Introduction to the theory of Algebraic $\mathcal{D}$-modules

$k = \mathbb{C}$ $\quad X$ irreducible smooth algebraic variety $/\mathbb{C}$

$n = \dim X$ $\qquad \mathcal{O}_X$ structural sheaf.

$\Delta : X \hookrightarrow X \times_{\mathbb{C}} X$ diagonal embedding

$\Omega^1_X := \dfrac{\mathcal{J}}{\mathcal{J}^2}$ $\qquad$ is a locally free sheaf of rank n (algebraic 1-forms)

$$0 \to \mathcal{J} \to \mathcal{O}_X \otimes_{\mathbb{C}_X} \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X \to 0$$

$1\otimes f - f \otimes 1 \in \mathcal{J}$ $\qquad f$ section in $\mathcal{O}_X$ $\qquad X = \mathbb{A}^n_{\mathbb{C}}$ $\quad \mathcal{O}_X = \mathbb{C}[x_1, \dots, x_n]$

$1 \otimes x_i - x_i \otimes 1$ $\qquad \Omega^1_X \ni \sum_{i=1}^n f_i dx_i \quad f_i \in \mathcal{O}_X$

$d : \mathcal{O}_X \longrightarrow \Omega^1_X$ $\qquad$ is a differential map

$f \longmapsto d(f) := 1\otimes f - f \otimes 1 \in \mathcal{J}/\mathcal{J}^2$ $\qquad dx_i = 1\otimes x_i - x_i \otimes 1$

$fg \longmapsto d(fg) = d(f)g + f d(g)$

(it is $\mathbb{C}_X$-linear it is <u>not</u> $\mathcal{O}_X$-linear).

Def: $\Theta_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ denoted also $\mathcal{D}er_X$ is the $\mathcal{O}_X$-module of derivations.

This is the $\mathcal{O}_X$-locally free sheaf generated in local coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$ it has a natural structure of Lie-algebra.

By composing with d

$$\Theta_X \xhookrightarrow{\circ d} \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$$

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "d"] \arrow[dr, "\eta d"'] & \Omega^1_X \arrow[d, "\eta"] \\ & \mathcal{O}_X \end{tikzcd}$$

and we can inject also

$$\begin{array}{ccc} \mathcal{O}_X & \hookrightarrow & \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X,\mathcal{O}_X) \\ f & \longmapsto & f\cdot - \end{array}$$ multiplication by f.

Definition: $\forall\, U \overset{\text{open}}{\subseteq} X$ $\mathcal{D}_X(U)$ is the ring of differential operators in U, it is the subring of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U,\mathcal{O}_U)$ generated by $\mathcal{O}_X(U)$ and $\Theta_X(U)$.

The sheaf $U \longmapsto \mathcal{D}_X(U)$ is called the **sheaf of differentian operators**

in local coordinates $P \in \mathcal{D}_X(U)$ can be written in a unique way as $\sum_{\underline{\alpha}} a_{\underline{\alpha}} \partial_x^{\underline{\alpha}}$ where $\underline{\alpha} = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$ m.i

$\partial_x^{\underline{\alpha}} := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$ $\quad a_{\underline{\alpha}} \in \mathcal{O}_X(U)$, or as $\sum_{\underline{\beta}} \partial_x^{\underline{\beta}} b_{\underline{\beta}}$

Example: $X = \mathbb{A}^1_{\mathbb{C}}$ $\quad \mathcal{D}(X) = \mathbb{C}[t]\langle \partial_t \rangle$ $\quad [\partial_t, t] = 1$ is the one-variable **Weyl algebra**.

Remark: the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{D}_X$ is a Ring map
$\Rightarrow$ $\mathcal{D}_X$ has two canonical structures of $\mathcal{O}_X$-module given by the multiplication on the left or on the right

Def: a left $\mathcal{D}_X$-module is a quasi coherent $\mathcal{O}_X$-mod. $\mathcal{M}$

$$\begin{array}{ccc} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M} & \longrightarrow & \mathcal{M} \\ P \otimes m & \longmapsto & P\cdot m \end{array}$$ multiplication on the left

s.t. $\mathcal{M}(U)$ is a $\mathcal{D}_X(U)$-module.

