Def A (linear) algebraic group is the datum of an aff. alg variety $G$ over $\bar{k} = k$ & a group structure on $G$ st

$\mu : G \times G \to G \quad (g, h) \mapsto gh$

$i : G \to G \quad g \mapsto g^{-1}$

are morphisms of varieties

( i.e. we have algebra maps

$\mu^* : k[G] \to k[G \times G] \cong k[G] \otimes k[G]$

$i^* : k[G] \to k[G]$ )

EXAMPLES

① $(k,+)$ $\qquad k[G]=k[T]$

$k=\mathbb{A}^1$ USUALLY DENOTED by $\mathbb{G}_a$

② $(k^*,\times)$ $\qquad k[G]=k[T,T^{-1}]$

$k^*$ closed $\mathbb{A}^1\times\mathbb{A}^1$ as

$\{(x,y)\in\mathbb{A}^1\times\mathbb{A}^1 \mid xy=1\}$

USUALLY DENOTED BY $\mathbb{G}_m$

③ $GL_n(k)\subset\mathbb{A}^{n^2+1}$

$$\begin{aligned} GL_n(k) &= \{(X,t)\in M_{n,n}(k)\times k \mid \det X\cdot t=1\} \\ &= \{(X,\det X^{-1}),\ X\in GL_n(k)\} \end{aligned}$$

4. all closed subgroups of $GL_n(k)$

$$B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \in GL_n(k) \right\}$$

$$= \left\{ X \in GL_n(k) \;\middle|\; \underset{i>j}{x_{ij} = 0} \right\}$$

$$U = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \in B \right\}$$

$$= \left\{ X \in B \;\middle|\; x_{ii} = 1 \right\}$$

$$T = \left\{ \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}, \ a_{ii} \in k^* \right\}$$

$$\begin{matrix} \wr\wr & & \wr\wr \\ D_n & & (\mathbb{G}_m)^n \end{matrix}$$

$$k[D_n] \simeq k[T_1, \ldots, T_n, T_1^{-1}, \ldots, T_n^{-1}]$$

5 $O_n(k) = \{ X \in GL_n(k) \mid {}^tX X = I \}$

$\det X = \pm 1$

If $\operatorname{char} k \neq 2$

$$SO_n(k) = \{ X \in O_n(k) \mid \det X = 1 \}$$

$$O_n(k) = SO_n(k) \sqcup \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} SO_n(k)$$

IRREDUCIBLE

G LAG : it is SMOOTH so

if G is not irreducable, its irreducible components are disjoint: $G = G^0 \sqcup \cdots$

irr. comp. containing 1,

$G^0 \trianglelefteq G$ of finite index

$$G = \bigsqcup_{\text{finite}} g G^0$$

is the decomposition into irr. (= connected) components of G

If $H \leq G$ closed, $[G:H] < \infty$

$\Rightarrow H \supseteq G^0$

**Def** $G, H$ LAG A morphism $\varphi : G \longrightarrow H$ is a map which is a group morphism & a morphism of alg varieties

PROPERTIES:

1.) $\varphi(G) \supset U$ open dense in $\overline{\varphi(G)}$

A topological argument shows that

$$U \cdot U = \overline{\varphi(G)} \text{ and so}$$

$$\varphi(G) = \overline{\varphi(G)} \text{ is closed.}$$

2.) $$\operatorname{Ker} \varphi = \varphi^{-1}(1) \text{ is closed}$$

3) $\varphi : G \longrightarrow H$

$$\underbrace{\varphi(G^\circ)}_{\text{has finite index in } \varphi(G)} \subseteq \varphi(G)^\circ$$

$$\Longrightarrow \quad \varphi(G^\circ) = \varphi(G)^\circ$$

4) $\dim G = \dim \operatorname{Ker} \varphi + \dim \operatorname{Im} \varphi$

$$\dim \operatorname{Ker} \varphi = \dim \varphi^{-1}(x) \quad \forall x \in G$$

(recall: $\dim G = \dim G^\circ$)

# ACTIONS

A variety $X$ is a $G$-variety if there is a $G$-action st

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g.x$$

is a morphism of varieties

It is called homogeneous if it is a single orbit

$$X = G \cdot x_0$$

REMARKS

① Orbits are locally-closed

Indeed they are images of

$$G \times \{x\} \xrightarrow{f} X$$
$$(g,x) \longmapsto g\cdot x$$

so $\operatorname{Im} f \supset A$, open dense in $\operatorname{Im} f$

$$\operatorname{Im} f \supset \underbrace{G\cdot A}_{\text{open in } \overline{\operatorname{Im} f}} \supset G.a = \operatorname{Im} f$$

② Stabilizers of points are closed:

