

From the BGG category \mathcal{O} to locally analytic representations

Abstract

¹These are lecture notes based on the presentation of Matthias Strauch "On Jordan-Hölder series of some locally analytic representations", and Sascha Orlik "On some local properties of the functor \mathcal{F}_P^G from Lie algebras to locally analytic representations" in the study group " p -adic representations and arithmetic \mathcal{D} -modules" carried out at the University of Padova.

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Notation: Throughout these notes L will be a finite extension of \mathbb{Q}_p and K a finite extension of L which will be our coefficient field. We will also assume that \mathbb{G} is a split connected reductive algebraic group over L , and we will take $\mathbb{T} \subseteq \mathbb{B} \subseteq \mathbb{P} \subseteq \mathbb{G}$ a maximal torus contained in a Borel subgroup, which in turns is contained in a parabolic subgroup of \mathbb{G} . We will use capital letters to denote the respective groups of L -points, for instance $G := \mathbb{G}(L)$, $P := \mathbb{P}(L)$, and so on. Moreover, we will denote by gothic letters the Lie algebra of the group concerned, this is $\mathfrak{g} := \text{Lie}(\mathbb{G})$, $\mathfrak{b} := \text{Lie}(\mathbb{B})$ and $\mathfrak{p} := \text{Lie}(\mathbb{P})$. Furthermore, once we fix the Levi decomposition [6, Part II, 1.8]

$$\mathbb{P} := \mathbb{L}_{\mathbb{P}} \mathbb{U}_{\mathbb{P}},$$

where $\mathbb{L}_{\mathbb{P}}$ is the **Levi factor** and $\mathbb{U}_{\mathbb{P}}$ is the **unipotent radical**, we set $\mathfrak{l}_{\mathbb{P}} := \text{Lie}(\mathbb{L}_{\mathbb{P}})$ and $\mathfrak{u}_{\mathbb{P}} := \text{Lie}(\mathbb{U}_{\mathbb{P}})$. Finally, the base change of an L -vector space (or an L -scheme) to K will always be denoted by the subscript K , in other words, $\mathfrak{g}_K := \mathfrak{g} \otimes_L K$ and $\mathbb{G}_K := \mathbb{G} \times_{\text{Spec}(L)} \text{Spec}(K)$.

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1 Review of the definition of the functor \mathcal{F}_P^G

The first part of these notes is dedicated to recall the definition of the functor (cf. [9, Section 4])

$$\mathcal{F}_P^G := \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \times \text{Rep}_K^{\infty, \text{adm}}(L_P) \rightarrow \text{Rep}_K^{\text{la}}(G) \quad (1)$$

which depends of the parabolic subgroup $P \subseteq G$. We recall for the reader that $\text{Rep}_K^{\infty, \text{adm}}(L_P)$ denotes the category of **smooth admissible representations** of the Levi subgroup $L_P \subseteq P$ on K -vector spaces. We also recall that $\text{Rep}_K^{\text{la}}(G)$ is the category of **locally analytic representations** of G on K -vector spaces.²

1.1 The algebraic BGG category \mathcal{O}

By $\mathcal{O}^{\mathfrak{b}_K}$ we will consider the following adaptation of the BGG category \mathcal{O} when the coefficient field is not algebraic [9, 2.5]:

- (1) M is a finitely generated $U(\mathfrak{g}_K)$ -module.
- (2) M decomposes as a direct sum of one-dimensional \mathfrak{t}_K -representations.
- (3) The action of \mathfrak{b}_K on M is locally finite in the usual sense (cf. [5, (1.1)]).³

We will also consider $\mathcal{O}^{\mathfrak{p}_K}$ the subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ consisting of those modules M in $\mathcal{O}^{\mathfrak{b}_K}$ on which \mathfrak{p}_K acts locally finitely. In [9] the authors were mainly interested in the following subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ (resp. $\mathcal{O}^{\mathfrak{p}_K}$). First of all, let us note that property (2) tells us that any object $M \in \mathcal{O}^{\mathfrak{b}_K}$ (resp. in $\mathcal{O}^{\mathfrak{p}_K}$) can be written as a direct sum on one-dimensional \mathfrak{t}_K -representations

$$M = \bigoplus_{\lambda \in \mathfrak{t}_K^*} M_\lambda \quad (2)$$

where $M_\lambda := \{m \in M \mid \forall \mathfrak{r} \in \mathfrak{t}_K, \mathfrak{r} \cdot m = \lambda(\mathfrak{r})m\}$ is the **eigenspace** associated to $\lambda \in \mathfrak{t}_K^* := \text{Hom}_K(\mathfrak{t}_K, K)$. Let $X^*(\mathbb{T}_K) := \text{Hom}_{\text{alg. gps}}(\mathbb{T}_K, \mathbb{G}_m)$ be the **group of (algebraic) characters** of the torus \mathbb{T}_K , which can be considered as a subgroup (lattice) of \mathfrak{t}_K^* via derivation. We have the following fundamental definition ([9, Definition 2.6]).

Definition 1.1.1. *We let $\mathcal{O}_{\text{alg}}^{\mathfrak{b}_K}$ be the full subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ whose objects are $U(\mathfrak{g}_K)$ -modules such that the decomposition (2) is algebraic. In other words, all λ appearing in (2), for which $M_\lambda \neq 0$, are contained in $X^*(\mathbb{T}_K) \subseteq \mathfrak{t}_K^*$.*

²To see the notes of the seminar of Andrés Sarrazola A. on "Fréchet-Stein algebras and coadmissible modules", and the notes of Minh Phuong Vu on "admissible locally analytic representations". See also [15, Section 3] and [16, Section 6].

³To see also the notes of Giovanna Carnovale on "The Bernstein-Gelfand-Gelfand category \mathcal{O} ".