$\mathcal{M}$ is **coherent** if $\forall x \in X$ $\exists$ affine open neighborhood

U: $\mathcal{D}_U^{m_1} \longrightarrow \mathcal{D}_U^{m_2} \longrightarrow \mathcal{M}_{|U} \to 0$ is exact in $\mathcal{D}_U$-mod.

(it has a finite presentation).

Example: $\mathcal{D}_X$ is a coherent $\mathcal{D}_X$-mod.

$\mathcal{O}_X$ is a left $\mathcal{D}_X$-module admitting the following presentation

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \Theta_X \to \mathcal{D}_X \twoheadrightarrow \mathcal{O}_X \to 0 \quad \Rightarrow \text{coherent}$$

$$P \longmapsto P(1)$$

$$\sum_{i=1}^n P_i \otimes \partial_i \longmapsto \sum_{i=1}^n P_i \partial_i \to \sum_{i=1}^n P_i(\partial_i(1)) = 0.$$

it can be extended in a simplicial way into the Spencer complex

$$0 \to \underbrace{\mathcal{D}_X \otimes_{\mathcal{O}_X} \Lambda^n \Theta_X \to \mathcal{D}_X \otimes \Lambda^{n-1} \Theta_X \to \dots \to \mathcal{D}_X \otimes \Theta_X \to \mathcal{D}_X}_{Sp^\bullet_X(\mathcal{D}_X)} \to \mathcal{O}_X \to 0$$

exact

Connections and left $\mathcal{D}_X$-modules

Given $\mathcal{M}$ an $\mathcal{O}_X$-module the following supplementary structures on $\mathcal{E}$ are equivalent:

ⓐ $\nabla : \mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ $\mathbb{C}_X$-linear satisfying the Leibniz rule

$\nabla(fm) = d(f) \otimes m + f\nabla(m)$ $\forall f \in \mathcal{O}_X$ $\forall m \in \mathcal{M}$

plus integrability condition $\nabla^2 \equiv 0$ where

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla} \Omega^2_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

$$\omega \otimes m \longmapsto d(\omega) \otimes m - \omega \wedge \nabla(m)$$

(b) a structure of left $\mathcal{D}_X$-module on $\mathcal{M}$: $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$

Remark: left coherent $\mathcal{D}_X$-modules which are $\mathcal{O}_X$-coherent $\Rightarrow$ are $\mathcal{O}_X$-locally free of finite rank + integrable connection $\nabla: \mathcal{F} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{F}$. They form an abelian category $\mathrm{Conn}(X)$.

Filtration of $\mathcal{D}_X$ by the order: increasing filtration

$$F_k \mathcal{D}_X \subset F_{k+1} \mathcal{D}_X \subset \cdots$$

- $F_k \mathcal{D}_X = 0$ if $k \leq -1$
- $F_0 \mathcal{D}_X = \mathcal{O}_X$, $F_1 \mathcal{D}_X = \mathcal{O}_X \oplus \Theta_X$
- the local sections P in $F_{k+1} \mathcal{D}_X$ satisfies $[P, f]$ is a local section of $F_k \mathcal{D}_X$ $\forall f$ in $\mathcal{O}_X$.

i.e. $F_k \mathcal{D}_X \ni \sum_{|\underline{\alpha}| \leq k} a_{\underline{\alpha}} \partial_x^{\underline{\alpha}}$.

Facts: • $\mathcal{D}_X = \bigcup_{k \in \mathbb{Z}} F_k \mathcal{D}_X$ any $F_k \mathcal{D}_X$ is locally $\mathcal{O}_X$-free of finite type $\Longrightarrow \mathcal{D}_X$ is a flat $\mathcal{O}_X$-module both left-right.

