Locally analytic manifolds & p-adic Lie groups

Throughout, K is a complete NA field with AV $|\cdot|$.

Some more functional analysis

* **direct limits:** $\{V_i\}_{i\in I}$ direct system of loc. convex K-vector spaces, make $\varinjlim_{i\in I} V_i$ into a loc. convex K-v.s defined by the (open) lattices $L \subseteq \varinjlim V_i$ s.t. $\varphi_j^{-1}(L) \subseteq V_j$ is open $\forall j \in I$ ($\varphi_j : V_j \to \varinjlim V_i$ is natural map)
* **direct products:** $\prod_{i\in I} V_i \xrightarrow{p_j} V_j \xrightarrow{q_{j_k}} \mathbb{R}_{\geq 0}$

$\{q_{j_k} \circ p_j \mid j\}$ defining the loc. convex top.

* Strong dual: V loc. convex K-v.s.

$$V' := \{\varphi \in V^* \mid \varphi \text{ continuous}\}$$

Say $B \subseteq V$ is bounded if for all open lattices $L \subseteq V$, $\exists a \in K$ s.t. $B \subseteq aL$.

e.g. if V normed then B bounded iff $\|B\| \subseteq \mathbb{R}$ is bounded.

For $B \subseteq V$ bounded, define seminorm p_B on V' by $p_B(\varphi) := \sup_{b \in B} |\varphi(b)|$.

$\{p_B : B \subseteq V \text{ bounded}\}$ defines a loc. convex top on V' called the strong top.

e.g. if V normed, V' is also a normed space w.r.t operator norm $\|\varphi\| := \sup_{0 \neq v \in V} \frac{\|\varphi(v)\|}{\|v\|}$.

Lattices: $\mathcal{L}(B, r) := \{\varphi \in V' \mid |\varphi(B)| < r\}$, $r > 0$ B bounded

Manifolds

Setup: • $L \subseteq K$ both complete NA, s.t. AV on K extends the one on L.

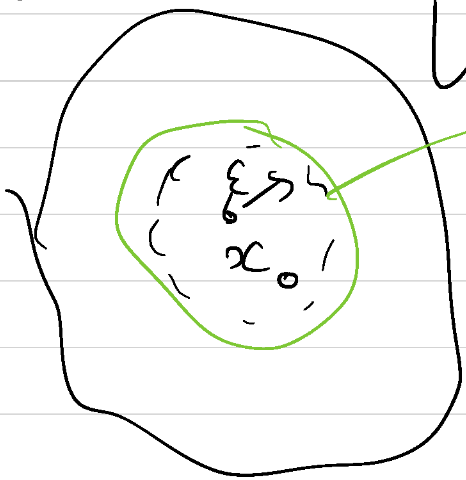
• $U \subseteq L^n$ open, V an L-Banach space (e.g. $V = K$)

Def^n: (1) $f : U \longrightarrow V$ is locally analytic if for all $x_0 \in U$, $\exists\, \varepsilon > 0$ and a power series

$$F(\underline{X}) = \sum_{\alpha \in \mathbb{N}^n} v_\alpha \underline{X}^\alpha \qquad (\underline{X}^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n})$$

with $v_\alpha \in V$ $\forall \alpha$ s.t. $\varepsilon^{|\alpha|} \cdot \|v_\alpha\| \longrightarrow 0$ as $|\alpha| := \alpha_1 + \dots + \alpha_n \longrightarrow \infty$ and for all $x \in B(x_0, \varepsilon) \cap U$

$$f(x) = F(x - x_0).$$



f = power series

(2) Given $x_0 \in L^n$, $\varepsilon >$, say $f : B(x_0, \varepsilon) \longrightarrow V$ is <u>holomorphic</u> if $f(x) = F(x - x_0)$ $\forall x \in B(x_0, \varepsilon)$ & F as in (1).