Let \mathcal{O}^P be the category whose objects are pairs $\underline{M} := (M, \tau)$, where $M \in \mathcal{O}^{\mathfrak{p}}$ and $\tau : P \rightarrow \text{End}_K(M)^*$ is locally analytic locally finite P -representation, e.g, $M = \cup_{i \in \mathbb{N}} M_i$ is an increasing union of finite-dimensional locally analytic P -stable subspaces, such that the derived action of \mathfrak{p}_k lifts the initial \mathfrak{p}_k -action, and such that the actions of P and \mathfrak{g}_k are compatible. The category \mathcal{O}^P is abelian ([10, Lemma 2.5]) and any object is of finite length ([10, Lemma 2.7]). It is clear that we have a forgetful functor

$$\begin{aligned} \omega : \mathcal{O}^P &\rightarrow \mathcal{O}^{\mathfrak{p}_k} \\ \underline{M} &\mapsto M. \end{aligned} \quad (3)$$

On the other hand, we will denote by $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_k}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}_k}$ formed by objects M such that in the weight decomposition $M = \bigoplus_{\lambda \in \mathfrak{t}_K^*} M_\lambda$ all occurring λ lie in the lattice of algebraic characters $X^*(\mathbb{T}_K) \subseteq \mathfrak{t}_K^*$. There exists a fully faithful embedding

$$\mathcal{O}_{\text{alg}}^{\mathfrak{p}_k} \rightarrow \mathcal{O}^P \quad (4)$$

whose composition with the forgetful functor equals the inclusion $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_k} \subseteq \mathcal{O}^{\mathfrak{p}_k}$.⁴ To see this, let us remark first that the algebraic \mathbb{T}_K -action on an object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_k}$ lifts uniquely to an algebraic $\mathbb{L}_{\mathbb{P}, K}$ -action on each finite dimensional simple $\mathbb{L}_{\mathfrak{p}, K}$ -constituent of M (see the proof of [9, Lemma 2.8]). Moreover, via the exponential map, the action of the Lie algebra $\mathfrak{u}_{\mathfrak{p}, K}$ integrates uniquely to give an algebraic action of $\mathbb{U}_{\mathbb{P}, K}$ on M (see the proof of [9, Lemma 3.2]). Both actions combined give an algebraic \mathbb{P}_K -action on M , and therefore $M \in \mathcal{O}^P$.

Example 1.1.2. (i) ([9, Example 2.7]) Let $\lambda \in \mathfrak{t}_K^*$ and K_λ be the one-dimensional \mathfrak{t}_K -representation whose \mathfrak{t}_K -action is given by λ . This action extends uniquely to a \mathfrak{b}_K -module structure. Let

$$M(\lambda) := U(\mathfrak{g}_K) \otimes_{U(\mathfrak{b}_K)} K_\lambda$$

be the corresponding Verma module, which is an object in $\mathcal{O}^{\mathfrak{b}_k}$. Denoting by $L(\lambda)$ its simple quotient, we have that $M(\lambda)$ and $L(\lambda)$ are objects in $\mathcal{O}^{\mathfrak{b}_k}$ if and only if $\lambda \in X^*(\mathbb{T}_K)$.

(ii) ([9, Example 2.10]) Let \mathcal{S} be the set of simple roots of \mathbb{G}_K with respect to $\mathbb{T}_K \subseteq \mathbb{B}_K$. Let $\lambda \in X^*(\mathbb{T}_K)$ be an algebraic character, and let us consider $I := \{\alpha \in \mathcal{S} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$. Let \mathbb{P}_I be the standard parabolic subgroup of \mathbb{G}_K attached to I . It is known that λ is dominant with respect to the Levi factor $\mathbb{L}_{\mathbb{P}_I}$. Denote by $V_I(\lambda)$ the corresponding irreducible finite-dimensional algebraic $\mathbb{L}_{\mathbb{P}_I}$ -representation ([6, Part II, 2.14] and [8, Page 4]), which we consider as a \mathbb{P}_I -module by letting act $\mathbb{U}_{\mathbb{P}}$ trivially on it. The **generalized parabolic Verma module** (in the sense of [8, Page 4]) attached to the weight λ is defined by

$$M_I(\lambda) := U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_I)} V_I(\lambda).$$

⁴The following reasoning has been taken from [2, 4.1].

Then $M_I(\lambda)$ is an object of $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Moreover, there exists a surjective map

$$M(\lambda) \rightarrow M_I(\lambda)$$

where the kernel is given by the image of $\bigoplus_{\alpha \in I} M(s_\alpha \cdot \lambda) \rightarrow M(\lambda)$ ([6, Theorem 9.4 (b)]). Given that λ is algebraic, it follows from [6, Theorem 9.4 (a)] and the first part of the example that $L(\lambda)$ is an object in $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$.

1.2 From \mathcal{O}_{alg} to locally analytic representations

The goal of this subsection is to show how to attach, in a natural way, to any object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ a locally analytic representation. This process will define a functor

$$\mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \rightarrow \text{Rep}_K^{\text{la}}(G),$$

which we will extend naturally to define the bi-functor (1). To start with, we remark for the reader that the defining properties (1) and (3) for $\mathcal{O}^{\mathfrak{p}_K}$ allow us to take a finite-dimensional \mathfrak{p}_K -representation $W \subseteq M$ which generates M as a $U(\mathfrak{g}_K)$ -module. In other words, for any $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$, we have a short exact sequence of $U(\mathfrak{g}_K)$ -modules

$$0 \rightarrow \mathfrak{d} \rightarrow U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M \rightarrow 0, \quad (5)$$

where \mathfrak{d} is the kernel of the canonical map $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M$. Furthermore, the same reasoning given just before the example 1.1.2 shows that W can be endowed with a structure of a locally analytic P -representation, and therefore its K -dual space W' is also a locally analytic P -representation. By [9, Lemma 2.4] the canonical map of $D(G, K)$ -modules