•• $F_\ell \mathcal{D}_X \cdot F_m \mathcal{D}_X = F_{\ell+m} \mathcal{D}_X$

$\because$ $P \in F_\ell \mathcal{D}_X$, $Q \in F_m \mathcal{D}_X$ $\quad [P, Q] \in F_{\ell+m-1} \mathcal{D}_X$

$\Longrightarrow$ we can take the graded sheaf

$$\mathrm{Gr} \mathcal{D}_X \quad \text{is commutative} \simeq \bigoplus_k \mathrm{Sym}^k \Theta_X$$

Example:

$X = \mathbb{A}^n_{\mathbb{C}}$ $\qquad \mathcal{D}_X = \mathbb{C}[x_1,\dots,x_n]\langle \partial_{x_1},\dots,\partial_{x_n}\rangle$ quasi coherent

$$Gr\,\mathcal{D}_X = \mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n] \quad \mathcal{O}_{T^*X}.$$

It is a noetherian ring of global hom. dimension $2n$.

Proposition: any algebraic coherent $\mathcal{D}_X$-module $\mathcal{M}$ admits a global good filtration i.e.

- $\mathcal{M} = \bigcup_{k\in\mathbb{Z}} F_k\mathcal{M} \qquad F_k\mathcal{M} \subset F_{k+1}\mathcal{M} \subset \dots \subset \mathcal{M}$

  any $F_k\mathcal{M}$ is $\mathcal{O}_X$-coherent $\quad F_i M = 0$ for $i \ll 0$
- $F_\ell\mathcal{D}_X \cdot F_k\mathcal{M} \subseteq F_{k+\ell}\mathcal{M} \quad \forall\, k,\ell$ and

  $F_\ell\mathcal{D}_X \cdot F_k\mathcal{M} = F_{k+\ell}\mathcal{M}$ for $k \gg 0$.

$\Longrightarrow$ $Gr_F\mathcal{M}$ is a coherent $Gr_F\mathcal{D}_X$-module.

Let T^*X be the cotangent bundle on X.

$$\begin{tikzcd} T^*X \arrow[d, shift left, "\pi"] \\ X \arrow[u, shift left, "\iota"] \end{tikzcd}$$

$\iota : X \to T^*X$ the zero section of π whose image is $T^*_X X$

$Gr_F\mathcal{D}_X \simeq \pi_*\mathcal{O}_{T^*X}$ in local coord. $\mathcal{O}_U[\xi_1,\dots,\xi_n]$.

We define $Ch(\mathcal{M})$ the characteristic variety of $\mathcal{M}$ as the support of the coherent $\mathcal{O}_{T^*X}$-module

$$\widetilde{gr_F\mathcal{M}} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1}Gr_F\mathcal{D}_X} \pi^{-1}(Gr_F\mathcal{M})$$

(subset of T^*X corresponding to the annihilator
$I_F(\mathcal{M}) = \mathrm{Ann}_{Gr_F \mathcal{D}_X} Gr_F \mathcal{M}$.)

$Ch(\mathcal{M})$ does not depend on the good filtration F and.

$\Rightarrow \mathrm{Supp}\,\mathcal{M} = \pi(Ch(\mathcal{M})) = Ch(\mathcal{M}) \cap T^*_X X$

Moreover $\forall$ s.e.s $0 \to \mathcal{M} \to \mathcal{L} \to \mathcal{N} \to 0$ in $\mathcal{D}_X$-coherent-mod

$$Ch(\mathcal{L}) = Ch(\mathcal{M}) \cup Ch(\mathcal{N}).$$

Examples:

- $\mathcal{D}_X \to \widetilde{Gr_F \mathcal{D}_X} = \mathcal{O}_{T^*X}$ $\quad Ch(\mathcal{D}_X) = T^*X$
- $\mathcal{O}_X$ $\quad F_k\mathcal{O}_X = \mathcal{O}_X \ \forall k \geq 0 \Rightarrow Gr_F \mathcal{O}_X = \mathcal{O}_X$ in degree 0
  $F_k \mathcal{O}_X = 0 \ \forall k < 0$

