LINEAR ALGEBRAIC GROUPS, part II

$\bar{k} = k$

$G \leq GL(V)$

$g = su$ $\quad$ s semisimple, u unipotent

$[s,u]=1$

$g \in G \Rightarrow s, u \in G$

<u>EXAMPLE</u> $\quad k = \overline{\mathbb{F}_p}$

$$GL_n(k) = \bigcup_{e \geq 0} GL_n(\mathbb{F}_{p^e})$$

$\Rightarrow$ every elt. has finite order

$$\boxed{\begin{array}{l} u \text{ unipotent} \Leftrightarrow \\ 0 = (u-1)^n = (u-1)^{p^M} = u^{p^M} - 1 \\ \Leftrightarrow \quad u \text{ is a } p\text{-element.} \end{array}}$$

s semis. $\Rightarrow s \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_e \end{pmatrix}$

$\lambda_i \in \mathbb{F}_{p^e} \Rightarrow (|\lambda_i|, p) = 1 \Rightarrow$

$(|s|, p) = 1$, viceversa holds

G LAG we set

$$\mathcal{U}_G := \{ g \in G \mid g \text{ is unipotent} \}$$

$$\mathcal{S}_G := \{ g \in G \mid g \text{ is semisimple} \}$$

For $G \leq GL_N(k)$

$$\mathcal{U}_{GL_N(k)} = \{ g \in GL_N(k) \mid (g-1)^N = 0 \}$$

hence

$$\mathcal{U}_G = G \cap \mathcal{U}_{GL_N(k)} \quad \text{is closed}$$

it not a subgp in general

$$\mathcal{S}_G = ?$$

EXAMPLE

$$G = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \; ab \neq 0 \right\}$$

$$\mathcal{S}_G = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in G \mid a \neq b \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G,\ a = b \right\}$$

neither open nor closed, DENSE in G

GROUPS WHERE ALL ELEMENTS ARE UNIPOTENT OR SEMISIMPLE

1) $G_a \cong (k,+) \cong U_2 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in k \right\}$

G_a IS UNIPOTENT (= all its elements are so)

$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\}$ PROTOTYPE UNIPOTENT

2) $\mathbb{D}_n = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}$

all its elements are ss

$\Downarrow$

$G_m \cong \mathbb{D}_1 \cong (k^*, \times)$

Def A LAG is a <u>TORUS</u> if it is isomorphic to $\mathbb{D}_n$ for some n.

What is special about TORI?

ABELIAN, ALL ELTS ARE SS $\Rightarrow$

For any $T \xrightarrow{\rho} GL(V)$ rational repn.

all $\rho(t)$ $t \in T$ are simultaneously diagonalizable

In particular: if

$T \leq GL(V)$ is a Torus

$\Rightarrow \exists g \in GL(V)$ st $gTg^{-1} \leq D_n$.

If T is maximal we have =

Hence:

Maximal tori (in $GL_n(k)$)

are all conjugate

(to D_n).

PROPERTIES

G LAG $\quad T \leq G$ torus

1. Any closed connected subgroup of T is a torus

2. $N_G(T) / C_G(T)$ is FINITE

$G = GL_4(k)$ $\quad T = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, ab \neq 0 \right\}$

$$N_G(T) = \left\{ \left(\begin{array}{cc|c} x & & \\ & y & 0 \\ \hline & 0 & A \end{array} \right) \right\} \cup \left\{ \left(\begin{array}{cc|c} 0 & x & \\ y & 0 & 0 \\ \hline & 0 & A \end{array} \right) \right\}$$

$C_G(T) =$ $\quad N_G(T) / C_G(T) \simeq C_2$

If T is maximal we set

$$\boxed{W = N_G(T) / C_G(T)}$$

called the WEYL GROUP

3) T maximal in G

$$\mathcal{S}_G = \bigcup_{g \in G} gTg^{-1}$$

- UNIPOTENT GROUPS -

PROP If $G \leq GL_n(k)$ is unipotent $\Rightarrow \exists\, g \in GL_n(k)$ $\quad gGg^{-1} \leq U_n$

