

LINEAR ALGEBRAIC GROUPS, part II

$$\bar{k} = k$$

$$G \leq GL(V)$$

$$\mathfrak{g} = \mathfrak{su}$$

s semisimple
 u unipotent

$$[s, u] = 1$$

$$g \in G \Rightarrow s, u \in G$$

EXAMPLE $k = \overline{\mathbb{F}_p}$

$$GL_n(k) = \bigcup_{e \geq 0} GL_n(\mathbb{F}_{p^e})$$

\Rightarrow every elt. has finite order

u unipotent \Leftrightarrow

$$\det(u-1)^n = (u-1)^{p^M} = u^{p^M} - 1$$

$\Leftrightarrow u$ is a p -element.

$$s \text{ semis.} \Rightarrow s \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_e \end{pmatrix}$$

$$\lambda_i \in \overline{\mathbb{F}_{p^e}} \Rightarrow (\lambda_i - 1, p) = 1 \Rightarrow$$

$$(|S|, p) = 1, \text{ viceversa holds}$$

G LAG we set

$$U_G := \{g \in G \mid g \text{ is unipotent}\}$$

$$S_G := \{g \in G \mid g \text{ is semisimple}\}$$

For $G \leq GL_N(k)$

$$U_{GL_N(k)} = \{g \in GL_N(k) \mid (g-1)^N = 0\}$$

hence

$$U_G = G \cap U_{GL_N(k)} \text{ is closed}$$

it is not a subgroup in general

$$S_G = ?$$

EXAMPLE

$$G = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid ab \neq 0 \right\}$$

$$S_G = \left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \in G \mid a \neq b \right\} \cup \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G, a = b \right\}$$

neither open nor closed, DENSE in G

GROUPS WHERE ALL ELEMENTS ARE UNIPOTENT OR SEMISIMPLE

$$1) G_a \cong (k, +) \cong U_2 = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in k \right\}$$

G_a IS UNIPOTENT (= all its elements are so)

$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \right\} \quad \text{PROTOTYPE UNIPOTENT}$$

$$2) D_n = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \right\}$$

all its elements are ss

\Downarrow

$$G_m \cong D_1 \cong (k^*, \cdot)$$

Def A LAG is a TORUS if it is isomorphic to D_n for some n .

What is special about TORI?

ABELIAN, ALL ELTS ARE SS \Rightarrow

For any $T \xrightarrow{P} GL(V)$ rational repr.
all $\rho(t)$ $t \in T$
are simultaneously diagonalizable

In particular: if

$T \subseteq GL(V)$ is a Torus

$\Rightarrow \exists g \in GL(V)$ st $gTg^{-1} \subseteq D_n$.

If T is maximal we have $=$.

Hence:

Maximal tori (in $GL_n(k)$)
are all conjugate
(to D_n).

PROPERTIES

G LAG

$T \leq G$
 T TORUS

1. Any closed connected subgroup of T is a torus

2. $N_G(T) / C_G(T)$ IS FINITE

$$G = GL_4(K) \quad T = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, ab \neq 0 \right\}$$

$$N_G(T) = \left\{ \left(\begin{array}{cc|cc} x & y & 0 & \\ \hline 0 & 1 & & \\ \hline 0 & & A & \end{array} \right) \right\} \cup \left\{ \left(\begin{array}{cc|cc} 0 & x & 0 & \\ y & 0 & 0 & \\ \hline 0 & & A & \end{array} \right) \right\}$$

$$C_G(T) = \quad \nearrow \quad N_G(T) / C_G(T) \cong C_2$$

If T is maximal we set

$$W = N_G(T) / C_G(T)$$

called the
WEYL GROUP

3) T maximal in G

$$\mathcal{I}_G = \bigcup_{g \in G} g T g^{-1}$$

- UNIPOTENT GROUPS -

PROP If $G \leq GL_n(k)$ is unipotent

$$\Rightarrow \exists g \in GL_n(k) \quad gGg^{-1} \subseteq U_n$$

(= U_n is really a prototype)

Pf By induction on n :

If $\exists 0 \subsetneq W \subsetneq k^n, G.W = W$ | If $\nexists W$

$$\exists h \quad hGh^{-1} \in \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

use induction on the blocks

$$\Downarrow$$

$$\text{span}_k G = \text{End}(k^n)$$

+ trick on traces

$$\forall g \in G \quad \text{Tr} g = n$$

$$\Downarrow$$

$$G = 1$$

□

CONSEQUENCE:

G unipotent

\leadsto essentially $G \leq U_n$ so

G is nilpotent. why?