$\Rightarrow \mathrm{Ann}_{Gr_F\mathcal{D}_X} Gr_F \mathcal{O}_X = \bigoplus_{i>0} Gr_F^i \mathcal{D}_X \Rightarrow Ch(\mathcal{O}_X) = T^*_X X$.

if $X = \mathbb{A}^n_{\mathbb{C}}$ $\quad (\xi_1,\dots,\xi_n) \trianglelefteq \mathbb{C}[x_1,\dots,x_n,\xi_1,\dots,\xi_n]$

- $\forall$ $\mathcal{F}$ integrable conn. ($\mathcal{O}_X$- loc. free finite rank) $Ch(\mathcal{F}) = T^*_X X$

- $X = \mathbb{A}^1_{\mathbb{C}}$ $\quad \mathcal{D}_X = \mathbb{C}[x]\langle \frac{d}{dx}\rangle$ $\quad \mathcal{L} := \frac{\mathcal{D}_X}{(x\frac{d}{dx}+1)}$

action of derivative

$$0 \to \mathrm{Ker}\,\varphi \to \mathcal{D}_X \overset{\varphi}{\twoheadrightarrow} \mathbb{C}[x,\tfrac{1}{x}] \simeq \mathcal{L} \qquad \mathcal{O}_{\mathbb{A}^1_{\mathbb{C}}}(*0)$$

$$1 \longmapsto \frac{1}{x}$$

$$\frac{d}{dx} \longmapsto \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

$$x\frac{d}{dx} + 1 \longmapsto x\frac{d}{dx}\left(\frac{1}{x}\right) + \frac{1}{x} = 0$$

$$x\frac{d}{dx}+1 \longmapsto x\frac{d}{dx}\left(\frac{1}{x}\right)+\frac{1}{x}=0$$

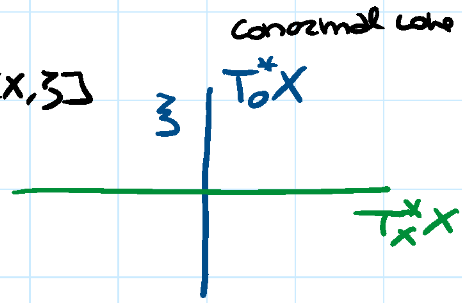
is coherent

$F_k\,\mathbb{C}[x,\tfrac{1}{x}] := 0$ se $k<0$; $F_0\,\mathbb{C}[x,\tfrac{1}{x}] = \mathbb{C}[x]$,

$F_k\,\mathbb{C}[x,\tfrac{1}{x}] := \bigoplus_{i=1}^{k} \mathbb{C}[x]\frac{1}{x^i} \oplus \mathbb{C}[x]$ order of the pole $\forall k \geq 1$.

$\Rightarrow \quad \mathrm{Ann}_{Gr_F \mathcal{D}_X}\, Gr_F \mathcal{L} = (x\xi) \trianglelefteq \mathbb{C}[x,\xi]$

$\Rightarrow \quad Ch(\mathcal{L}) = T^*_X X \cup T^*_0 X$



$$\begin{array}{ccccccccc}
 & & \mathcal{M} & & \mathcal{L} & & \mathcal{N} & & \\
 & & \wr\wr & & \| & & \| & & \\
0 & \to & \mathbb{C}[x] & \longrightarrow & \mathbb{C}[x,\tfrac{1}{x}] & \longrightarrow & \dfrac{\mathbb{C}[x,\frac{1}{x}]}{\mathbb{C}[x]} & \to & 0 \qquad (\bullet)\\
 & & \| & & \| & & \| & & \\
0 & \to & \dfrac{\mathcal{D}_X}{(\frac{d}{dx})} & \xrightarrow{x} & \dfrac{\mathcal{D}_X}{(x\frac{d}{dx}+1)} & \longrightarrow & \dfrac{\mathcal{D}_X}{(x)} & \to & 0
\end{array}$$

Recall that $\frac{d}{dx}x = x\frac{d}{dx}+1$

this is a torsion module supported on 0.

