Lie algebras: basics, nilpotency and solvability

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October 29, 2020

The main objects of study in this lecture are Lie algebras. First, we are going to see that they arise naturally as tangent spaces to linear algebraic groups at 1, and this will be a sort of motivation for the study of the structure of Lie algebras. Then we are going to focus on the main properties of nilpotent and solvable Lie algebras.

Notation

K	an algebraically closed field of zero characteristic;
$\mathbb{G}$	a linear algebraic group over K;
$I(\mathbb{G})$	the ideal of definition for $\mathbb{G}$;
$K[\mathbb{G}]$	the ring of regular functions on $\mathbb{G}$.

1 Lie algebras

Definition. A *Lie algebra* is a vector space L over K endowed with a bilinear operation (called *bracket* or *commutator*)

$$[\cdot,\cdot] : L \times L \longrightarrow L$$

that satisfies the following properties:

1. $[x,x] = 0$ for every $x \in L$.
2. $[x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$ for every $x,y,z \in L$.

The second property of the above definition is called *Jacobi identity.* One important result is that, given any associative algebra $(A,+,\cdot)$, we can turn it into a Lie algebra $(A,+,[\cdot,\cdot])$ by defining the commutator $[x,y] := xy - yx$ for every $x,y \in A$. During this notes we always assume that L is a finitely generated vector space over K.

2 The tangent space to a linear algebraic group

In this section we define the tangent space to a linear algebraic group at 1 in three ways, and eventually we see that they are equivalent definitions. Tangent spaces to algebraic groups will be our main example of Lie algebras.

2.1 First definition

Definition. Let $\mathbb{G}$ be a linear algebraic group over K, $x \in \mathbb{G}$. Then we define the *tangent space to* $\mathbb{G}$ *at* x as

$$T_x(\mathbb{G}) := \{\delta \in \operatorname{Hom}(K[\mathbb{G}], K) : \delta(fg) = (\delta f)g(x) + f(x)(\delta g) \text{ for every } f, g \in K[\mathbb{G}]\}.$$

2.2 Second definition

Let $D := K[t]/(t^2)$ and $\varepsilon := t + (t^2) \in D$. We have that

$$D \cong K + K\varepsilon$$

as vector spaces. Let also $\mathbb{G} \subseteq \mathbb{A}^n(K)$ be a linear algebraic group.

Definition. Let $x \in \mathbb{G}$. A vector $v \in K^n$ is *tangent to* $\mathbb{G}$ *at* x if $f(x + \varepsilon v) = 0$ in D for every $f \in I(\mathbb{G})$. We call $\operatorname{Tan}_x(\mathbb{G})$ the set of tangent vectors to $\mathbb{G}$ at x.

Remark 2.1. A vector $v \in K^n$ is tangent to $\mathbb{G}$ at x if and only if the map

$$\begin{aligned} \phi_v : K[\mathbb{G}] &\longrightarrow D \\ f &\longmapsto f(x + \varepsilon v) \end{aligned}$$

is an algebra map.

We can use this definition in order to compute $\operatorname{Tan}_1(\mathbb{G})$ for some famous algebraic groups $\mathbb{G}$.

Example 2.2. $\mathbb{G} = \mathrm{GL}_n(K)$. Remember the description of $\mathrm{GL}_n(K)$ as a linear algebraic group:

$$\mathbb{A}^{n^2+1} \supseteq \mathrm{GL}_n(K) := \{(A, t) \in \mathbb{A}^{n^2+1} : A = (a_{ij})_{i,j=1,\dots,n}, \quad (\det(A))t - 1 = 0\}.$$

We want to compute $\operatorname{Tan}_{(\mathbb{I}_n,1)} \mathrm{GL}_n(K)$. Let $v := ((a_{ij}), s) \in \mathbb{A}^{n^2+1}$ and consider the coordinate functions

$$\begin{aligned} x_{ij} : \mathrm{GL}_n(K) &\longrightarrow K \\ ((a_{ij}), s) &\longmapsto a_{ij} \end{aligned} \qquad \begin{aligned} t : \mathrm{GL}_n(K) &\longrightarrow K \\ ((a_{ij}), s) &\longmapsto s. \end{aligned}$$

