$k = \bar{k}$ char $k = 0$

G connected linear algebraic group over k

$\mathfrak{g} = \mathrm{Lie}(G)$ Lie algebra of G

<u>recall</u> :

(a) a (rational) representation of G on the k-vector space V is

a regular homomorphism

$$\rho : G \longrightarrow GL(V)$$

(b) a representation of the Lie algebra $\mathfrak{g}$ on the vector space V is

a Lie algebra homomorphism

$$\sigma : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

$\mathrm{Rep}(G)$, $\mathrm{Rep}(\mathfrak{g})$ abelian categories of representations of G and $\mathfrak{g}$ respectively

- <u>Prop</u>: Differentiation of the G-action gives a functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

which is exact and fully faithful.

- <u>rmk</u>: so $\mathrm{Rep}(G)$ may be regarded as a full subcategory of $\mathrm{Rep}(\mathfrak{g})$.

- <u>Example</u>: (tori) $G = T = (k^\times)^n$, $\mathfrak{g} = \mathfrak{h} = k^n$

characters (i.e. 1-dim. chars)

$$X(T) = \mathbb{Z}^n \subset \mathfrak{h}^* \cong k^n$$

Thm: The following hold

(a) G is semisimple $\iff$ $\mathfrak{g}$ is semisimple

(b) if $\mathfrak{g}$ is semisimple, then there exists a unique (up to isomorphism) $\tilde{G}$ with $\mathfrak{g} = \mathrm{Lie}(\tilde{G})$ and s.t.
$\forall G$ s.t. $\mathrm{Lie}(G) = \mathfrak{g}$, one has

$$G \cong \tilde{G}/K \qquad \text{where } K \subset Z(\tilde{G})$$

[$\tilde{G}$ is said to be simply connected]

(c) if G is semisimple and simply connected then

the functor

$$\mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(\mathfrak{g})$$

is an equivalence

<u>Example:</u> $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$

$\tilde{G} = SL_2(\mathbb{C}) \qquad Z(\tilde{G}) = \{\pm I_2\}$

$\tilde{G}/Z(\tilde{G}) = PGL_2(\mathbb{C}) \cong SO_3(\mathbb{C})$

$Rep(SL_2(\mathbb{C})) \cong Rep(\mathfrak{sl}_2(\mathbb{C}))$

is a semisimple abelian category

- irreducible representations of $SL_2(\mathbb{C})$:

$V_n \qquad n \in \mathbb{N}$

$V_0 = \mathbb{C}, \ V_1 = \mathbb{C}^2 \quad \cdots \quad V_n = Sym^n V_1 \cong \mathbb{C}[x,y]_n$

- irreducible representations of $SO_3(\mathbb{C})$:

$V_n \qquad n \in 2\mathbb{N}$

From now on

G is connected and simply connected semisimple linear algebraic group

Let us study $R = \mathrm{Rep}(G) \cong \mathrm{Rep}(\mathfrak{g})$

Thm: (Weyl) R is a semisimple abelian category.

remarks on the proof:

(a) in fact the thm holds for the larger class of reductive groups:

- sketch of the proof over $\mathbb{C}$
  - let $K \subset G$ be a maximal compact subgroup

    G reductive $\Longleftrightarrow$ $K \subset G$ Zariski dense

    $\Longrightarrow R \cong \mathrm{Rep}(K)$ which is semisimple (Weyl unitary trick)

(b) this may also be proved by Lie algebra methods

- uses nondegeneracy of Killing form

Problems:

(i) classify simple objects of R

(ii) decompose any rep. into its simple constituents

Lie algebra way

- the universal enveloping algebra

Def: Let $U(\mathfrak{g}) = T(\mathfrak{g})/_{I}$

where $I \subset T(\mathfrak{g})$ is the two-sided ideal of the tensor algebra $T(\mathfrak{g})$ generated by

$$x \otimes y - y \otimes x - [x,y] \qquad x, y \in \mathfrak{g}$$

Prop: there is an equivalence of categories

$$\mathrm{Rep}(\mathfrak{g}) \xrightarrow{\cong} U(\mathfrak{g})\text{-mod}^{(f)}$$

of $\mathrm{Rep}(\mathfrak{g})$ with the category of $U(\mathfrak{g})$-modules of finite dimension.