Involutivity of $Ch(\mathcal{M})$:

Let us consider the fundamental 2-form ω on T^*X in coord. $(x_1,\dots,x_n,\xi_1,\dots\xi_n)$ $\quad \omega = \sum_{i=1}^{n} d\xi_i \wedge dx_i$

$\forall\ (x,\xi) \in T^*X$, $\quad \omega$ defines on $T_{(x,\xi)}(T^*X)$ a non degenerate bilinear form

$\quad\ \ \downarrow \qquad \downarrow \pi$

$\quad\ \ x \ \in\ X$

Theorem (Gabber) Char $\mathcal{M}$ is involutive in T^*X

$\Longrightarrow \dim Ch(\mathcal{M}) \geq n$

Definition: a coherent $\mathcal{D}_X$-module is said to be **holonomic** if $\dim Ch(\mathcal{M}) = n$ (in this case $= \bigcup_\alpha T^*_{Y_\alpha} X$).

Ex. $0 \to \mathcal{M} \to \mathcal{L} \to \mathcal{N} \to 0$ of the previous example is a short exact sequence of holonomic $\mathcal{D}_X$-modules.

In $\mathbb{A}^2_{\mathbb{C}} = X$ $\quad \dfrac{\mathcal{D}_X}{(P)}$ is never holonomic its characteristic variety is a hypersurface of T^*X.

Prop: given a s.e.s. $0 \to \mathcal{M} \to \mathcal{L} \to \mathcal{N} \to 0$ of coherent $\mathcal{D}_X$-modules $\mathcal{L}$ is holonomic $\Longleftrightarrow$ $\mathcal{M}, \mathcal{N}$ are holonomic.

Using the algebraic properties of $Gr_F \mathcal{D}_X$ we get

Proposition: $\forall$ coherent $\mathcal{D}_X$-module $\mathcal{M}$

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \quad \forall \quad i \geq n+1$$

Moreover

$$2n - \dim_x Ch(\mathcal{M}) = \inf\{ i \in \mathbb{N} \mid \mathcal{E}xt^i_{\mathcal{D}_{X,x}}(\mathcal{M}_x, \mathcal{D}_{X,x}) \neq 0 \} =: j_x(\mathcal{M})$$

$\Rightarrow \dim Ch(\mathcal{M}) = 2n - \underbrace{j(\mathcal{M})}_{\text{grade of } \mathcal{M}}$

$\Rightarrow \mathcal{M}$ is holonomic $\Leftrightarrow \mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \ \forall i \neq n.$

This is a consequence of the fact that $Gr_F \mathcal{D}_X$ is Auslander regular.

Left $\mathcal{D}_X$-mod

$\mathcal{O}_X$

Right $\mathcal{D}_X$-mod.

$\omega_X := \bigwedge \Omega^1_X$ locally free of rk 1.

given $\xi \in \Theta_X \quad \omega \in \omega_X$

$\omega \cdot \xi = - L_\xi \omega$

Lie derivative

$\mathcal{O}_X \simeq \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \omega_X)$

In general:

given $\mathcal{M}_1, \mathcal{M}_2 \in \mathcal{D}_X$-mod

① $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_1, \mathcal{M}_2) \in \mathcal{D}_X$-mod

①' $\mathcal{M}_1 \otimes_{\mathcal{O}_X} \mathcal{M}_2 \in \mathcal{D}_X$-mod

while given $\mathcal{N}_1, \mathcal{N}_2 \in$ mod-$\mathcal{D}_X$

② $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}_1, \mathcal{N}_2) \in \mathcal{D}_X$-mod

$\mathcal{M} \in \mathcal{D}_X\text{-mod}$ $\quad \mathcal{N} \in \text{mod-}\mathcal{D}_X$

③ $\quad \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{M} \in \text{mod-}\mathcal{D}_X$

Equivalence left vs right $\mathcal{D}_X$ modules.