Then we have that

$$\begin{aligned} x_{ij}\big((\mathbb{I}, 1) + \varepsilon((a_{ij}), s)\big) &= \delta_{ij} + \varepsilon a_{ij} \\ t\big((\mathbb{I}, 1) + \varepsilon((a_{ij}), s)\big) &= 1 + \varepsilon s. \end{aligned} .$$

Remembering that $\varepsilon^2 = 0$ in D, we can compute

$$\begin{aligned}\phi_v(\det(A)t - 1) = \det(\delta_{ij} + \varepsilon a_{ij}) \cdot (1 + \varepsilon s) - 1 = (1 + \varepsilon s)\prod_{i=1}^{n}(1 + \varepsilon a_{ii}) - 1 = \\ = \varepsilon(\mathrm{Tr}(A) + s)\end{aligned}$$

Hence, by definition we have that (A, s) is tangent to $\mathrm{GL}_n(K)$ at 1 if and only if $s = -\mathrm{Tr}(A)$. Hence $T_1(\mathrm{GL}_n(K)) \cong M_n(K)$, the set of matrices of order n with entries in K.

Exercise 2.3. With similar computations, one can show that $T_1(\mathrm{SL}_n(K))$ is in one to one correspondence with the set of $n \times n$ traceless matrices (i.e. matrices with zero trace).

2.3 Third definition

Definition. Let $\mathbb{G}$ be a linear algebraic group. The set of *derivations* supported on the K-algebra $K[\mathbb{G}]$ is

$$\mathrm{Der}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{End}_K(K[\mathbb{G}]) : \tilde{\delta}(fg) = \tilde{\delta}(f)g + f\tilde{\delta}(g) \quad \text{for every } f, g \in K[\mathbb{G}]\}$$

where $\mathrm{End}_K(K[\mathbb{G}])$ is the set of K-linear maps from $K[\mathbb{G}]$ to itself.

Analizing the structure of $\mathrm{Der}(K[\mathbb{G}])$, one can easily see that whenever $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$ and $a, b \in K$ we have that $a\tilde{\delta} + b\tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$, but in general $\tilde{\delta} \cdot \tilde{\eta}$ does not lie in $\mathrm{Der}(K[\mathbb{G}])$.

Definition. Let $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$. We define the commutator of $\tilde{\delta}$ and $\tilde{\eta}$ as

$$[\tilde{\delta}, \tilde{\eta}] := \tilde{\delta}\tilde{\eta} - \tilde{\eta}\tilde{\delta}.$$

Lemma 2.4. *If $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$, then $[\tilde{\delta}, \tilde{\eta}] \in \mathrm{Der}(K[\mathbb{G}])$. Moreover, we have that $[\cdot, \cdot]$ is bilinear and*

(a) $[\tilde{\delta}, \tilde{\delta}] = 0$ for every $\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}])$.

(b) $[\tilde{\delta}, [\tilde{\eta}, \tilde{\theta}]] + [\tilde{\eta}, [\tilde{\theta}, \tilde{\delta}]] + [\tilde{\theta}, [\tilde{\delta}, \tilde{\eta}]] = 0$ for every $\tilde{\delta}, \tilde{\eta}, \tilde{\theta} \in \mathrm{Der}(K[\mathbb{G}])$.

In other words, $(\mathrm{Der}(K[\mathbb{G}]), [\cdot, \cdot])$ is a Lie algebra.

Proof. In order to prove that $[\tilde{\delta}, \tilde{\eta}] \in \mathrm{Der}(K[\mathbb{G}])$, let $f, g \in K[\mathbb{G}]$ and compute

$$\begin{aligned}[\tilde{\delta}, \tilde{\eta}](fg) &= \tilde{\delta}\tilde{\eta}(fg) - \tilde{\eta}\tilde{\delta}(fg) = \tilde{\delta}\big(\tilde{\eta}(f)g + f\tilde{\eta}(g)\big) - \tilde{\eta}\big(\tilde{\delta}(f)g + f\tilde{\delta}(g)\big) = \\ &= \tilde{\delta}\tilde{\eta}f \cdot g + \tilde{\eta}f \cdot \tilde{\delta}g + \tilde{\delta}f \cdot \tilde{\eta}g + f \cdot \tilde{\delta}\tilde{\eta}g - \tilde{\eta}\tilde{\delta}f \cdot g - \tilde{\delta}f \cdot \tilde{\eta}g - \tilde{\eta}f \cdot \tilde{\delta}g - f \cdot \tilde{\eta}\tilde{\delta}g = \\ &= [\tilde{\delta}, \tilde{\eta}](f) \cdot g + f \cdot [\tilde{\delta}, \tilde{\eta}](g).\end{aligned}$$