- $U(\mathfrak{g})$ is a filtered algebra

$$T(\mathfrak{g}) = \bigoplus_{n \in \mathbb{N}} \mathfrak{g}^{\otimes n}$$

is naturally graded $T(\mathfrak{g})_n = \mathfrak{g}^{\otimes n}$

it induces the filtration $T(\mathfrak{g})_{\leq n} = \bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i}$

the filtration on $U(\mathfrak{g})$ is

$$U(\mathfrak{g})_{\leq n} = T(\mathfrak{g})_{\leq n} \Big/ I_{\leq n}$$

let $(x_1, \ldots, x_\ell)$ be a k-basis of $\mathfrak{g}$

- I is the two-sided ideal generated by

$$x_i x_j - x_j x_i - [x_i, x_j]$$

pf: by bilinearity. $\square$

- let $gr\, U(\mathfrak{g}) = \bigoplus_i \dfrac{U(\mathfrak{g})_{\leq i}}{U(\mathfrak{g})_{\leq i-1}}$ be

the associated graded algebra, then

there is a surjective map of graded algebras

$$Sym(\mathfrak{g}) \longrightarrow gr(U(\mathfrak{g}))$$

- Consider the following $k[h]$-algebra

$$\tilde{U}(\mathfrak{g}) = \frac{T(\mathfrak{g}) \otimes_k k[h]}{(x \otimes y - y \otimes x - [x,y]h)}$$

$$\tilde{U}(\mathfrak{g}) / (h) = \mathrm{Sym}(\mathfrak{g})$$

$$\tilde{U}(\mathfrak{g}) \cong \mathrm{Sym}(\mathfrak{g}) \otimes_k k[h]$$

as $k[h]$-modules

- Consequences :

  - <u>thm</u> (PBW) : $\{ x_1^{a_1} \cdots x_\ell^{a_\ell} \ , a_1, \dots, a_\ell \in \mathbb{N} \}$ is a k-basis of $U(\mathfrak{g})$.

  - <u>thm</u> : $gr\left(U(\mathfrak{g})\right) \cong Sym(\mathfrak{g})$

recall:

$\mathfrak{g}$ semisimple Lie algebra

$\mathfrak{b}$ Borel subalgebra

$\mathfrak{h}$ Cartan subalgebra

$\Phi \subset E \subset \mathfrak{h}^*$ root system

Φ^+ positive roots

$\Lambda = \{\alpha_1, \dots, \alpha_\ell\}$ simple roots

$\Lambda^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset \mathfrak{h}$ simple coroots

$$W = \frac{N_G(\mathfrak{h})}{Z_G(\mathfrak{h})} \quad \text{Weyl group}$$

$$\mathfrak{g} = \mathfrak{h} \oplus_{\alpha \in R} \mathfrak{g}_\alpha \quad \text{root space decomposition}$$

$\mathfrak{g}_\alpha$ root space relative to the root α

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

$$\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \quad , \quad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$$

$$\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \quad , \quad \mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$$

- <u>weight</u> <u>space decomposition</u> <u>of</u> <u>a</u> <u>representation</u>

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

$$V_\lambda = \{ v \in V \mid \forall h \in \mathfrak{h},\ hv = \lambda(h) v \}$$

weight space w.r.t. the weight λ

- $\operatorname{ch} V = \sum_{\lambda \in \mathfrak{h}^*} \dim(V_\lambda) e^\lambda \in \mathbb{Z}[P]$

  - $\operatorname{ch}(V \oplus W) = \operatorname{ch}(V) + \operatorname{ch}(W)$
  - $\operatorname{ch}(V \otimes W) = \operatorname{ch}(V) \cdot \operatorname{ch}(W)$

- $x_\alpha \in \mathfrak{g}_\alpha$, $v \in V_\lambda$, then

$$x_\alpha v \in V_{\lambda+\alpha}.$$

- <u>Def</u>: a weight vector $v \in V_\lambda \subset V$ is a highest weight vector (of weight λ) if

$$\forall \alpha \in \Phi^+, \quad x_\alpha v = 0$$

- <u>Prop</u>: any fin. dim. $\mathfrak{g}$-rep. V has highest weight vectors.

So $\mathrm{Rep}^{(f)}(\mathfrak{g})$ may be embedded in a bigger category of $\mathfrak{g}$-reps.

