Lie algebras: basics, nilpotency and solvability

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November 5, 2020

Lie algebras

Definition

A *Lie algebra* is a vector space L over $K = \bar{K}$ ($\mathrm{char}(K) = 0$) endowed with a bilinear operation

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

that satisfies the following properties:

1. $[x, x] = 0$ for every $x \in L$.
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for every $x, y, z \in L$.

Example

Any associative algebra $(A, +, \cdot)$ is a Lie algebra when defining $[x, y] := xy - yx$ for every $x, y \in A$.

The tangent space at the identity

Definition

Let $\mathbb{G}$ be a linear algebraic group. The set of *derivations* supported on the K-algebra $K[\mathbb{G}]$ is

$$\mathrm{Der}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{End}_K(K[\mathbb{G}]) : \tilde{\delta}(fg) = \tilde{\delta}(f)g + f\tilde{\delta}(g) \quad \forall f, g \in K[\mathbb{G}]\}.$$

Lemma

If $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$, *then* $[\tilde{\delta}, \tilde{\eta}] := \tilde{\delta}\tilde{\eta} - \tilde{\eta}\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}])$. *Moreover, we have that* $[\cdot, \cdot]$ *is bilinear and*

(a) $[\tilde{\delta}, \tilde{\delta}] = 0$ *for every* $\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}])$.

(b) $[\tilde{\delta}, [\tilde{\eta}, \tilde{\theta}]] + [\tilde{\eta}, [\tilde{\theta}, \tilde{\delta}]] + [\tilde{\theta}, [\tilde{\delta}, \tilde{\eta}]] = 0$ *for every* $\tilde{\delta}, \tilde{\eta}, \tilde{\theta} \in \mathrm{Der}(K[\mathbb{G}])$.

In other words, $(\mathrm{Der}(K[\mathbb{G}]), [\cdot, \cdot])$ *is a Lie algebra.*

The tangent space at the identity

Definition

Let $x \in \mathbb{G}$. Consider $\lambda_x : K[\mathbb{G}] \to K[\mathbb{G}]$ the algebra map defined as $(\lambda_x(f))(h) := f(x^{-1}h)$ for every $f \in K[\mathbb{G}]$ and $h \in \mathbb{G}$. Then the set of *left invariant* $\mathbb{G}$*-derivations* is

$$\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}]) : \tilde{\delta}\lambda_x = \lambda_x\tilde{\delta} \quad \text{for every } x \in \mathbb{G}\}.$$

$\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}])$ is closed under $[\cdot,\cdot]$, and so it is a Lie algebra, often called $\mathrm{Lie}(\mathbb{G})$.

Theorem

We have a 1:1 correspondence between the elements of $\mathrm{Lie}(\mathbb{G})$ *and the elements of the tangent space to* $\mathbb{G}$ *at the identity, namely*

$$T_1(\mathbb{G}) := \{\delta \in \mathrm{Hom}(K[\mathbb{G}], K) : \delta(fg) = (\delta f)g(1) + f(1)(\delta g) \ \ \forall f, g \in K[\mathbb{G}]\}.$$

The tangent space at the identity

- The tangent space to $\mathbb{G}$ at the identity is naturally a Lie algebra.
- $\mathrm{Lie}(\mathrm{GL}_n(K)) \cong \mathfrak{gl}_n(K)$ as Lie algebras, where $\mathfrak{gl}_n(K)$ is the vector space of the $n \times n$ K-matrices endowed with the structure of a Lie algebra using the commutator operation.
- $\mathrm{Lie}(\mathrm{SL}_n(K)) \cong \mathfrak{sl}_n(K)$ as Lie algebras, where $\mathfrak{sl}_n(K)$ is the vector space of the $n \times n$ K-matrices with zero trace, endowed with the structure of a Lie algebra using the commutator operation.

Lie algebras

From now on, L will denote a finitely dimensional Lie algebra over K.

Definition

If E a subspace of L such that $[E,E] \subseteq E$. Then E is called a *subalgebra* of L. If, moreover, $[E,L] \subseteq E$, then E is called an *ideal* of L.

Definition

Let L and M be two Lie algebras. A vector spaces morphism $\phi : L \to M$ is called a *Lie algebra map* if $[\phi(x), \phi(y)] = \phi([x,y])$ for every $x, y \in L$.

- One can define quotients and prove the usual isomorphism theorems.

Definition

A *representation* of L is a Lie algebra map $L \to \mathfrak{gl}_n(K)$ for some $n \in \mathbb{N}$.

The adjoint representation

Definition

The *adjoint representation of L* is the representation

$$\begin{aligned} \mathrm{ad}_L : L &\longrightarrow \mathfrak{gl}(L) \\ x &\longmapsto [x, \cdot], \end{aligned}$$

where $\mathfrak{gl}(L)$ is the group of K-endomorphisms of L as a vector space, endowed with the structure of a Lie algebra by using the commutator.