Point (a) is clear from the definition of $[\cdot, \cdot]$ and point (b) is just another matter of computations. ☐

Definition. Let $x \in \mathbb{G}$. Consider $\lambda_x : K[\mathbb{G}] \to K[\mathbb{G}]$ to be the algebra map defined as $(\lambda_x(f))(h) := f(x^{-1}h)$ for every $f \in K[\mathbb{G}]$ and $h \in \mathbb{G}$. Then the set of *left invariant* $\mathbb{G}$*-derivations* is

$$\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}]) : \tilde{\delta}\lambda_x = \lambda_x\tilde{\delta} \quad \text{for every } x \in \mathbb{G}\}.$$

It is easy to see that $\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}])$ is closed with respect to the commutator $[\cdot,\cdot]$, hence it is a Lie algebra. For this reason it is usually called the *Lie algebra of* $\mathbb{G}$ and we write $\mathrm{Lie}(\mathbb{G}) := \mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}])$.

2.4 Identification theorem

Theorem 2.5. *Let* $\mathbb{G}$ *be a linear algebraic group. We have a* $1:1$ *correspondence between the elements of* $T_1(\mathbb{G})$, $\mathrm{Tan}_1(\mathbb{G})$ *and* $\mathrm{Lie}(\mathbb{G})$. *In particular, the tangent space at the identity is naturally a Lie algebra.*

Example 2.6. We call $\mathfrak{gl}_n(K) := \mathrm{Lie}(\mathrm{GL}_n(K))$ and $\mathfrak{sl}_n(K) := \mathrm{Lie}(\mathrm{SL}_n(K))$. These are the tangent spaces we computed in example 2.2 and exercise 2.3, hence $\mathfrak{gl}_n(K)$ is the set of $n \times n$ matrices and $\mathfrak{sl}_n(K)$ the set of $n \times n$ traceless matrices. It is just a matter of computations to show that the Lie algebra structure on $\mathfrak{gl}_n(K)$ and $\mathfrak{sl}_n(K)$ that comes from the correspondence in theorem 2.5 is exactly the Lie algebra structure that comes from the usual commutator operator.

3 First properties of Lie algebras

Definition. Let L and M be two Lie algebras. A vector spaces morphism $\phi : L \to M$ is called a *Lie algebra map* if $[\phi(x), \phi(y)] = \phi([x,y])$ for every $x, y \in L$.

We have seen that to any linear algebraic group $\mathbb{G}$ we can associate the Lie algebra $\mathrm{Lie}(\mathbb{G})$, and this is the main motivation that will lead us to the systematic study of Lie algebras. Moreover, if we have $\phi : \mathbb{G} \longrightarrow \mathbb{H}$ a morphism of linear algebraic groups, we can always define a morphism of Lie algebras $d\phi : \mathrm{Lie}(\mathbb{G}) \longrightarrow \mathrm{Lie}(\mathbb{H})$ in the following way. Let $f \in \mathrm{Tan}_1(\mathbb{G}) \cong \mathrm{Lie}(\mathbb{G})$ and consider the pull-back $\phi^* : K[\mathbb{H}] \to K[\mathbb{G}]$. The we define $d\phi(f) := f \circ \phi^*$. It can be proved that $d\phi$ is well defined and that it is a Lie algebra map. This means also that if we have a linear representation

$$\rho : \mathbb{G} \longrightarrow \mathrm{GL}_n(K)$$

than we can always canonically construct the Lie algebra map

$$d\rho : \mathrm{Lie}(\mathbb{G}) \longrightarrow \mathfrak{gl}_n(K).$$

This fact leads us to the following definition.

Definition. Let L be a Lie algebra. A *representation* of L is a Lie algebra map $L \to \mathfrak{gl}_n(K)$ for some $n \in \mathbb{N}$.

Definition. Let L be a Lie algebra. The *adjoint representation* is the representation

$$\begin{aligned} \mathrm{ad}_L : L &\longrightarrow \mathfrak{gl}(L) \\ x &\longmapsto [x, \cdot], \end{aligned}$$

where $\mathfrak{gl}(L)$ is the group of K-endomorphisms of L as a vector space, endowed with the structure of a Lie algebra by using the commutator.