($\Leftarrow$ U_n is really a prototype)

Pf By induction on n:

If $\exists\, 0 \subsetneq W \subsetneq k^n$, $G.W = W$

$\exists h$

$$hGh^{-1} \in \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

use induction on the blocks

If $\nexists$ W

$\Downarrow$

$\mathrm{span}_k G = \mathrm{End}(k^n)$

+ trick on traces

$\forall g \in G \quad \mathrm{Tr}\, g = n$

$\Downarrow$

$G = 1$

$\square$

CONSEQUENCE:

G unipotent $\rightsquigarrow$ essentially $G \leq U_n$ so

G is nilpotent. why?

Def H a group, we set
$H^1 = [H,H]$, $H^i = [H, H^{i-1}]$,
H is nilpotent if the series terminates to 1

EXAMPLE

$$[U_n, U_n] = \left\{ \begin{pmatrix} 1 & 0 & & * \\ & \ddots & \ddots & \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix} \right\} = U_n'$$

$$[U_n^i, U_n] = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ & \ddots & & \\ & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix} \right\} \text{ terminates}$$

So G unipotent $\Rightarrow$ nilpotent
$\Rightarrow$ solvable

Def H a group. We set
$H^{(i)} = [H^{(i-1)}, H^{(i-1)}]$, $H^{(1)} = [H,H]$
H is solvable if the series terminates to 1

– SOLVABLE GROUPS –

Prototype

→ $B_n = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

Indeed

$[B_n, B_n] = U_n \Rightarrow$ the series terminates

which is nilpotent

B_n stabilizes the flag in k^n

$\operatorname{span} e_1 \subset \operatorname{span}(e_1, e_2) \leq \ldots$

On the other hand if $G \leq GL_n(k)$ stabilizes a flag $0 \leq V_1 \leq \cdots \leq k^n$ then $\exists \; g \in GL_n(k)$ st

$g G g^{-1} \leq B_n \Rightarrow$

G is solvable

The converse also holds

LIE KOLCHIN THEOREM

If $G \leq GL(V)$ is solvable, connected $\Rightarrow$ it stabilizes a flag.

Pf (idea)

Step 1 Enough to show that G stabilizes a line

$$\left(\overset{\text{so}}{\Rightarrow} \quad G \overset{\text{conj}}{\sim} \leq \left\{ \left(\begin{array}{c|c} * & * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right) \right\} \right)$$

then use induction

Step 2 Use induction on the derived length of G, or on $\dim V$, using $\underbrace{[G,G]}_{\text{solvable, closed connected}} \subsetneq G$

If $\exists$ $0 \subsetneq W \subsetneq V$

$G\underbrace{W}_{} \subseteq \underbrace{W}_{}$

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

use induction on blocks

If $\boxed{\nexists\ W}$

$[G,G] \subsetneq G$

solv, connected

$V' = $ Span lines stabilized by $[G,G]$

is G-stable

$V' = V$,

$[G,G]$ acts diagonally on V

+ use connectedness to show

$\Downarrow$

$[G,G] \subseteq Z(G)$ Schur's lemma

$\Rightarrow [G,G]$ finite

use again connectedness

$\Rightarrow G$ abelian

$\Rightarrow \dim V = 1$

$\square$

STRUCTURE OF SOLVABLE GROUPS!

G solvable

$$G \leq \underbrace{B_n = D_n \ltimes U_n}_{\text{use}}$$

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

In this case: $\mathcal{U}_{B_n} = U_n \trianglelefteq B_n$

So $\mathcal{U}_G \triangleleft G$, $T = G/\mathcal{U}_G \hookrightarrow B_n/U_n = D_n$ is ss.