$$\begin{aligned} D(G, K) \otimes_{D(P, K)} W &\rightarrow (\text{Ind}_P^G(W'))' \\ \delta \otimes w &\mapsto [f \mapsto \delta(f)(w)] \end{aligned}$$

is an isomorphism (I think that the definition of the function on the right hand side uses the fact that W' is of compact type and therefore we have the identification $C^{\text{an}}(G, W') = C^{\text{an}}(G, K) \hat{\otimes}_{K, \pi} W'$) and we dispose of a canonical paring

$$\langle \cdot, \cdot \rangle : (D(G, K) \otimes_{D(P, K)} W) \otimes_K \text{Ind}_P^G(W') \rightarrow K \quad (6)$$

which identifies the left hand side with the topological dual of the right hand side and vice versa. We can give a more explicit description of the preceding paring (cf. [9, (3.2.2)]). First of all, let us considering the $C^{\text{an}}(G, K)$ -valued paring

$$\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, K)} : \begin{aligned} (D(G, K) \otimes_{D(P, K)} W) \otimes_K \text{Ind}_P^G(W') &\rightarrow C^{\text{an}}(G, K) \\ (\delta \otimes w) \otimes f &\mapsto [g \mapsto (\delta_r(f(\cdot)(w)))(g)] \end{aligned}$$

where $(\delta_r(f(\cdot)(w)))(g) := \delta(x \mapsto f(gx)(w))$. On the other hand, using the arguments exhibited in [11, 3.4.1], it is possible to prove that the canonical map

$$U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow D(G, K) \otimes_{D(P, K)} W$$

is injective, and we can suppose that $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W$ is a subspace of $(\text{Ind}_P^G(W'))'$. We then denote by $\text{Ind}_P^G(W')^\mathfrak{d}$ the closed subspace of $\text{Ind}_P^G(W')$ annihilated by \mathfrak{d} via the paring $\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, K)}$:

$$\text{Ind}_P^G(W')^\mathfrak{d} := \{f \in \text{Ind}_P^G(W') \mid \forall \delta \in \mathfrak{d}, \langle \delta, f \rangle_{C^{\text{an}}(G, K)} = 0\}.$$

This is, by construction, a G -invariant subspace of $\text{Ind}_P^G(W')$ (cf. [9, Comment before the proposition 3.3]). Moreover, by [9, Lemma 2.4] the representation $\text{Ind}_P^G(W')$ is strongly admissible (in the sense of [15]), and therefore $\text{Ind}_P^G(W')^\mathfrak{d}$ is also a strongly admissible locally analytic G -representation, being a closed invariant subspace of a strongly admissible representation [15, Lemma 3.5]. Finally, it is not very hard to prove that the annihilator, under the canonical paring defined in (6), of $\text{Ind}_P^G(W')^\mathfrak{d}$ in $D(G, K) \otimes_{D(P, K)} W$ is equal to $D(G, K)\mathfrak{d}$ ([9, Proposition 3.3 (ii)]). Therefore we have a canonical isomorphism of coadmissible $D(G, K)$ -modules⁵

$$(\text{Ind}_P^G(W')^\mathfrak{d})' \simeq (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d} \quad (7)$$

In practice, we are interested in the following description of $(\text{Ind}_P^G(W')^\mathfrak{d})'$, cf. (8) below.

In what follows, we will denote by \mathfrak{o} the ring of integers of the finite extension $L|\mathbb{Q}_p$. We will also take smooth integral models $\mathbb{T}_0 \subseteq \mathbb{B}_0 \subseteq \mathbb{P}_0 \subseteq \mathbb{G}_0$, which are by definition groups \mathfrak{o} -schemes, of $\mathbb{T} \subseteq \mathbb{B} \subseteq \mathbb{P} \subseteq \mathbb{G}$, respectively, and we will denote by $G_0 := \mathbb{G}_0(\mathfrak{o})$, which is a maximal open compact subgroup G .

Let $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. By definition, M is a union of finite dimensional \mathfrak{p}_K -modules. Let us denote by M_0 one of these finite-dimensional submodules. As we have remarked, M_0 lifts to a locally analytic P -representation which extends to a unique $D(P, K)$ -module structure, in the sense that the Dirac distributions acts as group element on P . cf. [15, Proposition 3.2] and the paragraph before the lemma 3.1 in [15].

The algebra $D(\mathfrak{g}_K, P)$: In what follows, we will denote by $D(\mathfrak{g}_K, P)$ the subring⁶ of $D(G, K)$ generated by $U(\mathfrak{g}_K)$ and $D(P, K)$. Actually, this ring is equal to $U(\mathfrak{g}_K)D(P, K)$, which means that every element in $D(\mathfrak{g}_K, P)$ can be written as a finite sum $\sum_j \mathfrak{z}_j \delta_j$ with $\mathfrak{z}_j \in U(\mathfrak{g}_K)$ and $\delta_j \in D(P, K)$ ([9, Proposition 3.5]). From this description and the discussion in the previous paragraph, we can conclude that any object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ carries a unique $D(\mathfrak{g}_K, P)$ -structure, such that the $U(\mathfrak{p}_K)$ -action, as a subring of $U(\mathfrak{g}_K)$, coincides with the $U(\mathfrak{p}_K)$ -action as a subring of $D(P, K)$. Furthermore, the Dirac distributions $\delta_p \in D(P, K)$ act like the group

⁵The coadmissible structure on the right hand side becomes from the fact that $\mathfrak{d} \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Hence $D(G, K)\mathfrak{d}$ is finitely generated as $D(G, K)$ -module and therefore closed. The assertion now follows from [16, Lemma 3.6].