The adjoint representation allows us find a bridge between our abstract algebra L and an algebra of linear transformations over a (finitely dimensional) vector space, that can be represented by an algebra of matrices.

Definition. Let L be a Lie algebra, E a subspace of L such that $[E, E] \subseteq E$. Then E is called a *subalgebra* of L. If, moreover, $[E, L] \subseteq E$, then E is called an *ideal* of L.

Exercise 3.1. Prove that kernels of Lie algebra morphisms are ideals, and that quotients of Lie algebras by Lie ideals are still canonically Lie algebras. Also, the usual isomorphism theorems hold.

Definition. The *center of* L is $\zeta(L) := \ker(\mathrm{ad}_L)$. If $\zeta(L) = L$, we say that L is *abelian*.

Exercise 3.2. Every 1-dimensional Lie algebra is abelian.

Lemma 3.3. $\mathrm{Im}(\mathrm{ad}_L) \subseteq \mathrm{Der}(L)$ *and we have an injection*

$$L/\zeta(L) \longrightarrow \mathrm{Der}(L) \subseteq \mathfrak{gl}(L).$$

Proof. Let $x \in L$: we see that $\mathrm{ad}_L(x) = [x, \cdot]$ is a derivation. Take $y, z \in L$. Then

$$\begin{aligned} \mathrm{ad}_L(x)([y,z]) = [x,[y,z]] = -[y,[x,z]] - [z,[x,y]] = [y,[x,z]] + [[x,y],z] = \\ = [y, \mathrm{ad}_L(x)(z)] + [\mathrm{ad}_L(x)(y), z]. \end{aligned}$$

Since $\zeta(L) = \ker(\mathrm{ad}_L)$, we have that

$$L/\zeta(L) \cong \mathrm{ad}_L(L) \subseteq \mathrm{Der}(L) \subseteq \mathfrak{gl}(L).$$

□

Definition. A Lie algebra L is *simple* if it has no nontrivial ideals and it is not abelian.

Example 3.4. Consider the Lie algebra $\mathfrak{sl}_2(K)$ of traceless 2×2 matrices. Define

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathfrak{sl}_2(K)$ is generated by e, f, h as a vector space over K. By a straight computation, one finds that

$$[e,f] = h, \quad [h,e] = 2e, \quad [h,f] = -2f. \tag{1}$$

We prove now that $\mathfrak{sl}_2(K)$ is simple. Suppose that $I \subseteq \mathfrak{sl}_2(K)$ is an ideal and let $x \in I$. Then there exist $a, b, c \in K$ such that $x = ae + bf + ch$. Using bilinearity and the relations in equation (1), we obtain that $[x,e] = 2ec - bh$. Hence $[[x,e],e] = -2be$. If $b \neq 0$, since I is an ideal, we have that $-2be \in I$, hence $e \in I$. But, using relations in equation (1), we can easily deduce that $h, f \in I$ and then $I = \mathfrak{sl}_2(K)$. If $b = 0$ and $c \neq 0$, then we obtain the same result by considering $[x,e] = 2ec \in I$. If $b = c = 0$ and $a \neq 0$, we directly have that $ae \in I$ and we can conclude as before. This means that either $I = \mathfrak{sl}_2(K)$ or $I = 0$.

4 Nilpotent Lie algebras

Definition. Let L be a Lie algebra and consider the sequence

$$L^1 := [L, L], \quad L^j := [L, L^{j-1}] \quad \text{for } j > 1.$$

L is called *nilpotent* if $L^j = 0$ for some $j > 0$.

Example 4.1. Here there are two important examples of nilpotent Lie algebras.

1. Abelian Lie algebras are nilpotent.
2. The subalgebra $\mathfrak{n}_n(K)$ of $\mathfrak{sl}_n(K)$ consisting of strictly upper triangular matrices, i.e. matrices of the form

   $$\begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

   is nilpotent.

Proposition 4.2. *Let L be a Lie algebra.*

(a) If L is nilpotent, then its subalgebras and homomorphic images are so.

(b) If $L/\zeta(L)$ is nilpotent, then L is nilpotent.

(c) If L is nilpotent, then $\zeta(L) \neq 0$.

Proof. (a) If E is a subalgebra of L, we have that $E^j \subseteq L^j$, hence if L is nilpotent then E is so. If $f : L \to N$ is a morphism of Lie algebras, then $f(L^j) = f(L)^j$. Hence if L is nilpotent, then $f(L)$ is so.