$F(x_0, \varepsilon, V) := \{ f : B(x_0, \varepsilon) \to V \mid f \text{ holom.} \}$

an L-Banach space via $\| \sum v_\alpha \underline{X}^\alpha \| := \sup_{\alpha \in \mathbb{N}^n} (\varepsilon^{|\alpha|} \cdot \|v_\alpha\|)$

(if V is a K-Banach space, so is $F(x_0, \varepsilon, V)$)

<u>Def^n:</u> (1) A locally L-analytic manifold of dim d is a Hausdorff space M equipped with an atlas $\mathcal{A} = \{(U_i, \varphi_i)\}_{i \in I}$ s.t.:

- $M = \bigcup_{i \in I} U_i$ open cover
- $\varphi_i : U_i \longrightarrow L^d$ homeo onto an open subset
- for all $i, j \in I$, $\varphi_i(U_i \cap U_j) \underset{\varphi_i \circ \varphi_j^{-1}}{\overset{\varphi_j \circ \varphi_i^{-1}}{\rightleftarrows}} \varphi_j(U_i \cap U_j)$ are locally analytic

(2) V is an L-Banach space, say $f: M \to V$ is locally analytic if $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to V$ is loc. an. $\forall i$.

$C^{an}(M, V) := \{ f: M \to V \mid f \text{ loc. an.} \}$.

<u>Example</u>: $\mathbb{P}^d(L)$ is a loc. L-an. manifold.

$\left(\mathbb{P}^d(L) = \left(L^{d+1} \setminus \{\underline{0}\} \right) / \sim \right)$

$$U_j = \{ (a_0 : \ldots : a_d) \in \mathbb{P}^d(L) \mid |a_i| \leq |a_j| \ \forall i \}$$

$$\varphi_j : U_j \xrightarrow{\sim} B(0,1) = (L^o)^d \subseteq L^d$$

$$(a_0 : \ldots : a_d) \longmapsto \left(\frac{a_0}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_d}{a_j} \right)$$

Just need to check (for $j<k$, $k<j$ similar)

$$\{\underline{x} \in (L^0)^d \mid |x_{k-1}| = 1\} \xrightarrow{\varphi_j^{-1}} U_j \cap U_k \xrightarrow{\varphi_k} \{y \in (L^0)^d \mid |y_j| = 1\}$$

V ''

$$(x_1, \dots, x_d) \longmapsto \left(\frac{x_1}{x_{k-1}}, \dots, \frac{x_{j-1}}{x_{k-1}}, \frac{1}{x_{k-1}}, \frac{x_j}{x_{k-1}}, \dots, \frac{x_{k-2}}{x_{k-1}}, \frac{x_k}{x_{k-1}}, \dots, \frac{x_d}{x_{k-1}}\right)$$

is loc. an.

Pick $a \in V$, $0 < \varepsilon < 1$ ($\Rightarrow B(a, \varepsilon) \subseteq V$)

$$F_j(\underline{X}) = \frac{1}{a_{k-1}} \sum_{n \geq 0} \left(\frac{-1}{a_{k-1}}\right)^n X_{k-1}^n$$

Easy calculation: $x \in B(a, \varepsilon) \Rightarrow F_j(x - a) = \frac{1}{x_{k-1}}$

For $i \neq j$, $\quad F_i(\underline{X}) := F_j(\underline{X}) \cdot \begin{cases} (X_i + a_i) & \text{o/w} \\ (X_{i-1} + a_{i-1}) & \text{if } j < i < k \end{cases}$

$$F(\underline{X}) = (F_1(\underline{X}), \ldots, F_d(\underline{X})) \in \mathcal{F}(a, \varepsilon, L^d)$$

& $f(x) = F(x-a) \quad \forall x \in B(a, \varepsilon)$.

<u>Def^n:</u> A (loc. L-an.) Lie group G is a manifold & a group s.t. mult: $G \times G \longrightarrow G$ is a loc. an.

<u>e.g.</u> (1) $(L, +)$, $(L^\times, \cdot)$, $GL_n(L)$, $GL_n(L^\circ)$, etc

(2) More generally, the L-points of any conn. alg. gp over L.