$$\begin{array}{ccc} \mathcal{D}_X\text{-mod} & & \text{mod-}\mathcal{D}_X \\ \mathcal{M} & \longmapsto & \omega_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \omega_X^{-1} \otimes_{\mathcal{O}_X} \mathcal{N} := \mathcal{H}om_{\mathcal{O}_X}(\omega_X, \mathcal{N}) & \longleftarrow & \mathcal{N} \end{array}$$

Grothendieck operations:

formalism of 6 functors between derived categories

In $D^b_{R}(\mathcal{D}_X)$

$R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_1^\bullet, \mathcal{M}_2^\bullet)$ $\quad \mathcal{M}_1 \otimes^{L}_{\mathcal{O}_X} \mathcal{M}_2$ $\quad \mathbb{D}_X$

$f_+ \quad f^+ \quad f_! \quad f^!$

① **Duality:** if $\mathcal{M}$ is left holonomic $\mathcal{D}_X$-mod.

$\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is a right holonomic $\mathcal{D}_X$-mod

$$\mathbb{D}_X(\mathcal{M}) := \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}$$

In general

$\swarrow$ derived categ. with coherent cohomology.

$\mathbb{D}_X : D_c(\mathcal{D}_X) \longrightarrow D_c(\mathcal{D}_X)$ contravariant

$\mathcal{M} \longmapsto R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \omega_X^{-1}[n]$

Transfer modules:

Let consider $f: X \longrightarrow Y$ finite type morphism between smooth algebraic varieties.

$\boxed{\mathcal{D}_{X\to Y}} := f^*(\mathcal{D}_Y)$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule (see below).

Reversing the structures using left-right trasformation on both sides

$$\boxed{\mathcal{D}_{Y\leftarrow X}} := \mathcal{H}om_{f^{-1}\mathcal{O}_Y}(f^{-1}\omega_Y, \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y}) =$$

$$= \omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}(\omega_Y^{-1}).$$

② Inverse image for left $\mathcal{D}$ modules:

we have a canonical s.e.s $0 \to f^*\Omega^1_Y \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$

Given $\mathcal{M}$ a left $\mathcal{D}_Y$-mod: $\mathcal{M} \xrightarrow{\nabla} \Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{M}$

$$f^*\mathcal{M} \longrightarrow f^*(\Omega^1_Y \otimes_{\mathcal{O}_Y} \mathcal{M}) \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} f^*\mathcal{M}$$

induce a left $\mathcal{D}_X$-module structure on

$$f^*\mathcal{M} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}.$$

$$\boxed{\mathcal{D}_{X\to Y} := f^*\mathcal{D}_Y}$$ is a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$-bimodule

transfer module.

$$f^*\mathcal{M} \simeq f^*(\mathcal{D}_Y \otimes_{\mathcal{D}_Y} \mathcal{M}) \simeq \mathcal{D}_{X\to Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$$

passing to $D^b(\mathcal{D}_Y) \xrightarrow{f^+} D^b(\mathcal{D}_X)$

$$f^+(\mathcal{M}^\cdot) := Lf^*(\mathcal{M})[d_X - d_Y]$$

③ Direct Image for right $\mathcal{D}$ modules

$\mathcal{N} \in \text{mod-}\mathcal{D}_X$

$$f_+(\mathcal{N}) := Rf_*(\mathcal{N} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} \mathcal{D}_{X\to Y}) \in \text{mod-}\mathcal{D}_Y.$$

Ex: $\{0\} \overset{f}{\hookrightarrow} \mathbb{A}^1_{\mathbb{C}}$

$$\mathcal{D}_{\{0\}\to \mathbb{A}^1_{\mathbb{C}}} = f^*(\mathcal{D}_{\mathbb{A}^1_{\mathbb{C}}}) = \mathbb{C}[x]\langle \tfrac{d}{dx}\rangle \otimes_{\mathbb{C}[x]} \frac{\mathbb{C}[x]}{(x)} \simeq \frac{\mathcal{D}_{\mathbb{A}^1_{\mathbb{C}}}}{(x)}$$

it is not coherent as $\mathcal{D}_0\text{-mod} = \mathbb{C}\text{-vect}$. $\quad \mathcal{O}_{\{0\}} = \mathcal{D}_{\{0\}} = \mathbb{C}$

$$f_*(\mathbb{C}) = f_*(\mathcal{D}_{\{0\}\to \mathbb{A}^1_{\mathbb{C}}}) = \frac{\mathcal{D}_{\mathbb{A}^1_{\mathbb{C}}}}{(x)} \simeq \mathbb{C}[\tfrac{d}{dx}] \qquad (\cdot).$$