Definition

The *center of L* is $\zeta(L) := \ker(\mathrm{ad}_L)$. If $\zeta(L) = L$, we say that L is *abelian*.

Lemma

$\mathrm{Im}(\mathrm{ad}_L) \subseteq \mathrm{Der}(L)$ *and we have an injection*

$$L/\zeta(L) \longrightarrow \mathrm{Der}(L) \subseteq \mathfrak{gl}(L).$$

Examples

- We call $\mathfrak{n}_n(K)$ the set of strictly upper triangular $n \times n$ matrices. It is a subalgebra of $\mathfrak{gl}_n(K)$.
- We call $\mathfrak{b}_n(K)$ the set of upper triangular $n \times n$ matrices. It is a subalgebra of $\mathfrak{gl}_n(K)$. Also, $\mathfrak{n}_n(K)$ is an ideal inside $\mathfrak{b}_n(K)$ and we have $[\mathfrak{b}_n(K), \mathfrak{b}_n(K)] = \mathfrak{n}_n(K)$.
- The set of all diagonal matrices is an abelian subalgebra of $\mathfrak{gl}_n(K)$.
- The center of $\mathfrak{gl}_n(K)$ is the Lie algebra of scalar matrices.

Nilpotent Lie algebras

Definition

Let L be a Lie algebra and consider the sequence

$$L^1 := [L, L], \quad L^j := [L, L^{j-1}] \quad \text{for } j > 1.$$

L is called *nilpotent* if $L^j = 0$ for some $j > 0$.

1. Abelian Lie algebras are nilpotent.
2. The subalgebra $\mathfrak{n}_n$ of $\mathfrak{sl}_n(K)$ consisting of strictly upper triangular matrices is nilpotent.

$$\begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix} \begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}$$

Nilpotent Lie algebras

Proposition

Let L be a Lie algebra.

(a) *If L is nilpotent, then its subalgebras and homomorphic images are so.*

(b) *If $L/\zeta(L)$ is nilpotent, then L is nilpotent.*

(c) *If L is nilpotent, then $\zeta(L) \neq 0$.*

Proof

(a) If E is a subalgebra of L, then $E^j \subseteq L^j$. If $f : L \to M$ is an algebra map, then $f(L^j) = f(L)^j$.

(b) If $L/\zeta(L)$ is nilpotent, then $(L/\zeta(L))^j = L^j/(\zeta(L) \cap L^j) = 0$ for some j. This means that $L^j \subseteq \zeta(L)$, therefore $L^{j+1} \subseteq [L, \zeta(L)] = 0$.

(c) Consider j minimal such that $L^j = 0$. Then $L^{j-1} \neq 0$ and the condition $0 = L^j = [L, L^{j-1}]$ implies that $L^{j-1} \subseteq \zeta(L)$.

Nilpotent Lie algebras

Definition

Let $x \in L$. Then x is ad-*nilpotent in* L if $\mathrm{ad}_L(x)$ is a nilpotent element of $\mathfrak{gl}(L)$, i.e. if there exists a $m \in \mathbb{N}$ such that $(\mathrm{ad}_L(x))^m \equiv 0$.

Lemma

If L is nilpotent, then any element of L is ad_L*-nilpotent.*

Proof

$L^j = 0$ implies that $(\mathrm{ad}_L(x))^j(y) = [x, [x, \dots, [x, y]]] = 0$ for every $x, y \in L$.

Nilpotent Lie algebras

Lemma

Let $x \in \mathfrak{gl}(V)$ for some vector space V. If x is a nilpotent endomorphism, then x is ad*-nilpotent in $\mathfrak{gl}(V)$.*

Proof

Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism. Define

$$\begin{aligned} \lambda_x : \mathrm{End}_K(V) &\longrightarrow \mathrm{End}_K(V) \\ y &\longmapsto x \circ y \end{aligned} \qquad \begin{aligned} \rho_x : \mathrm{End}_K(V) &\longrightarrow \mathrm{End}_K(V) \\ y &\longmapsto y \circ x \end{aligned}$$

$\lambda_x, \rho_x \in \mathrm{End}_K(\mathrm{End}_K(V))$, they commute and they are nilpotent. Hence we have that $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \lambda_x - \rho_x$ is nilpotent.

Nilpotent Lie algebras

Theorem

Let $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra. If $V \neq 0$ and L consists of nilpotent endomorphisms, then there exists $v \in V$, $v \neq 0$, such that $Lv = 0$.

Proof Use induction on $\dim L$.

- If $\dim L = 1$, then $L = Kx$ for some $x \in L$. Take $v \in \ker(x)$.
- Assume now that the statement holds for every L' subalgebra of $\mathfrak{gl}(V)$ with $\dim L' < \dim L$.
- A subalgebra $M \subset L$ acts on L/M via the adjoint representation.
- By inductive hypothesis, there exists a vector $x + M \in L/M$, $x \notin M$, killed by the image of M in $\mathfrak{gl}(L/M)$.
- The normalizer $N_L(M) := \{y \in L : [y, z] \in M \text{ for every } z \in M\}$ properly contains M.