The topology on $C^{an}(M, K)$

Setup: • $[L : \mathbb{Q}_p] < \infty$, K spherically complete

• M d-dim^l loc. L-analytic manifold, assume M is paracompact i.e. any open cover can be refined to a cover where the opens are disjoint (e.g. true if $M = G$ is a Lie group cf. Cor 18.8 in P. Schneider's book on p-adic Lie Groups).

Suppose $f \in C^{an}(M, K)$, (U_i, φ_i) charts

$\rightsquigarrow$ WLOG U_i are disjoint

$\left(\Rightarrow C^{an}(M, K) = \prod C^{an}(U_i, K) \right)$

We can cover $\varphi_i(U_i) \subseteq L^d$ by disjoint open balls B_{ij} s.t on each B_{ij}, $f \circ \varphi_i^{-1}$ is holomorphic

Summary: Given f, there's a family of charts $(U_i', \varphi_i')_{i \in I}$ of M s.t.:

(1) $M = \coprod_{i \in I} U_i'$

(2) $\varphi_i'(U_i') = B(x_i, \varepsilon_i) \subseteq L^d$ some $x_i \in L^d$, $\varepsilon_i > 0$

(3) $f \circ \varphi_i^{-1}$ is holomorphic.

Defn: An index $\mathcal{I}$ on M is a family of charts $(U_i, \varphi_i)_{i \in I}$ satisfying (1) & (2).

- $\mathcal{F}_I(M, K) := \prod_{i\in I} \underbrace{\mathcal{F}(x_i, \varepsilon_i, K)}_{K\text{-Banach space}}$

with product loc. convex top.

Then $C^{an}(M, K) := \varinjlim_{I} \mathcal{F}_I(M, K)$ with direct limit loc. conv. top.

<u>Defn</u>: $D(M, K) := C^{an}(M, K)'$ with strong top. is the space of <u>K-valued distributions</u>

(all these constructions are due to Féaux de Lacroix for the top. on $C^{an}(M, K)$, & study of $D(M, K)$ is due to Schneider & Teitelbaum)

Facts: • M compact $\Rightarrow D(M, K)$ is Fréchet

• $M = \coprod_{i \in I} M_i$, M_i open $\Rightarrow D(M, K) = \bigoplus_{i \in I} D(M_i, K)$ as loc. convex spaces.

• $M = G$ is a Lie group $\Rightarrow D(G, K)$ is a K-algebra. (F. de Lacroix)

Example: $M = (\mathbb{Z}_p, +)$.

Write for $b \in \mathbb{Z}_p$, $\varepsilon > 0$, $\mathcal{O}_{b,\varepsilon} :=$ K-Banach space of holom. functions on $B(b, \varepsilon)$

Then $C^{an}(\mathbb{Z}_p, K) = \varinjlim_{j \geq 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}_{b, p^{-j}}$

& $D(\mathbb{Z}_p, K) = \varprojlim_{j \geq 1} \prod_{b \in \mathbb{Z}/p^j\mathbb{Z}} \mathcal{O}'_{b, p^{-j}}$

<u>Thm:</u> (Amice) $D(\mathbb{Z}_p, K) \cong \mathcal{O}(X)$ = ring of rigid analytic functions on open unit disk over K

If $M = G$ Lie group:

$$G \hookrightarrow D(G, K) = C^{an}(G, K)'$$

$$g \longmapsto \delta_g : f \longmapsto f(g)$$

(Dirac distributions)

Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ $(= T_1(G))$

$$\mathfrak{g} \longrightarrow D(G, K)$$

$$x \longmapsto \mathrm{dist}(x) : f \longmapsto \frac{d}{dt}\left(f(\exp(tx))\right)\Big|_{t=0}$$

$\rightsquigarrow$ $D(G,K)$ contains images of KG & $\mathcal{U}(\mathfrak{g})$