(b) If $L/\zeta(L)$ is nilpotent, then there exists a $j \in \mathbb{N}$ with the property that $(L/\zeta(L))^j = L^j/(\zeta(L) \cap L^j) = 0$. This means that $L^j \subseteq \zeta(L)$, therefore $L^{j+1} \subseteq [L, \zeta(L)] = 0$.

(c) Let L be nilpotent and consider j minimal such that $L^j = 0$. Then $L^{j-1} \neq 0$ and the condition $0 = L^j = [L, L^{j-1}]$ implies that $L^{j-1} \subseteq \zeta(L)$. □

Definition. Let L be a Lie algebra and $x \in L$. Then x is ad-*nilpotent in L* if $\mathrm{ad}_L(x)$ is a nilpotent element of $\mathfrak{gl}(L)$, i.e. if there exists a $j \in \mathbb{N}$ such that $(\mathrm{ad}_L(x))^j \equiv 0$.

Lemma 4.3. *Let $x \in \mathfrak{gl}(V)$ for some vector space V. If x is a nilpotent endomorphism, then x is* ad-*nilpotent in $\mathfrak{gl}(V)$.*

Proof. Let $x \in \mathfrak{gl}(V)$ be a nilpotent endomorphism. Define

$$\begin{aligned} \lambda_x : \mathrm{End}_K(V) &\longrightarrow \mathrm{End}_K(V) & \qquad \rho_x : \mathrm{End}_K(V) &\longrightarrow \mathrm{End}_K(V) \\ y &\longmapsto x \circ y & y &\longmapsto y \circ x \end{aligned}$$

Clearly we have that $\lambda_x, \rho_x \in \mathrm{End}_K(\mathrm{End}_K(V))$, they commute and they are nilpotent (since x is so). Then also $\lambda_x - \rho_x$ is nilpotent (just use the binomial formula to prove it). Hence we have that $\mathrm{ad}_{\mathfrak{gl}(V)}(x) = \lambda_x - \rho_x$ is nilpotent. □

Theorem 4.4. *Let $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra, with V a finitely dimensional vector space. If $V \neq 0$ and L consists of nilpotent endomorphisms, then there exists $v \in V$, $v \neq 0$, such that $Lv = 0$.*

Proof. Use induction on $\dim L$. If $\dim L = 1$, then $L = Kx$ for some $x \in L$, and $\ker(x) = \ker(ax) \neq 0$ for every $a \in K \setminus \{0\}$, since x is nilpotent. Taking $v \in \ker(x)$, we are done.

Assume now that the statement holds for every L' subalgebra of $\mathfrak{gl}(V)$ with $\dim L' < \dim L$. Let $M \neq L$ be a subalgebra of L. According to the previous lemma, M acts via ad_L as a lie algebra of nilpotent endomorphisms on the vector space L, and hence it acts also on the vector space L/M since $[M, M] \subseteq M$. Since $\dim M < \dim L$, the inductive hypothesis guarantees the existence of a vector $x + M \in L/M$, $x \notin K$, killed by the image of M in $\mathfrak{gl}(L/M)$. This just means that $[y, x] \in M$ for every $y \in M$, whereas $x \notin M$. In other words, M is properly included in

$$N_L(M) := \{y \in L : [y, z] \in M \quad \text{for every } z \in M\},$$

the normalizer of M in L.

Consider now M to be a maximal proper subalgebra of L. The preceding argument forces $N_L(M) = L$, i.e. M is an ideal of L. If $\dim L/M$ were greater that one, then the inverse image in L of a one dimensional subalgebra of L/M (which always exists) would be a proper subalgebra properly containing M, which is absurd. Therefore, M has codimension one and so we can write $L = M + Kz$ for any $z \in L \setminus M$.

By induction, $W := \{v \in V : Mv = 0\}$ is nonzero. Since M is an ideal, W is stable under the action of L: for every $x \in L$, $y \in M$, $w \in W$ we have

$$yx(w) = xy(w) - [x,y]w = 0.$$

Choose $z \in L \setminus M$. Then the nilpotent endomorphism z, acting on the subspace W, has an eigenvector of eigenvalue 0, i.e there exists a nonzero $v \in W$ for which $zv = 0$. Therefore $Lv = 0$ as desired. ☐

Theorem 4.5. *(Engel) Let L be a Lie algebra. Then L is nilpotent if and only if every element of L is ad_L-nilpotent.*

Proof. If L is nilpotent, then there exists a $j \in \mathbb{N}$ such that $L^j = 0$, and so for every $x \in L$ we have that $(\mathrm{ad}_L(x))^j = 0$, hence x is ad-nilpotent.