⁶We recall for the reader that we have an inclusion $U(\mathfrak{g}_K) \rightarrow D(G, K)$ via the exponential map, cf. discussion after the proposition 2.3 in [15]. The same reasoning applies for the compact subgroup $G_0 \subseteq G$.

elements $p \in P$ and, by uniqueness of the $D(\mathfrak{g}_K, P)$ -module structure, any morphism $M_1 \rightarrow M_2$ in $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ is in particular a homomorphism of $D(\mathfrak{g}_K, P)$ -modules.⁷

Remark 1.2.1. If $D(\mathfrak{g}_K, P_0)$ denotes the subring of the **Fréchet-Stein algebra** $D(G_0, K)$ generated by $U(\mathfrak{g}_K)$ and $D(P_0, K)$, then we have the same description $D(\mathfrak{g}_K, P_0) = U(\mathfrak{g}_K)D(P_0, K)$ than for $D(\mathfrak{g}_K, P)$.

Let us take, as before, $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Since, by definition, M is a finitely generated $U(\mathfrak{g}_K)$ -module, we see that M is a finitely generated $D(\mathfrak{g}_K, P)$ -module, and therefore $D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M$ is a finitely generated $D(G, K)$ -module. It is clear that the same reasoning applies for the algebra $D(\mathfrak{g}_K, P_0)$, i.e. $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$ is also a finitely generated $D(G_0, K)$ -module.

Remark 1.2.2. Using the Iwasawa decomposition⁸ $G = G_0P$, it is possible to prove that the canonical map

$$D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M \rightarrow D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M$$

is an isomorphism of $D(G_0, K)$ -modules ([18, Lemma 6.1 (i)]).

We have introduced the preceding information because we pretend to prove that we have a canonical isomorphism of $D(G_0, K)$ -modules

$$D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M \simeq (\text{Ind}_P^G(W')^{\mathfrak{d}})' , \quad (8)$$

which in particular implies that $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$ is a coadmissible $D(G_0, K)$ -module (**in fact strongly coadmissible**). By using the previous remark and the isomorphism (7), we only need to exhibit a canonical isomorphism of $D(G, K)$ -modules

$$D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M \simeq (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d},$$

which we construct as follows (this is exactly as in [9, Proposition 3.7]). We start by considering the canonical map

$$\iota : M = (U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W) / \mathfrak{d} \rightarrow (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d}$$

and we point out that if this map is in fact $D(\mathfrak{g}_K, P)$ -linear, then we can define a homomorphism of $D(G, K)$ -modules

$$\Phi : D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M \rightarrow (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d}$$

by extending the relation⁹ $\Phi(\delta \otimes w) := \delta \iota(w)$, for $w \in W$. It turns out that this map is in fact an isomorphism of $D(G, K)$ -modules whose inverse Ψ is given by

⁷This reasoning has been taken from the proof of the corollary 3.6 in [9].

⁸This has been explained by Stefano Morra in his talk. The interested reader can also find a discussion about this decomposition in [4, 3.5].

⁹ $\Phi(\delta \otimes m) = \Phi(\delta \otimes \sum \mathfrak{z}_i w_i) = \Phi(\sum \delta \mathfrak{z}_i \otimes w_i) = \sum \delta \mathfrak{z}_i \iota(w_i) = \sum \delta \iota(\mathfrak{z}_i w_i) = \delta \iota(m)$, the last equality by $D(\mathfrak{g}_K, P)$ -linearity.

$\Psi((\delta \otimes w) + D(G, K)\mathfrak{d}) := \delta \otimes (w + \mathfrak{d})$. The $D(\mathfrak{g}_K, P)$ -linearity of ι follows from the fact that the $U(\mathfrak{p}_K)$ -action on W is compatible as a subalgebra of both $U(\mathfrak{g}_K)$ and $D(P, K)$. This clearly implies that the natural map

$$U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow D(G, K) \otimes_{D(P, K)} W$$

is $D(\mathfrak{g}_K, P)$ -linear and so is ι .

All in all, we can pass to define the functor \mathcal{F}_P^G ([9, 4.1]). For $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$, we have constructed a coadmissible¹⁰ $D(G, K)$ -module in (8). This motivates the definition of the functor

$$\begin{aligned} \mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} &\rightarrow \text{Rep}_K^{\text{la}}(G) \\ M &\rightsquigarrow \mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)' \end{aligned} \quad (9)$$

Proposition 1.2.3. *The functor \mathcal{F}_P^G is exact.*

Remark 1.2.4. We will follow the arguments given by Matthias Strauch in his talk, which in turn are inspired by the appendix B in [1]. These arguments introduce interesting relations between the algebra of locally analytic distributions and Kohlhaase's ring of distributions supported in a closed subset. The interested reader can also find an alternative proof of the proposition 1.2.3 in [9, Proposition 4.2].

To explain the ideas given by Matthias Strauch, we will need the following notions introduced in [1, B.5, B.6, B.7 and B.8]. Let us denote by $D(G, K)_P$ the **Kohlhaase's ring of distributions supported in P** [7, 1.2.1 - 1.2.6]. This can be considered as the topological closure of $D(\mathfrak{g}_K, P)$ inside $D(G, K)$ [7, Lemma 1.2.10].¹¹ In particular, $D(G_0, K)_{P_0}$ is Fréchet. The same constructions apply for the maximal compact open subgroup $G_0 \subset G$, and we want to prove that, in this case, $D(G_0, K)_{P_0}$ is a Fréchet-Stein algebra, where $P_0 := P \cap G_0$. To do that, let us consider the Fréchet-Stein structure

$$D(G_0, K) = \varprojlim_{r \in (1/p, 1) \cap p^{\mathbb{Q}}} D_r(G_0, K)$$

of the distribution algebra $D(G_0, K)$ constructed in [16, Theorem 5.1], and let us denote by $D_r(G_0, K)_{P_0}$ the closure of $D(G_0, K)_{P_0}$ inside $D_r(G_0, K)$. If $\mathfrak{z}_1, \dots, \mathfrak{z}_d$ is an L -basis of \mathfrak{g} and

$$U_r(\mathfrak{g}_K) := \left\{ \sum_{\beta \in \mathbb{N}^d} s_{\beta} \mathfrak{z}^{\beta} \mid s_{\beta} \in K \text{ and } \|s_{\beta} \mathfrak{z}^{\beta}\|_r \xrightarrow{|\beta| \rightarrow \infty} 0 \right\}$$

¹⁰Let us recall that in the non compact case, admissibility is tested over a open compact subgroup. In this case $G_0 \subset G$.