<u>Def</u>: category $\mathcal{O}$

- full subcategory of $\mathrm{Rep}(\mathfrak{g})$
- $V \in \mathcal{O}$ if

  (a) it is finitely generated

  (b) $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ it is a weight module

  (c) $U(\mathfrak{n})$ acts locally nilpotently

  i.e. $\forall x \in U(\mathfrak{n}), \forall v \in V$

  $x^n v = 0$ for $n \gg 0$

<u>rmk</u>: Equivalently, $\mathcal{O}$ is the category of finitely generated $U(\mathfrak{g})$-modules s.t. the $\mathfrak{n}$-action integrates to a (rational) U-action

<u>Def</u>: Verma modules

$\lambda \in \mathfrak{h}^*$ $\qquad U(\mathfrak{b}) \twoheadrightarrow U(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}$

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$$

where $\mathbb{C}_\lambda$ is the $U(\mathfrak{b})$-rep. given by λ

M_λ is the Verma module of highest weight λ

<u>Prop</u>:

- $M_\lambda = U(\mathfrak{n}_-) v_\lambda$, with v_λ highest weight vector of weight λ.
- there is an isomorphism of functors

$$\mathrm{Hom}(M_\lambda, V) \cong V_\lambda^{\mathfrak{n}}$$

- consequences :

(a) $U(\mathfrak{n}_-) \longrightarrow M_\lambda$ is an iso of $U(\mathfrak{n}_-)$-modules

$x \longmapsto x v_\lambda$

(b) $M', M'' \subsetneq M_\lambda \Rightarrow M' + M'' \subsetneq M_\lambda$

(c) M_λ is indecomposable

and contains a maximum proper

submodule R_λ

(d) $L_\lambda = M_\lambda / R_\lambda$ is irreducible

of highest weight λ

(g) $L_\lambda \cong L_\mu \iff \lambda = \mu$

and

L simple in $\mathcal{O} \Rightarrow L = L_\lambda$ for some

$\lambda \in \mathfrak{h}^*$

Def: P^+ is the set of dominant integral weights

i.e. $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}$

for all simple coroots $\alpha_i^\vee \in \Lambda^\vee$

Prop: $\dim L_\lambda < \infty \iff \lambda \in P^+$.

proof: (sketch)
Follows by restricting the $\mathfrak{g}$-action to the copies of $sl_2(k)$ inside $\mathfrak{g}$ corresponding to the simple roots. $\square$

remarks on characters:

- $ch(L_\lambda)$ for $\lambda \in P^+$ is known by the Weyl character formula

- $ch(M_\lambda) = ch(U(n_-))e^\lambda$

- one way to determine $ch(L_\lambda)$ is to express L_λ in terms of Verma modules M_μ

- for $\lambda \in P^+$, the BGG resolution is a resolution of L_λ in terms of (sums of) Verma modules.

- knowledge of the composition multiplicities

$$[M_\mu : L_\lambda] \quad \forall \lambda, \mu \in \mathfrak{h}^*$$

is equivalent to the knowledge of the irreducible characters

$$ch(L_\lambda) \quad \lambda \in \mathfrak{h}^*$$

the Kazhdan-Lusztig conjecture gives an expression of $[M_\mu : L_\lambda]$ in terms of combinatorial objects (the Kazhdan-Lusztig polynomials)

steps in the proof of the KL conjecture
(for regular integral weights)

(KL I '79)
- definition of KL polynomials in terms of combinatorics of Coxeter group
- statement of KL conjecture

(KL II '80)
- interpretation of KL polynomials in terms of intersection cohomology of Schubert varieties

(Kashiwara, Mebkhout '80)
RH-correspondence perverse sheaves $\leftrightsquigarrow$ reg.-hol. $\mathcal{D}$-modules

(Beilinson–Bernstein
Brylinski–Kashiwara '81)
- equivalence between $\mathfrak{g}$-modules (with a fixed central character) and $\mathcal{D}$-modules on G/B

KL conj. (KL I '79)

KL pol. $[M_\mu : L_\lambda]$

IC cmplxs $\mathcal{D}$-modules BB-loc.

(KL II '80) (BB, BK '81)

RH corr.