Conversely, let L be composed of ad-nilpotent elements, and use induction on $\dim L$. If $\dim L = 1$, then L is abelian and we are done. Let now $\dim L > 1$ and work with $\mathrm{ad}_L(L) \subseteq \mathfrak{gl}(L)$. We can use the above theorem to find a nonzero $x \in L$ such that $[x, L] = 0$, i.e. $x \in \zeta(L)$. Hence $L/\zeta(L)$ has smaller dimension than L, and its elements are again ad-nilpotent. Using the inductive hypothesis, we obtain that $L/\zeta(L)$ is nilpotent. Thanks to point (c) of proposition 4.2, we conclude that L is nilpotent. ☐

Definition. Let V be a finitely dimensional vector space of dimension $n \in \mathbb{N}$. A *flag* on V is a collection of subspaces $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that $\dim V_i = i$.

Corollary 4.6. *Let V be a vector space of dimension n. If $L \subseteq \mathfrak{gl}(V)$ is a Lie algebra consisting of nilpotent elements, then there exists a flag $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that $xV_i \subseteq V_{i-1}$ for every $x \in L$, $i = 1, \dots, n$.*

Proof. Using theorem 4.4, we find a vector $v_1 \in V$ such that $Lv_1 = 0$. Hence we define $V_1 := Kv_1$. Work on V/V_1: L acts on V/V_1 by nilpotent endomorphisms, hence by theorem 4.4 there exists a $v_2 \in V \setminus V_1$ such that $L(v_2) \subseteq V_1$. Define $V_2 := Kv_1 + Kv_2$. Proceeding inductively, we conclude. ☐

Corollary 4.7. *Let $L \subseteq \mathfrak{gl}(V)$ be a Lie algebra consisting of nilpotent elements, V a vector space of dimension n. Then there exists a basis of V such that, with respect to this basis, L is a subalgebra of the algebra $\mathfrak{n}$ of strictly upper triangular matrices.*

5 Solvable Lie algebras

Definition. Let L be a Lie algebra and consider the sequence

$$L^{(1)} := [L,L], \quad L^{(j)} := [L^{(j-1)}, L^{(j-1)}] \quad \text{for } j > 1.$$

L is called *solvable* if $L^j = 0$ for some $j > 0$.

Example 5.1. Here there are three important examples of nilpotent Lie algebras.

1. Abelian Lie algebras are solvable.
2. Nilpotent Lie algebras are solvable, as $L^j \subseteq L^{(j)}$.
3. The subalgebra $\mathfrak{b}_n(K)$ of $\mathfrak{sl}_n(K)$ consisting of all upper triangular matrices is solvable.

Proposition 5.2. *Let L be a Lie algebra.*

(a) If L is solvable, then its subalgebras and homomorphic images are so.

(b) If an ideal I of L is solvable and L/I is solvable, then L is solvable.

(c) If I, J are solvable ideals of L, then $I + J$ is solvable.

Proof. (a) If E is a subalgebra of L, we have that $E^{(j)} \subseteq L^{(j)}$, hence if L is solvable then E is so. If $f : L \to N$ is a morphism of Lie algebras, then $f(L^{(j)}) = f(L)^{(j)}$. Hence if L is solvable, then $f(L)$ is so.

(b) Consider the projection $\pi : L \to L/I$. Then there exists a j such that $0 = (\pi(L))^{(j)} = \pi(L^{(j)})$, i.e. $L^j \subseteq I$. There exists also a k such that $I^{(k)} = 0$, hence $L^{(j+k)} \subseteq I^{(k)} = 0$.

(c) We have that $(I+J)/I \cong J/(I \cap J)$. Since I is solvable and $J/(I \cap J)$ is solvable (by point (a)), then by point (b) we have that $I + J$ is solvable. □

Point (c) of the above proposition implies that inside a Lie algebra L there is always a maximal solvable ideal, namely the sum of all solvable ideals.