¹¹We recall for the reader that the set of Dirac distributions is dense in $D(P, K)$ [15, Lemma 3.1].

where $\|\cdot\|_r$ is the so-called r -norm [16, Section 4], it is proved in [7, Theorem 1.4.2 and corollary 1.4.3] (see also [13, Theorem 2.3]) that there exists a cofinal system (which we fix from now on) defining the Fréchet-Stein structure of $D(G_0, K)$, such that $U_r(\mathfrak{g}_K)$ is noetherian and $D_r(G_0, K)_{P_0}$ is a finitely free $U_r(\mathfrak{g}_K)$ -module. In other words $D_r(G_0, K)_{P_0}$ is noetherian. Furthermore, it is not so hard to prove, by using the results cited before ([7, 1.4.2 and 1.4.3]), that $D_r(G_0, K)$ is finite free as a right module over $D_r(G_0, K)_{P_0}$ -module, cf. [1, Lemma B.5.2]. From these facts Agrawal-Strauch conclude that $D(G_0, K)_{P_0}$ is a **Fréchet-Stein algebra** by comparing the transition morphisms via the commutative diagram

$$\begin{array}{ccc} D_r(G_0, K)_{P_0} & \longrightarrow & D_{r'}(G_0, K)_{P_0} \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \longrightarrow & D_{r'}(G_0, K). \end{array} \quad (10)$$

Indeed, the horizontal map on the bottom being flat and both vertical maps faithfully flat (by the arguments in the previous paragraph), imply that the morphism on the top is also flat. To see [1, Proposition B.5.3] for more details.

Remark 1.2.5. As we have remarked before, in the noncompact case coadmissibility over the distribution algebra can be tested over any open compact subgroup $H \subset G$. By [7, (1.7)] and [1, Corollary B.7.2] the same holds for the algebra $D(G, K)_P$. The following definition is given in [1, Definition B.7.3].

Definition 1.2.6. A $D(G, K)_P$ -module M is *coadmissible* if M is coadmissible as a module over the Fréchet-Stein algebra $D(G_0, K)_{P_0}$.

Remark 1.2.7. (i) Let $M \in \mathcal{O}_{\text{alg}}^{\text{pK}}$. In particular M is a finitely generated $U(\mathfrak{g}_K)$ -module. We already know that M can be endowed with a structure of $D(\mathfrak{g}_K, P)$ -module, and with this action it is possible to prove that

$$D(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M$$

is a coadmissible $D(G, K)_P$ -module, cf. [14, Lemma 4.3].

(ii) ([14, Lemma 4.6]) For a $D(\mathfrak{g}_K, P)$ -module M , the natural map

$$U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} M \rightarrow D_r(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M$$

is an isomorphism of left $U_r(\mathfrak{g}_K)$ -modules.

(iii) ([1, Corollary B.8.2]) If M is a $D(\mathfrak{g}_K, P)$ -module, then we have a canonical isomorphism

$$D(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M = D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$$

of $D(G_0, K)_{P_0}$ -modules.

¹² Now, let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad (11)$$

be an exact sequence in $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. By (i) and (iii) in the previous remark and [1, Proposition 4.1.5], we know that

$$D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (11) \quad (12)$$

is a sequence of coadmissible modules over the Fréchet-Stein algebra $D(G_0, K)_{P_0}$, so it is exact if and only if

$$\begin{aligned} D_r(G_0, K)_{P_0} \otimes_{D(G_0, K)_{P_0}} \left(D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (11) \right) \\ = U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} (11) \end{aligned}$$

is exact, where we have used (ii) in the previous remark, and r in the cofinal set that we have fixed. But $U_r(\mathfrak{g}_K)$ is flat over $U(\mathfrak{g}_K)$, cf. [14, Theorem 3.13] and [16, Remark 3.2], thus $U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} (11)$ is exact for all r . This clearly says that the functor

$$D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet) : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \rightarrow \mathcal{C}_{D(G, K)_P}$$

is exact. Here $\mathcal{C}_{D(G, K)_P}$ denotes the category of coadmissible $D(G, K)_P$ -modules.

Remark 1.2.8. This fact illustrates one of the motivations of the authors in [1] to introduce Kohlhaase' ring of distributions $D(G, K)_P$. With this ring we somehow have a *good algebraic control*¹³ over the modules in the category $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$.

Of course, if we now want to deal with the proof of the proposition 1.2.3, the naive reasoning that we can carry out is to consider a short exact sequence like in (11) and then tensoring $D(G_0, K) \otimes_{D(G_0, K)_{P_0}} (\bullet)$ the (short exact) sequence (12). Unfortunately, we don not currently know if $D(G_0, K)$ is right flat over $D(G_0, K)_{P_0}$, cf. [1, Remark B.6.3]. To deal with this gap, Agrawal-Strauch have attacked the problem from a *topological point of view* by introducing Ardakov-Wadsley notion of **c-fatness**, [3]. It turns out that this is the right version of flatness for Fréchet-Stein algebras. Let us recall the definition.