Proof

- Let M be a maximal proper subalgebra of L. Then $N_L(M) = L$, i.e. M is an ideal.
- M has codimension one in L, so $L = M + Kz$ for any $z \in L \setminus M$.
- By induction, $W := \{v \in V : Mv = 0\}$ is nonzero.
- For every $x \in L$, $y \in M$, $w \in W$ we have $yx(w) = xy(w) - [x, y]w = 0$, i.e. W is stable under the action of L.
- Let $z \in L \setminus M$ act on W and choose $0 \neq v \in W$ such that $zv = 0$. Then $Lv = 0$.

Theorem (Engel)

The Lie algebra L is nilpotent if and only if every element of L is ad_L-nilpotent.

Proof

- Let L be composed of ad-nilpotent elements, and use induction on $\dim L$.
- If $\dim L = 1$, then L is abelian.
- Work with $\mathrm{ad}_L(L) \subseteq \mathfrak{gl}(L)$. The previous theorem allows us to find a nonzero $x \in L$ such that $[x, L] = 0$.
- $L/\zeta(L)$ has smaller dimension than L. By inductive hypothesis, $L/\zeta(L)$ is nilpotent.
- L is nilpotent.

Nilpotent Lie algebras

Definition

Let V be a vector space of dimension $n \in \mathbb{N}$. A *flag* on V is a collection of subspaces $0 \subset V_1 \subset \cdots \subset V_n = V$ such that $\dim V_i = i$.

Corollary

If $L \subseteq \mathfrak{gl}(V)$ is a Lie algebra consisting of nilpotent elements, then there exists a flag $0 \subset V_1 \subset \cdots \subset V_n = V$ such that $xV_i \subseteq V_{i-1}$ for every $x \in L$, $i = 1, \ldots, n$.

Proof

- Take a vector $v_1 \in V$ such that $Lv_1 = 0$, and define $V_1 := Kv_1$.
- Work on V/V_1: there exists a vector $v_2 \in V \setminus V_1$ such that $Lv_2 \subseteq V_1$.
- Define $V_2 := Kv_1 + Kv_2$ and proceed inductively.

CONCLUSIONS

- Any Lie algebra $L \subseteq \mathfrak{gl}(V)$ consisting of nilpotent elements can be seen as a subalgebra of the algebra $\mathfrak{n}$ of strictly upper triangular matrices.
- A Lie algebra L is nilpotent if and only if $\mathrm{ad}_L(L) \subseteq \mathfrak{gl}(L)$ consists of nilpotent elements, and so it can be seen as a subalgebra of the algebra $\mathfrak{n}$.

Solvable Lie algebras

Definition

Let L be a Lie algebra and consider the sequence

$$L^{(1)} := [L, L], \quad L^{(j)} := [L^{(j-1)}, L^{(j-1)}] \quad \text{for } j > 1.$$

L is called *solvable* if $L^{(j)} = 0$ for some $j > 0$.

1. Abelian Lie algebras are solvable.
2. Nilpotent Lie algebras are solvable, since $L^j \subseteq L^{(j)}$.
3. The subalgebra $\mathfrak{b}_n$ of $\mathfrak{sl}_n(K)$ consisting of all upper triangular matrices is solvable, since $[\mathfrak{b}_n, \mathfrak{b}_n] = \mathfrak{n}_n$.

Solvable Lie algebras

Proposition

Let L be a Lie algebra.

(a) *If L is solvable, then its subalgebras and homomorphic images are so.*

(b) *If an ideal I of L is solvable and L/I is solvable, then L is solvable.*

(c) *If I, J are solvable ideals of L, then $I + J$ is solvable.*

Definition

Let L be a Lie algebra. The maximal solvable ideal of L is called $\mathrm{Rad}(L)$, the radical ideal of L.

Solvable Lie algebras

Theorem

Let $L \subseteq \mathfrak{gl}(V)$ a solvable Lie algebra, with V a finite dimensional vector space. If $V \neq 0$ then there exists $v \in V$, $v \neq 0$ such that $Lv = Kv$.

Theorem (Lie)

Let V be a vector space of dimension n. If $L \subseteq \mathfrak{gl}(V)$ is a solvable Lie algebra, then there exists a flag $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that $xV_i \subseteq V_i$ for every $x \in L$, $i = 1, \ldots, n$.

Corollary

Let $L \subseteq \mathfrak{gl}(V)$ be a solvable Lie algebra, V a vector space of dimension n. Then there exists a basis of V such that, with respect to this basis, L is a subalgebra of the algebra $\mathfrak{b}$ of upper triangular matrices.

CONCLUSION

The following are equivalent for a Lie algebra L:

- L is solvable (nilpotent).
- $\mathrm{ad}_L(L)$ is solvable (consists of nilpotent elements).
- There exists a basis of L such that $\mathrm{ad}_L(L)$ is contained in $\mathfrak{b}$ (in $\mathfrak{n}$).