Definition. Let L be a Lie algebra. The maximal solvable ideal of L is called $\mathrm{Rad}(L)$, the *radical ideal* of L.

Following the ideas of the nilpotent case, we give a description of every solvable Lie algebra $L \subseteq \mathfrak{gl}(V)$, for some finitely dimensional vector space V.

Theorem 5.3. *Let $L \subseteq \mathfrak{gl}(V)$ a solvable Lie algebra, with V a finitely dimensional vector space. If $V \neq 0$ then there exists $v \in V$, $v \neq 0$, such that $Lv = Kv$.*

Proof. Use induction on $\dim L$, the case of $\dim L = 0$ being trivial. The proof follows now some steps similar the proof of theorem 4.4, but here the initial hypotheses that $\operatorname{char} K = 0$ and $K = \bar{K}$ play a fundamental role.

Step 1: Find an ideal M of L of codimension 1. Since L is solvable of positive dimension, L properly contains $[L, L]$. Since $L/[L, L]$ is abelian, any subspace is automatically an ideal. Call M the inverse image in L of a subspace of codimension 1 in $L/[L, L]$. Then M is an ideal of L of codimension 1.

Step 2: Common eigenvectors exist for M. Use induction to find a common eigenvector $v \in V$ for M. This means that for every $x \in M$ we have that $xv = \lambda(x)v$, with $\lambda : M \to K$ some linear function. Fix this λ and define the nonzero subspace

$$W := \{w \in V : xw = \lambda(x)w \quad \text{for every } x \in M\}.$$

Step 3: L stabilizes W. Let $w \in W$, $x \in L$. To test wether or not xw lies in W, we must take an arbitrary $y \in M$ and examine

$$yxw = xyw - [x, y]w = \lambda(y)xw - \lambda([x, y])w.$$

In particular, we have to prove that $\lambda([x, y]) = 0$. For this, let $n > 0$ be the smallest integer for which $w, xw, \dots, x^{n-1}w$ are linearly independent. Let W_i be the subspace of V generated by $w, xw, \dots, x^{i-1}w$, so that $\dim W_i = i$, $W_n = W_{n+j}$ for every $j \geq 0$ and clearly x maps W_n to W_n. It is easy to prove by induction on i that each $y \in M$ leaves each W_i invariant. This implies that, with respect to the basis $w, xw, \dots, x^{n-1}w$, every $y \in M$ is represented by an upper triangular matrix. Moreover, an induction similar to the previous one shows that the diagonal entries of those matrices are equal to $\lambda(y)$. Hence we have that $\operatorname{Tr}_{W_n}(y) = n\lambda(y)$. In particular, this is true for elements of M of the special form $[x, y]$, with x as above, $y \in K$. But x and y both stabilize W_n, so $[x, y]$ acts on W_n as the commutator of two endomorphisms of W_n and hence its trace is equal to 0. We conclude that $n\lambda([x, y]) = 0$. Since $\operatorname{char} K = 0$, this implies that $\lambda([x, y]) = 0$ as required.

Step 4: Conclusion. Write $L = M + Kz$ for any $z \in L \smallsetminus M$. Since $K = \bar{K}$ and W is stable under the action of L, we find an eigenvector $v \in W$ for the action of z. Then v is obviouly an eigenvector for the whole L. ☐

Theorem 5.4 (Lie). *Let V be a vector space of dimension n. If $L \subseteq \mathfrak{gl}(V)$ is a solvable Lie algebra, then there exists a flag $0 \subseteq V_1 \subseteq \cdots \subseteq V_n = V$ such that $xV_i \subseteq V_i$ for every $x \in L$, $i = 1, \dots, n$.*

Proof. Using the above theorem, we find a $v_1 \in V$ such that $Lv_1 = Kv_1$, and we set $V_1 = Kv_1$. Then we can work on V/V_1 and find $v_2 \in V \smallsetminus V_1$ such that $L(v_2 + V_1) = K(v_2 + V_1)$. This means that, if we call $V_2 := Kv_1 + Kv_2$, then $Lv_2 \subseteq V_2$. Proceeding inductively, we conclude. ☐

Corollary 5.5. *Let* $L \subseteq \mathfrak{gl}(V)$ *be a solvable Lie algebra,* V *a vector space of dimension* n*. Then there exists a basis of* V *such that, with respect to this basis,* L *is a subalgebra of the algebra* $\mathfrak{b}$ *of upper triangular matrices.*