Definition 1.2.9. Let $\varphi : A \rightarrow B$ be a continuous morphism of Fréchet-Stein algebras. There exists a right-exact functor $B \otimes_A^c (\bullet)$ from coadmissible left A -modules to coadmissible left B -modules [3, Section 7]. We say that B is **right c-flat** if this functor is exact, and **right faithfully c-flat** if this functor is faithfully exact.

¹²This reasoning is an extract of the proof of the proposition B.8.5 in [1].

¹³Shishir Agrawal has indicated to me that we don't have a *good algebraic control* over the ring $D(\mathfrak{g}_K, P)$, so they required a good algebraic property from the modules over this ring, namely, finitely presented $D(\mathfrak{g}_K, P)$ -modules. A good source of examples of such a modules are those $D(\mathfrak{g}_K, P)$ -modules arising from the category $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$, cf. [1, Proposition 4.1.5].

Remark 1.2.10. Under the notation of the previous definition, if M is a finitely presented A -module, then $B \otimes_A^c M = B \otimes_A M$.

Now, by construction (see for example the diagram (10)) we know that the inclusion $D(G_0, K)_{P_0} \rightarrow D(G_0, K)$ is a continuous morphism of Fréchet-Stein algebras whose transition maps $D_r(G_0, K)_{P_0} \rightarrow D_r(G_0, K)$ are right faithfully flat (paragraph right before the diagram (10)). From this fact, it is not hard to see that $D(G_0, K)_{P_0} \rightarrow D(G_0, K)$ is right faithfully c-flat, cf. [1, Lemma B.6.1] for more details.

All in all, we can finally give the proof of the proposition 1.2.3.

Proof of the proposition 1.2.3. Let us recall that for $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ we have

$$\mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)'$$

Furthermore, given that $(\bullet)'$ induces an equivalence of categories

$$\mathcal{C}_G \xrightarrow{(\bullet)'} \text{Rep}_K^{\text{adm}}(G)$$

between the category of admissible locally analytic G -representations and the category \mathcal{C}_G of coadmissible $D(G, K)$ -modules, cf. [16, Theorem 6.3], we only need to prove, using (iii) in the remark 1.2.7, that $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet)$ defines an exact functor between $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ and \mathcal{C}_G . But this functor equals the composition of the functors

$$\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \xrightarrow{D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet)} \mathcal{C}_{D(G, K)_P} \xrightarrow{D(G_0, K) \otimes_{D(G_0, K)_{P_0}}^c (\bullet)} \mathcal{C}_G.$$

The exactness now follows from c-flatness of $D(G_0, K)$ over $D(G_0, K)_{P_0}$, the reasoning given after the remark 1.2.7 and the remark 1.2.10. \square

Remark 1.2.11. The reasoning given in the proof of the proposition 1.2.3 gives a second motivation to introduce the ring $D(G, K)_P$. From a topological point of view, we have a good knowledge of the morphism $D(G, K)_P \rightarrow D(G, K)$, in the sense it is faithfully c-flat.

1.3 Extending the functor \mathcal{F}_P^G

The goal of this subsection will be to extend the functor defined in (9) in order to obtain the announced bi-functor in (1). We will follow word by word the arguments and definitions given in [9, 4.4]. To start with, let us first recall the definition of a **smooth representation** and an **admissible smooth representation**, cf. [18, Section 2]. We will give the definition for the group G , but we warn the reader that in this subsection we will be interested in L_P -representations.

First of all, we say that a K -vector space V is a **smooth representation** of G , if V is endowed with a K -linear G -action such that the stabilizer of each vector in V is open in G . Furthermore, we say that V is an **admissible smooth representation** if, for any compact open subgroup $H \subset G$, the vector space V^H of H -invariants vectors in V is finite dimensional.

Remark 1.3.1. Let V be an admissible smooth G -representations. Given that the unit element in G has a countable fundamental system of open compact neighbourhoods $\{H_n\}_{n \in \mathbb{N}}$, we have

$$V = \varinjlim_{n \in \mathbb{N}} V^{H_n}.$$

This implies that V is of compact type, being the countable locally convex inductive limit of the finite dimensional spaces V^{H_n} .

From now on, we will assume that V is a smooth admissible representation of the Levi subgroup $L_P \subseteq P$, and we regard it as a representation of P . We will always consider on V the finest locally convex topology exhibited in the previous remark.

Let $M \in \mathcal{O}_{\text{alg}}^{\text{pk}}$ and write it as a quotient of a generalized Verma module

$$0 \rightarrow U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M \rightarrow 0.$$

We have¹⁴

$$W' \otimes_K V = \varinjlim_{n \in \mathbb{N}} W' \otimes_K V^{N_n},$$

where $\{N_n\}_{n \in \mathbb{N}}$ runs through a countable fundamental system of open compact subgroups of P . Equipped with the diagonal action $W' \otimes_K V$ is a locally analytic representation. By using the pairing $\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, V)}$ defined in the subsection 1.2, we can consider

$$\begin{aligned} \mathcal{F}_P^G(M, V) &:= \text{Ind}_P^G(W \otimes V')^{\mathfrak{d}} \\ &:= \{f \in \text{Ind}_P^G(W' \otimes_K V) \mid \forall \mathfrak{z} \in \mathfrak{d}, \langle \mathfrak{z}, f \rangle_{C^{\text{an}}(G, V)} = 0\}. \end{aligned}$$

Remark 1.3.2. Strictly speaking we should have written $\mathcal{F}_P^G(M, W, V)$ instead of $\mathcal{F}_P^G(M, V)$, because it could depend of the chosen P -representation W . In [9, Proposition 4.5] the authors prove that if $W_1 \subseteq W_2$ are two finite-dimensional $U(\mathfrak{g}_K)$ -representations which generates M as a $U(\mathfrak{g}_K)$ -module, then there exists a canonical isomorphism

$$\mathcal{F}_P^G(M, W_2, V) \simeq \mathcal{F}_P^G(M, W_1, V).$$

We can henceforth identify all representations $\mathcal{F}_P^G(M, W, V)$ and to write just $\mathcal{F}_P^G(M, V)$.