Direct image for left $\mathcal{D}$ modules:

$f: X \to Y \quad \mathcal{M} \;\; \mathcal{D}_X\text{-mod}$

$$\mathbb{R}f_* \left(\mathcal{D}_{Y \leftarrow X} \otimes^{\mathbb{L}}_{\mathcal{D}_X} \mathcal{M} \right)$$

Derived categories

The use of derived categories is an essential tool in the theory of $\mathcal{D}_X$-modules: if we consider

$\mathcal{A} = \mathcal{D}_X$-modules, it is an abelian category

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Ker} f \arrow[r] & \mathcal{M} \arrow[r] \arrow[d] & \mathcal{N} \arrow[r] & \operatorname{Coker} f \arrow[r] & 0 \\ & & \operatorname{Im} f \arrow[ur] & & & \end{tikzcd}$$

We construct $C^b(A)$ the category of complexes:

$$\begin{tikzcd} M^\bullet & \cdots \arrow[r] & M^{i-1} \arrow[r] \arrow[d, "\varphi^{i-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & M^i \arrow[r, "d_M^i"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & M^{i+1} \arrow[r] \arrow[d] & \cdots \\ N^\bullet & \cdots \arrow[r] & N^{i-1} \arrow[r] & N^i \arrow[r, "d_N^i"'] & N^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

$i \in \mathbb{Z}$ $\quad d_M^{i+1} d_M^i = 0$

$M^i = 0$ for $i >> 0$, $i << 0$

we are interested in $H^i(M^\bullet) := \dfrac{\operatorname{Ker} d_M^i}{\operatorname{Im} d_M^{i-1}}$ gap of exactness

we would like to say that a morphism of complexes $L^\bullet \xrightarrow{\varphi^\bullet} M^\bullet$ is an "isomorphism" (= quasi-isomorphic) in $D(A)$ if $H^i(L^\bullet) \xrightarrow{H^i(\varphi^\bullet)} H^i(M^\bullet)$ is an isomorphism $\forall i \in \mathbb{Z}$.

Hence a complex $N^\bullet \xrightarrow{\sim} 0^\bullet \iff H^i(N^\bullet) = 0$ exact complex.

Inside exact complexes there is a special class

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[rr] & & N^{i-1} \arrow[rr] \arrow[dr, two heads] & & N^{i} \arrow[rr] \arrow[dr, two heads] & & N^{i+1} \arrow[rr] \arrow[dr] & & \cdots \\
\cdots & \operatorname{Ker} d^{i-1} \arrow[ur, hook, "\oplus"] & & \operatorname{Ker} d^{i} \arrow[ur, hook, "\oplus"] & & \operatorname{Ker} d^{i+1} \arrow[ur, hook, "\oplus"] & & \cdots &
\end{tikzcd} \qquad (\circledast)$$

such that

$0 \to \operatorname{Ker} d^i \to N^i \to \operatorname{Ker} d^{i+1} \to 0$ is a split s.e.s.