(K, M, '80)

Example: $G = SL_2(k)$

irreducible reps: L_n, $n \in \mathbb{N}$

$L_n = \mathrm{Sym}^n(k^2)$

observe:

$$L_n = H^0(\mathbb{P}^1, \mathcal{O}(n))$$

and $\mathbb{P}^1 = G/B$

Moreover, by Serre duality,

$$H^i(\mathbb{P}^1, \mathcal{O}(n)) \cong H^{1-i}(\mathbb{P}^1, \mathcal{O}(-2-n))$$

this gives the cohomology spaces for all line bundles on $\mathbb{P}^1$.

the example generalizes to the

Borel–Weil–Bott thm:

<u>thm</u>: (Borel–Weil–Bott)
Let $\rho \in X(T)$ s.t. $\forall \alpha^\vee \in \Delta^\vee$, $\langle \rho, \alpha^\vee \rangle = 1$.
For $\lambda \in X(T)$, let $\mathcal{L}_\lambda$ be corresponding line bundle on G/B.

Suppose $\lambda \in X(T)^+$ (roots of B being negative) and $w \in W$. Then

- $$H^{\ell(w)}(G/B, \mathcal{L}_{w \cdot \lambda}) \cong L_\lambda$$

  and

- $$H^i(G/B, \mathcal{L}_{w\cdot\lambda}) = 0 \text{ for } i \neq \ell(w)$$

G - equivariant vector bundles

<u>Def</u>: a vector bundle on Y is a map

$$\begin{array}{c} X \\ \downarrow \pi \\ Y \end{array}$$

with compatible local trivializations

$$\pi^{-1}(U) \xrightarrow{\cong} U \times V$$

for V a fixed vector space and U small enough open subset of Y.

<u>Def</u>: a section of π on U is a map

$$s : U \longrightarrow \pi^{-1}(U)$$

s.t. $\pi \circ s = id_U$

Def: • $\mathcal{L}(X)$ is the sheaf of sections of $X \to Y$.

• $H^0(Y, \mathcal{L}(X)) = \{ s : Y \to X \mid \pi \circ s = id_Y \}$

is the space of global sections of $\mathcal{L}(X)$.

The functor H^0 is not exact in general,
it is only left-exact

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \mapsto 0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C})$$

Such left-exact short exact sequence may be extended to a long exact sequence by considering higher cohomology groups:

$$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to$$

$$\cdots \to H^i(\mathcal{A}) \to H^i(\mathcal{B}) \to H^i(\mathcal{C}) \to H^{i+1}(\mathcal{A})$$

$$\cdots$$

<u>Def</u>: a G-equivariant vector bundle is the datum of:

(a) a vector bundle $\pi : X \to Y$

(b) a G-action on X and Y such that

(i) π is G-equivariant

(ii) G acts linearly on the fibers of π (which are vector spaces)

G acts on $H^0(Y, \mathcal{L}(X))$ by

$$(g \cdot s)(y) = g(s(g^{-1}y)) \quad \text{for } g \in G, \; s \in H^0(Y, \mathcal{L}(X)), \; y \in Y$$

<u>Fact</u>: In good cases, this gives representations of G on the cohomology spaces. (e.g. $Y = G/B$)

associated bundles

$H \subset G$ subgroup

G/H is a homogeneous G-space

Def: Let $V \in \mathrm{Rep}(H)$, then

$$G \times^H V = G \times V /_H$$

where the H-action is given by

$$h(g,v) = (gh^{-1}, hv)$$

the map
$$G \times^H V \longrightarrow G/H$$
$$(g,v) \longmapsto gH$$

is a G-equivariant vector bundle on G/H with fiber V over H.

Prop: The functor

$$\mathrm{Rep}(H) \longrightarrow \mathrm{Vect}_G(G/H)$$
$$V \longmapsto G \times^H V$$

is an equivalence of abelian categories.