¹⁴Given that W is finite-dimensional, the injective tensor products $W' \otimes_{K, \pi} V = W' \otimes_{K, \iota} V$ coincide, and we can just write $W' \otimes_K V$.

Proposition 1.3.3. ([9, Proposition 4.7]) $\mathcal{F}_P^G(\cdot, \cdot)$ is a bi-functor

$$\begin{array}{ccc} \mathcal{O}_{alg}^{\mathfrak{p}_K} \times \text{Rep}_K^{\infty, adm}(L_P) & \rightarrow & \text{Rep}_K^{la}(G) \\ (M, V) & \mapsto & \mathcal{F}_P^G(M, V), \end{array}$$

which is contravariant in M and covariant in V .

Proof. Let $\alpha : M_1 \rightarrow M_2$ be a morphism in $\mathcal{O}_{alg}^{\mathfrak{p}_K}$ and $\beta : V_1 \rightarrow V_2$ be a morphism in $\text{Rep}_K^{\infty, adm}(L_P)$. Let us choose $W_1 \subseteq M_1$ a finite-dimensional $U(\mathfrak{p}_K)$ -submodule which generates M_1 as a $U(\mathfrak{g}_K)$ -module. Then let us choose $W_2 \subseteq M_2$ a finite-dimensional $U(\mathfrak{p}_K)$ -submodule which generates M_2 as a $U(\mathfrak{g}_K)$ -module and which contains $\alpha(W_1)$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{d}_1 & \longrightarrow & U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W_1 & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{d}_2 & \longrightarrow & U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W_1 & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

This shows that the map

$$\text{Ind}_P^G(W_2' \otimes_K V_1) \rightarrow \text{Ind}_P^G(W_1' \otimes_K V_2)$$

induced by $\alpha' \otimes \beta$ maps $\mathcal{F}_P^G(M_2, V_1)$ into $\mathcal{F}_P^G(M_1, V_2)$. \square

Proposition 1.3.4. (i) For all $M \in \mathcal{O}_{alg}^{\mathfrak{p}_K}$, and for all smooth admissible L_P -representation V , the G -representation $\mathcal{F}_P^G(M, V)$ is admissible.

(ii) If V is of finite length, then $\mathcal{F}_P^G(M, V)$ is strongly admissible.

Proof. The first item follows from [9, Lemma 2.4 (i)] and [16, Proposition 6.4 (iii)]. The second item follows from [9, Lemma 2.4 (ii)] and [15, Lemma 3.5]. \square

The reader can find the proof of the following proposition in [9, Proposition 4.9].

Proposition 1.3.5. (i) The bi-functor $\mathcal{F}_P^G(\cdot, \cdot)$ is exact in both arguments.

(ii) If $P \subseteq Q$ are parabolic subgroups, $\mathfrak{q}_K := \text{Lie}(Q)$, and $M \in \mathcal{O}_{alg}^{\mathfrak{q}_K}$, then

$$\mathcal{F}_P^G(M, V) = \mathcal{F}_Q^G(M, \text{Ind}_P^Q(V)).$$

Remark 1.3.6. (i) It is clear that if $\mathbb{1}$ denotes the trivial L_P -representation, then

$$\mathcal{F}_P^G(M, \mathbb{1}) = \mathcal{F}_P^G(M)$$

for all $M \in \mathcal{O}_{alg}^{\mathfrak{p}_K}$.

- (ii) Let us recall the category \mathcal{O}^P , [10], whose objects consist of those pairs $\underline{M} := (M, \tau)$, where M is an object of¹⁵ $\mathcal{O}^{\mathfrak{p}_K}$ and $\tau : P \rightarrow \text{End}_K(M)^*$ is a locally finite-dimensional locally analytic representation on M which lifts the Lie algebra representation of \mathfrak{p}_K on M . As we have remarked, any object $M \in \mathcal{O}^P$ can be endowed with a structure of $D(\mathfrak{g}_K, P)$ -module which allows us to consider

$$\mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)'$$

This is an extension of the functor (9) to the category \mathcal{O}^P , in the sense of the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} & \xrightarrow{\quad} & \mathcal{O}^P \\ & \searrow \mathcal{F}_P^G & \swarrow \mathcal{F}_P^G \\ & \text{Rep}_K^{\text{la}}(G) & \end{array}$$

where we have used the fully faithful embedding (4).

2 Local properties of the functor \mathcal{F}_P^G

In this section we will review the main ideas introduced by S. Orlik in his talk. We will consider the aspects of faithfulness, projective and injective objects and we will compute some Ext-groups. The main reference for this section is [12].

2.1 Jacquet functors

Let us recall the Levi decomposition $P = L_P U_P$. For any locally analytic P -representation, we let $V(U_P)$ be the subspace generated by the expressions $uv - v$, with $u \in U_P$, $v \in V$ and let $\overline{V(U_P)}$ be its topological closure which is a P -stable subspace of V . We will denote by

$$\overline{H}_0(U_P, V) := V_{U_P} := V / \overline{V(U_P)}$$

the corresponding naive Jacquet module,¹⁶ which is the largest Hausdorff quotient of V on which U_P acts trivially. Furthermore, since $\overline{V(U_P)}$ is a closed subspace of V , the quotient $\overline{H}_0(U_P, V)$ is of compact type. Moreover, the orbit maps $P \rightarrow \overline{H}_0(U_P, V)$ are clearly locally analytic since they are induced via the locally analytic orbit maps $P \rightarrow V$, cf. [12, Lemma 4.1]. In other words, $\overline{H}_0(U_P, V)$ has a canonical structure of a locally analytic P -representation.