$G_x = f^{-1}(\{x\})$, $f$ as above

# IMPORTANT ACTIONS

① $G \curvearrowright G$

- left multiplication
- right "
- conjugation

② $G \curvearrowright k[G]$ $\infty$-dim$^{e}$ vector space

For $f \in k[G]$ we define

$$(\rho_g f)(-) = f(-g), \; g \in G$$

$$\mu^* f = \sum_{\text{finite}} f_i \otimes \tilde{f}_i \in k[G] \otimes k[G]$$

$$(\rho_x f)(y) = f(yx) = \sum_{\text{finite}} f_i(y) \underbrace{\tilde{f}_i(x)}_{\in k}$$

$\rho_x f \in \operatorname{span}\{f_i\}_{\text{finite}}, \; \forall x \in G$

$\forall f \in k[G]$

$\exists$ $V$ fd$^e$ vector space st

$\rho_x V = V$ $\quad \forall x \in G$ <u>and</u>

$\rho : G \longrightarrow GL(V)$

morphism of LAG

Any morphism

$G \longrightarrow GL(W)$

is called a RATIONAL REPRESENTATION.

Theorem (Chevalley)

If $G$ LAG, $H \underset{\text{closed}}{\leq} G$

$\exists$ fd vector space $V$, a rational representation

$\rho : G \to GL(V)$

and a 1-dim$^{l}$ $W \subseteq V$

s.t. $H = \mathrm{Stab}_G(W)$.

CONSEQUENCES

1) If we take $H = \{1\}$, then

$\ker \rho \subseteq H = 1$, so

$\rho : G \underset{\text{closed}}{\hookrightarrow} GL(V)$

2)

$G \circlearrowleft X$

$\underline{\underline{G.x_0}}$ is bijection with $G/G_x$

In general

$H \leq G$ $G/H$ can be

"identified" with a

quasi-projective variety: how?

we have $\rho: G \longrightarrow GL(V)$

so $G$ acts on $\mathbb{P}(V)$

by: $\rho(g).[v] = [\rho(g).v]$

and $H = Stab_G(w)$ (for $\rho$)

$W = Kw$ implies

$H = Stab_G([w])$, for the action

the orbit of $[w]$ is locally closed in $\mathbb{P}(V)$ and is in bijection with $G/H$. $\Rightarrow$ quasi-projective

CAN BE PROVED:

The quasi-projective variety structure does <u>not</u> depend on $V, W$, only on $G$ and $H$

Pf of Chevalley's theorem:

$G, H$

$I(H) = I \subseteq k[G]$

Check: $H = \{x \in G \mid \rho_x I = I\}$

$I(H) = (F_1, \dots, F_n)$

$\exists$
$V_1$ $G$-stable containing
$F_1, \dots, F_n$

$W_1 := V_1 \cap I$ is $H$-stable,
contains $F_1, \dots, F_n$

If $g \in G$ and
$\rho_g(V_1 \cap I) = V_1 \cap I$, then

$\rho_g(I) = \rho_g( W_1 \quad k[G]) =$

$= W_1 \quad k[G] = I, \Rightarrow g \in H$

$$\boxed{H = \mathrm{Stab}_G(V_1 \cap I)}$$

If $\dim(V_1 \cap I) = d$, then we take

$V := \Lambda^d V_1$

$W := \Lambda^d W_1$ is 1-dim$^e$

$G$ acts linearly on $V$

it is a rational

repn.

$H \subseteq \mathrm{Stab}_G W$ by construction

If $g$ stabilizes $W = \Lambda^d W_1$

$\Rightarrow$ $g$ stabilizes $W_1$

$\Rightarrow$ $g \in H$

$\Rightarrow$ $\boxed{H = \mathrm{Stab}_G [W]}$. $\square$

# JORDAN DECOMPOSITION

$$G \leq GL(V)$$

$$\psi$$

$$g = s\,u$$

diagonalizable

$\shortparallel$

semisimple

unipotent = all its eigenvalues are $=1$.

e.g.

$$g = \begin{pmatrix} \boxed{\begin{matrix}\lambda & & \\ & \lambda & \\ & & \lambda\end{matrix}} & \\ & \boxed{\begin{matrix}\mu & \\ & \mu\end{matrix}} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix}1 & 1 & \\ & 1 & \\ & & 1\end{matrix}} & \\ & \boxed{\begin{matrix}1 & 1 \\ & 1\end{matrix}} \end{pmatrix}$$

$$[s,u] = 1$$

FACT!

If $g \in G \Rightarrow s, u \in G$

# NO SOCIAL MOBILITY PRINCIPLE

The Jordan decomposition in $G$ does <u>not</u> depend on the embedding

If $G \leq GL(V)$

$g = su$ $\qquad$ $s, u$ are sem, unip

$\Updownarrow$

they are so in $GL(V') \geq G$

for any other embedding of $G$ in some $GL(V')$.