Consequence: $\mathrm{Pic}_G(G/B) \cong X(T)$

i.e. the G-equivariant line bundles on G/B are of the form

$$\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda \qquad \lambda \in X(T)$$

rmk: in fact, one has, for G simply connected,

$$\mathrm{Pic}_G(G/B) = \mathrm{Pic}(G/B)$$

Outline of the proof of the BWB thm

(i) for $s \in S$ simple reflection,

$\lambda \in X(T)$ s.t. $s \cdot \lambda < \lambda$,

there are short exact sequences

$$0 \to \mathcal{L}_{s\cdot\lambda} \to M \to N'' \to 0$$

$$0 \to M \to N' \to \mathcal{L}_\lambda \to 0$$

where $H^i(N') = H^i(N'') = 0 \ \forall i$

hence

$$H^i(\mathcal{L}_\lambda) \cong H^{i+1}(M) \cong H^{i+1}(\mathcal{L}_{s\cdot\lambda})$$

(ii) iterating, one gets

for $\lambda \in X(T)^+$, $w \in W$

$$H^{i+\ell(w)}(\mathcal{L}_{w\cdot\lambda}) \cong H^i(\mathcal{L}_\lambda)$$

(iii) $H^0(\mathcal{L}_\lambda)$ if nonzero is irreducible:

indeed, Bruhat decomposition implies

$$\dim H^0(\mathcal{L}_\lambda)^N = 1$$

so $H^0(\mathcal{L}_\lambda)$ is irreducible by highest weight theory

<u>rmk:</u>

step (i) is proved by using the SL_2 case and the fact that

$$H^i(\mathbb{P}^1, \mathcal{O}(-1)) = 0 \quad \forall i$$

Appendix A: equality with the Rees construction

$$U(\mathfrak{g}) = T(\mathfrak{g}) / I$$

$$R = \bigoplus_{n \in \mathbb{N}} U(\mathfrak{g})_{\leq n} t^n$$

$$R' = T(\mathfrak{g}) \otimes k[t] \Big/ x \otimes y - y \otimes x - t[x,y]$$

$$\begin{aligned} T(\mathfrak{g}) \otimes k[t] &\longrightarrow R \\ x &\longmapsto xt \end{aligned}$$

$$xt\,yt - yt\,xt = [x,y]t^2$$

so the map factorizes through

$$R' \longrightarrow R$$

$$R' \longrightarrow R$$

is a graded map of graded $k[t]$-algebras.

it is an isomorphism by the PBW thm

Appendix B:
Outline:

- $G \to GL(V)$, $\mathfrak{g} \to \mathfrak{gl}(V)$
- $\mathrm{Rep}(G) \xhookrightarrow{d} \mathrm{Rep}^{(f)}(\mathfrak{g})$ fully faithful
- $\mathrm{Rep}(G)$ and $\mathrm{Rep}^{(f)}(\mathfrak{g})$ are in general not s.s.

$\mathrm{Rep}(G)$, $\mathrm{Rep}^{(f)}(\mathfrak{g})$ semisimple $\Longleftrightarrow$ G, $\mathfrak{g}$ semisimple

- unitary trick, $\mathrm{Rep}(K)$ K maximal compact
- nondegeneracy of Killing form

- $\tilde{G}$ simply connected
  $\mathrm{Rep}(\tilde{G}) \cong \mathrm{Rep}^{(f)}(\mathfrak{g})$ for G simply conn. s.s.

- $SL_2(k) \twoheadrightarrow PGL_2(k) \cong SO_3(k)$, $sl_2(k)$

- Pb: classification and structure of irreducibles

Appendix C:

BGG resolution of L_λ (BGG '75)

$\lambda \in P^+$

$$\cdots \to C_1 \to C_0 \to L_\lambda \to 0$$

where $C_i = \bigoplus_{\substack{w \in W \\ \ell(w)=i}} M_{w\cdot\lambda}$

e.g. $C_0 = M_\lambda$

---

appendix D:

Weyl character formula $\lambda \in P^+$

$$ch(L_\lambda) = \frac{\sum_{w \in W} \varepsilon(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

Kazhdan-Lusztig conjecture

$\cdot \lambda \quad w_0 \cdot \lambda'$

$\lambda \in P^+$

$w \cdot \lambda$

$$ch(M_{w\cdot\lambda}) = \sum_{y \in W} P_{w,y}(1)\, ch(L_{y\cdot\lambda}) \quad y\cdot\lambda$$

may be inverted

$$ch(L_{w\cdot\lambda'}) = \sum_{y \in W} (-1)^{\ell(w)-\ell(y)} P_{y,w}(1)\, ch(M_{y\cdot\lambda'})$$

$$\boxed{[M_{w\cdot\lambda} : L_{y\cdot\lambda}] = P_{w,y}(1)}$$

Jordan-Holder multiplicity
$\shortparallel$
value of a Kazhdan-Lusztig polynomial