¹⁵We remark for the reader that the motivation in considering the full subcategory $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ was to have a control over the root decomposition of certain objects in $\mathcal{O}^{\mathfrak{p}_K}$, that allowed us to lift the algebraic \mathbb{T}_K -action to a P -action.

¹⁶The interested reader can take a look to the notes of Gabriel Dospinescu "A review of Emerton's Jacquet functors".

Now, by the very definition, we know that V' is a K -Fréchet space endowed with a continuous P -action. We let

$$H_0(U_P, V) := \{v' \in V' \mid U_P \cdot v' = v'\} \subseteq V'.$$

Being a closed subspace $H_0(U_P, V)$ inherits the structure of a K -Fréchet space equipped with a P -action, as well. More exactly, we have the following topological isomorphism ([12, Lemma 4.2])

$$H_0(U_P, V) \simeq \left(\overline{H_0(U_P, V)}\right)'$$

of P -representations.

Finally, if M is a \mathfrak{g}_K -representation, then we can consider the subspace $H^0(\mathfrak{u}_P, M)$ of vectors killed by \mathfrak{u}_P , and the quotient $H_0(\mathfrak{u}_P, M) = M/\mathfrak{u}_P M$. Both are $U(\mathfrak{p}_K)$ -modules.

We will need the following technical result whose proof can be founded in [12, Proposition 4.20].

Proposition 2.1.1. *Let $\underline{M} = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \in \mathcal{O}^P$ be a generalized Verma module. Then*

$$\overline{H_0(U_P, \mathcal{F}_P^G(\underline{M}))} = H^0(\mathfrak{u}_P, \omega(\underline{M}))',$$

where $\omega(\cdot)$ denotes the forgetful functor (3).

Remark 2.1.2. ([12, Remark 4.22]) The same statement holds true for objects $\underline{M} \in \mathcal{O}^P$ of the shape $\underline{M} = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W$ where W is a finite-dimensional locally analytic P -representation. In particular, it holds if W is a finite dimensional algebraic representation of the levi factor $\mathbb{L}_{\mathfrak{p}}$.

2.2 Functorial properties

In the first part of this subsection, we want to discuss whether the functors \mathcal{F}_P^G are faithful resp. fully faithful. More exactly, we want to prove that the functor

$$\begin{aligned} \mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} &\rightarrow \text{Rep}_K^{\text{la}}(G) \\ M &\mapsto \mathcal{F}_P^G(M, \mathbb{1}) \end{aligned}$$

is fully faithful.

Theorem 2.2.1. ([12, Theorem 5.1]) *Let $\underline{M}_1, \underline{M}_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Then the map*

$$\begin{aligned} \text{Hom}_{\mathcal{O}^P}(\underline{M}_1, \underline{M}_2) &\rightarrow \text{Hom}_G(\mathcal{F}_P^G(\underline{M}_2), \mathcal{F}_P^G(\underline{M}_1)) \\ f &\mapsto \mathcal{F}_P^G(f) \end{aligned}$$

is bijective.

As S. Orlik did in his presentation, we will prove the preceding theorem in the special case when $M_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ and M_1 is the generalized Verma module $M_1 = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} Z$, where Z is a finite-dimensional locally analytic L_P -representation. We remark for the reader that in this particular case we do not have a set of *differential equations* in the sense that $\mathfrak{d} = 0$ (in (5)) and therefore

$$\mathcal{F}_P^G(M_1) = \text{Ind}_P^G(Z'), \quad (13)$$

and U_P acts trivially on Z .

Proof of the theorem 2.2.1. The authors have divided this proof into several steps. We will give the proof in the particular case when M_1 is the generalized Verma module of the preceding paragraph and $M_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. The interested reader can find a complete proof in [12, Theorem 5.1].

By proposition 2.1.1 we have $\overline{H}_0(U_P, \mathcal{F}_P^G(M_2))' = H^0(\mathfrak{u}_{\mathfrak{p}}, M_2)$. From this fact we have the following identities

$$\begin{aligned} \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) &= \text{Hom}_P(\mathcal{F}_P^G(M_2), Z') \\ &= \text{Hom}_{L_P}(\overline{H}_0(U_P, \mathcal{F}_P^G(M_2)), Z') \\ &= \text{Hom}_{D(L_P, K)}(Z, H^0(\mathfrak{u}_{\mathfrak{p}}, M_2)) \\ &= \text{Hom}_{D(P, K)}(Z, M_2) \\ &= \text{Hom}_{D(\mathfrak{g}_K, P)}(M_1, M_2). \end{aligned}$$

The first relation follows from (13) and Frobenius reciprocity, the second one uses the fact that Z is a representation of the Levi factor L_P and by definition U_P acts trivially on $\overline{H}_0(U_P, \mathcal{F}_P^G(M_2))$. The third relation is the well-known Schneider-Teitelbaum equivalence of categories [16, Theorem 6.3]. The fourth one has already explained and the last one is just by definition of M_1 . \square

Remark 2.2.2. The following example given by S. orlik in his talk shows that the functor obtained by fixing M and letting vary V is not fully faithful.

Example 2.2.3. Let us take $G = \text{GL}_2(\mathbb{Q}_p)$, $B = P$ the Borel subgroup of upper triangular matrices, $M = \mathbb{1}$ and $V = \chi$ a character of the diagonal torus, such that the induced representation splits

$$V := \text{ind}_P^G(\chi) = V_1 \oplus V_2.$$

By the Schur's lemma is known that

$$\dim(\text{Hom}_B(V, V)) = 1,$$

but

$$\dim(\text{Hom}_G(\mathcal{F}_B^G(M, V), \mathcal{F}_B^G(M, V))) = 2.$$

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