$\mathcal{U}_G = U_n \cap G$ is a (normal) subgroup

in fact

$$\boxed{G \cong T \ltimes \mathcal{U}_G}$$

IDEAS

S solvable $\rightsquigarrow$ stabilizes a flag $\mathcal{F}$

$S \leq G$

G/S quasiprojective

$G/S \longleftrightarrow G \cdot \mathcal{F}$

hope : for some S G/S IS PROJECTIVE

idea want G/S closed

$\Updownarrow$

closed $G \cdot \mathcal{F}$

$\Updownarrow$

$G \cdot \mathcal{F}$ of minimal dimension

$\Updownarrow$

S maximal dimension

This motivates the following

Def A Borel subgroup of G is a maximal subgp among the closed connected solvable subgroups of G.

Ex: $G = GL_n(k)$

If B Borel subgroup $\Longrightarrow$ (solv. conn)

$$\exists\, g \in GL_n(k) \quad gBg^{-1} \subseteq B_n$$

$=$ by maximality

All Borel subgps are conj in $GL_n(k)$

(G/B will be our flag variety)

We will need

BOREL FIXED POINT THEOREM

G connected, solvable $G \curvearrowright X$

X projective variety $\Rightarrow$ It has a fixed point.

LIE KOLCHIN $X = \mathbb{P}(V)$

RADICAL OF G

B a Borel

$$R(G) = \Big(\bigcap_{g \in G} (gBg^{-1}) \Big)^{\circ} \text{ is}$$

maximal among the NORMAL solvable connected subgroups of G

$$\mathcal{U}_{R(G)} \triangleleft R(G) \text{ characteristic}$$

$$\mathcal{U}_{R(G)} \trianglelefteq G$$

called unipotent radical

EX:

$$R(GL_n(k)) \subseteq \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\} \cap \left\{ \begin{pmatrix} * & & \\ * & \ddots & \\ & & * \end{pmatrix} \right\}$$

$$\subseteq \mathbb{D}_n \quad \text{is a torus}$$

Also one may prove that

$$R(GL_n(k)) = Z(GL_n(k))$$

Groups for which
$R(G)$ is a TORUS are called
REDUCTIVE
($GL_n(K)$ is reductive)

Groups for which $R(G) = 1$
are called SEMISIMPLE
(eg. $SL_n(K)$)

$$GL_n(K) = Z(GL_n(K))\, SL_n(K)$$
$$= Z(GL_n(K))\, [GL_n(K), GL_n(K)]$$

IS A PROTOTYPE

in general: G reductive then

$$\boxed{\begin{aligned} G &= R(G)\, [G, G'] \\ &= Z(G)^{\circ}\, [G, G] \end{aligned}}$$

always semisimple

REPRESENTATION THEORY

LAG G

$$\rho : G \longrightarrow GL(V)$$

Assume V is IRREDUCIBLE
($\nexists$ G-stable subspace)

Then
$\mathcal{U}_{R(G)}$ fixes a line pointwise
(is unipotent) so

$0 \neq V^{\mathcal{U}_{R(G)}}$. Also $\mathcal{U}_{R(G)} \trianglelefteq G$

$\Rightarrow$ $V^{\mathcal{U}_{R(G)}}$ is
G-stable

so $V = V^{\mathcal{U}_{R(G)}}$, i.e.,

$\mathcal{U}_{R(G)} \subseteq \ker \rho$

$$\bar{\rho} : G/\mathcal{U}_{R(G)} \longrightarrow GL(V)$$

G_{red} is reductive

i.e. we reduced to a reductive gp

Use $G_{red} = Z(G_{red})^{\circ}\,[G_{red}, G_{red}]$
to reduce to semisimple groups

G . LAG , B Borel sbgp

G/B Flag variety

$$R(G) \subseteq B \implies$$

$$G/B \simeq \frac{G/R(G)}{B/R(G)}$$

is semisimple

"ALSO AT THE LEVEL OF FLAG VARIETY WE CAN REDUCE TO SEMISIMPLE (OR REDUCTIVE) GROUPS."