Since we want $N^\bullet \simeq 0^\bullet$ we have to pass to

$K(\mathcal{A}) :=$ category of complexes up to homotopy

objects of $K(\mathcal{A})$ = objects of $\mathcal{C}(\mathcal{A})$

$$\operatorname{Hom}_{K(\mathcal{A})}(M^\bullet, N^\bullet) = \frac{\operatorname{Hom}_{\mathcal{C}(\mathcal{A})}(M^\bullet, N^\bullet)}{Ht(M^\bullet, N^\bullet)}$$

$\varphi \in Ht(M^\bullet, N^\bullet)$

$$\Longleftrightarrow \quad \begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r] \arrow[d] & M^{i} \arrow[r] \arrow[d, "\varphi^i"] \arrow[dl, dashed, "s^i"'] & M^{i+1} \arrow[r] \arrow[d] \arrow[dl, dashed, "s^{i+1}"'] & \cdots \\
\cdots \arrow[r] & N^{i-1} \arrow[r] & N^{i} \arrow[r] & N^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

such that $\varphi^i = d_N^{i-1} s^i + s^{i+1} d_M^i$.

$\Longrightarrow$ any complex $\circledast$ is zero since $id_{N^\bullet} \in Ht(N^\bullet, N^\bullet)$.

If any s.e.s. splits we have done:

Ex: $X = \operatorname{Spec} \mathbb{C}$ $\quad \mathcal{D}_X = \mathcal{O}_X = \mathbb{C}$ $\quad \mathcal{D}_X\text{-mod} = \mathbb{C}\text{-vect}$

Since any s.e.s. of vector spaces splits the complexes qis to $0^\bullet$ are that of the form $(\circledast)$ and

$K(\mathbb{C}\text{-vect}) \simeq \mathbb{Z}$-graded vector spaces since $\forall$ complex $V^\bullet$

$$\begin{tikzcd}
& H^i(V^\bullet) \arrow[d, equal] & H^{i+1}(V^\bullet) \arrow[d, equal] & \\
\cdots \arrow[r, "0"] & \dfrac{\operatorname{Ker} d_V^i}{\operatorname{Im} d_V^{i-1}} \arrow[r, "0"] \arrow[d, "\varphi^i"] & \dfrac{\operatorname{Ker} d_V^{i+1}}{\operatorname{Im} d_V^{i}} \arrow[r, "0"] \arrow[d] & {} \\
& \cdots & \cdots &
\end{tikzcd}$$

$$\begin{tikzcd} & \operatorname{Im} d_V \arrow[d, "\varphi^i"'] & & \operatorname{Im} d_V \arrow[d] \\ \arrow[r] & V^i \arrow[rr, "d^i"] & & V^{i+1} \end{tikzcd}$$

since any $\operatorname{Ker} d^i_V \twoheadrightarrow H^i(V)$ splits (α_i) $\quad V^\cdot \simeq \bigoplus_{i\in\mathbb{Z}} H^i(V^\cdot)$.

In general it is not true that any s.e.s splits, but it is true if the first term of the sequence is injective

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "id"'] & L \arrow[r] \arrow[dl, dotted] & M \arrow[r] & 0 \\ & J \end{tikzcd}$$

or the third one projective

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] & M \arrow[r] & P \arrow[r] & 0 \\ & & & P \arrow[u, "id"'] \arrow[ul, dotted] \end{tikzcd}$$

$\mathcal{D}^b(A) := K^b(A)/qis$

$$\begin{tikzcd} & M' \arrow[dl, "\varphi"', "qis"] \arrow[dr, "f"] & \\ M^\cdot & & N^\cdot \end{tikzcd} \quad f\varphi^{-1}$$

$$\begin{tikzcd} M^\cdot \arrow[dr, "g"'] & & N^\cdot \arrow[dl, "\varphi"] \\ & N'' & \end{tikzcd} \quad \varphi^{-1}g$$

but if we think at the previous examples $D^b(A) \simeq K^b(J)$ where J = injective $\mathcal{D}_X$-modules.

$\mathcal{D}^b(A)$ is additive and triangulated

$(M^\cdot[1])^i = M^{i+1}$ shift one step on the left $\quad \to \overset{0}{M^1} \to \overset{1}{M^2} \to \dots$

Any s.e.s of complexes $\longrightarrow$

$$\begin{tikzcd} M^\cdot \arrow[rr] & & L^\cdot \arrow[dl] \\ & N^\cdot \arrow[ul, "{[1]}"] & \end{tikzcd}$$

distinguished triangles + axioms.