Def If a group, we set

$$H^1 = [H, H], \quad H^i = [H, H^{i-1}]$$

H is nilpotent if the series terminates to 1

EXAMPLE

$$[U_n, U_n] = \left\{ \begin{pmatrix} 1 & 0 & & * \\ & \ddots & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \right\} = U_n^1$$

$$[U_n^i, U_n] = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ & \ddots & & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \right\} \text{ terminates}$$

So G unipotent \Rightarrow nilpotent

\Rightarrow solvable

Def If a group. We set

$$H^{(i)} = [H^{(i-1)}, H^{(i-1)}], \quad H^{(1)} = [H, H]$$

H is solvable if the

series terminates to 1

- SOLVABLE GROUPS -

Prototype

$$\rightarrow B_n = \left\{ \begin{pmatrix} * & & \\ & \circ & \\ & & * \end{pmatrix} \right\}$$

Indeed

$$[B_n, B_n] = U_n \quad \Rightarrow \quad \begin{array}{l} \text{which} \\ \text{is nilpotent} \end{array} \quad \Rightarrow \quad \begin{array}{l} \text{the series} \\ \text{terminates} \end{array}$$

B_n stabilizes the flag in k^n

$$\text{span } e_1 \subset \text{span}(e_1, e_2) \subset \dots$$

On the other hand if $G \leq GL_n(k)$

stabilizes a flag $0 \subset V_1 \subset \dots \subset k^n$

then $\exists g \in GL_n(k)$ st

$$g G g^{-1} \subset B_n \Rightarrow$$

G is solvable

The converse also holds

LIE KOLCHIN THEOREM

If $G \leq GL(V)$ is solvable,
connected \Rightarrow it stabilizes a
flag.

Pf (idea)

Step 1 Enough to show
that G stabilizes a line

$$\left(\begin{array}{l} \text{so} \\ \Rightarrow \end{array} G \cong \left(\begin{array}{c|c} * & * \\ \hline 0 & * \\ \vdots & \\ 0 & \end{array} \right) \right)$$

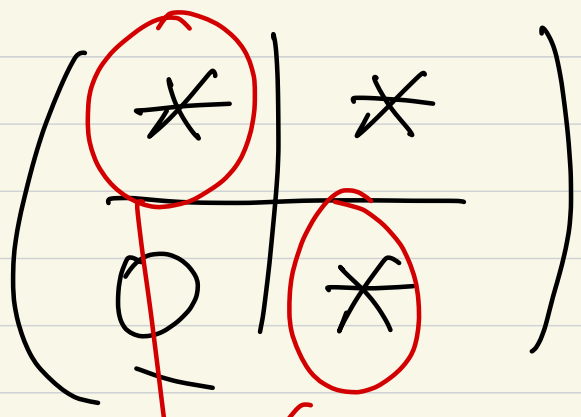
then
use
induction

Step 2 Use induction on the
derived length of G , or on
 $\dim V$, using $[G, G] \subsetneq G$

|
solvable, closed
connected

If $\mathbb{F} \subsetneq W \subsetneq V$

$$G \cdot W \subseteq W$$



use induction on blocks

If $Z = W$

$[G, G] \subsetneq G$
Solv, connected

$V' = \text{Span lines stabilized by } [G, G]$

is G -stable

$$V' = V$$

$[G, G]$ acts diagonally on V

+ use connectedness to show

$$[G, G] \subseteq Z(G) \text{ Schur's lemma}$$

$\Rightarrow [G, G]$ finite
use again connectedness

$\Rightarrow G$ abelian

$$\Rightarrow \dim V = 1$$

□

STRUCTURE OF SOLVABLE GROUPS!

G solvable



$$G \leq B_n = D_n \rtimes U_n$$

use

$$\begin{pmatrix} a_1 & & * \\ & \ddots & \\ & & a_n \end{pmatrix} = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix}$$

In this case : $U_{B_n} = U_n \trianglelefteq B_n$

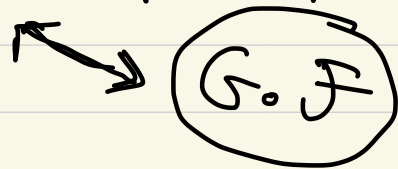
So $U_G \triangleleft G$, $T = G/U_G \hookrightarrow B_n/U_n \cong D_n$
is ss.

$U_n \cap G$ is a subgroup
In fact

$$G \cong T \rtimes U_G$$

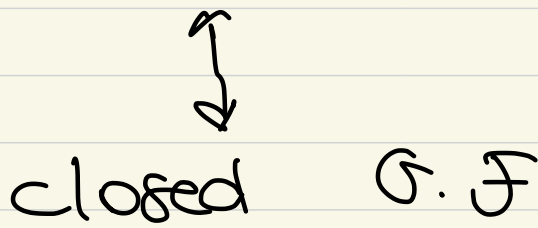
IDEAS

S Solvable \rightsquigarrow stabilizes a flag F
 \wedge
 G G/S quasiprojective

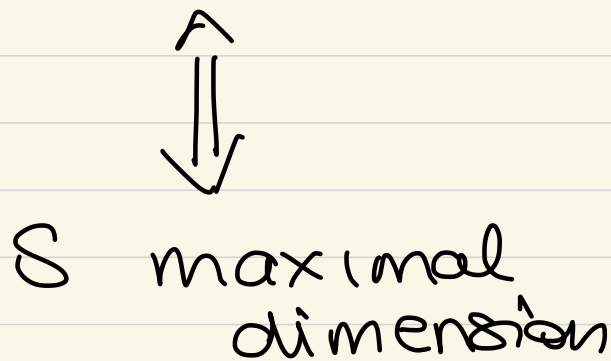


hope : for some S G/S IS PROJECTIVE

idea want G/S closed



$G.F$ of minimal dimension



This motivates the following

Def A Borel subgroup of G is a maximal ^{among the} closed connected solvable subgroups of G .

Ex: $G = GL_n(k)$

If B Borel subgroup \Rightarrow
Solv. conn

$\exists g \in GL_n(k) \quad gBg^{-1} \subseteq B_n$
by maximality

All Borel subgps are conj
in $GL_n(k)$

$(G/B$ will be our flag variety)

We will need

BOREL FIXED POINT THEORY

G connected, solvable $G \curvearrowright X$

X projective variety \Rightarrow It has
a fixed point.

LIE KOLCHIN $X = \mathbb{P}(V)$

RADICAL OF G

B a Borel

$$R(G) = \left(\bigcap_{g \in G} (gBg^{-1}) \right)^{\circ}$$
 is

maximal among the NORMAL solvable connected subgroups of G

$$\bigcup_{R(G)} \triangleleft R(G)$$

characteristic

$\triangleleft G$ called unipotent radical

EX:

$$R(GL_n(k)) = \left\{ \begin{pmatrix} * & & \\ & * & \\ 0 & & * \end{pmatrix} \cap \left\{ \begin{pmatrix} * & & \\ & * & \\ * & & * \end{pmatrix} \right\} \right\}$$

$\subseteq ID_n$ is a torus

Also one may prove that

$$R(GL_n(k)) = Z(GL_n(k))$$

Groups for which $R(G)$ is a Torus are called **REDUCTIVE** ($GL_n(k)$ is reductive)

Groups for which $R(G) = 1$ are called **SEMISIMPLE** (eg. $SL_n(k)$)

$$\begin{aligned} GL_n(k) &= Z(GL_n(k)) SL_n(k) \\ &\rightarrow = Z(GL_n(k)) [GL_n(k), GL_n(k)] \end{aligned}$$

ISA PROTOTYPE

in general: G reductive then

$$\begin{aligned} G &= R(G) [G, G'] \\ &= Z(G)^\circ [G, G] \end{aligned}$$

always semisimple

REPRESENTATION THEORY

LAG G

$$\rho: G \longrightarrow GL(V)$$

Assume V is IRREDUCIBLE
(\nexists G -stable subspace)

Then

$U_{R(G)}$ fixes a line pointwise
(is unipotent) so

$0 \neq V^{U_{R(G)}}$. Also $U_{R(G)} \triangleleft G$

$\Rightarrow V^{U_{R(G)}}$ is G -stable
 $\infty V = V^{U_{R(G)}}$, i.e.,

$$U_{R(G)} \subseteq \ker f$$

$$\bar{\rho}: \left(\frac{G}{U_{R(G)}} \right) \longrightarrow GL(V)$$

G_{red} is reductive i.e. we reduced to a reductive gp

Use $G_{red} = Z(G_{red})^\circ [G_{red}, G_{red}]$
to reduce to semisimple groups

G - LAG, B Borel subgroup
 G/B Flag variety

$$R(G) \subseteq B \Rightarrow$$
$$G/B \cong \frac{G/R(G)}{B/R(G)}$$

is semisimple

ALSO AT THE LEVEL OF
FLAG VARIETY WE CAN
REDUCE TO SEMISIMPLE (OR REDUCTIVE)
GROUPS -