

Lecture notes for the seminar

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 \mathcal{D} -modules

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1 Grassmannian varieties

By definition, projective spaces parametrize one dimensional subspaces in affine spaces. The **Grassmannian varieties** are a generalization of the projective space in the sense that they parametrize higher-dimensional subspaces.

Let V be an n -dimensional complex space. We can set-theoretically define the Grassmannian $G(k, V)$ as follows:

$$G(k, V) := \{U \subseteq V \mid \dim(U) = k\}.$$

This is a coordinate-free definition. On the other hand, we can also fix a basis $\{v_1, \dots, v_n\}$ of V and identify $V \simeq \mathbb{C}^n$. In this situation we have

$$G(k, n) := \{U \subseteq \mathbb{C}^n \mid \dim(U) = k\}.$$

Our goal in this section is to realize Grassmannians as projective varieties, but let us assume for a while that we have accomplished this objective and let us try to compute $\dim(G(k, n))$. To do that, let us start by recalling that the multiplicative group \mathbb{G}_m acts on the punctured affine space $\mathbb{A}^{n+1} \setminus \{0\}$ by scalar multiplication and \mathbb{P}^n is nothing more than the orbit space of this action¹, in other words

$$\mathbb{P}^n = (\mathbb{A}^{n+1} \setminus \{0\}) / \mathbb{G}_m.$$

Following our preliminary discussion, it is natural to think in generalize the preceding presentation of \mathbb{P}^n for Grassmannian varieties. Every k -dimensional subspace of \mathbb{C}^n is spanned by k vectors. So, we can look at the space of all k -tuples of linearly independent vectors, which we think of as the rows of $k \times n$ -matrices. The group GL_k acts on this space by left multiplication and two $k \times n$ -matrices have the same row span if and only if they are in the same orbit under this group action. This means that we can identify

$$G(k, n) = M_{k \times n}^{\mathrm{rank} \ k} / \mathrm{GL}_k.$$

Let us look a little deeper at the action

$$\begin{pmatrix} \lambda_{1,1} & \cdots & \lambda_{1,k} \\ \vdots & \ddots & \vdots \\ \lambda_{k,1} & \cdots & \lambda_{k,k} \end{pmatrix} \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & \ddots & \vdots \\ a_{k,1} & \cdots & a_{k,k} \end{pmatrix}$$

If the first $k \times k$ -minor of the matrix of the right is non-zero, the orbit contains a unique element of the form

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & b_{1,1} & \cdots & b_{1,n-k} \\ 0 & 1 & \cdots & 0 & b_{2,1} & \cdots & b_{2,n-k} \\ \vdots & & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & b_{k,1} & \cdots & b_{k,n-k} \end{pmatrix}.$$

¹In order to soft the notation we let implicit the fact that we are in fact considering the space of k -points.

Conversely, we can always obtain a matrix of rank k for any $k \times (n - k)$ -matrix. In other words, the row span of matrices of the preceding form are in bijection with an affine space $\mathbb{A}^{k(n-k)}$. This result involves a choice coming from the assumption that the first $k \times k$ -minor is non-zero. In general, we have to permute columns first. So, we see in this way that the Grassmannian $G(k, n)$ is covered by $\binom{n}{k}$ affine spaces $\mathbb{A}^{k(n-k)}$. In particular, if the Grassmannian is a variety, it must be of dimension $k(n - k)$.

1.1 Grassmannians and the Plücker embedding

Let V be a finite dimensional complex vector space, say of dimension n . We know that $\mathbb{P}(V)$ is the set of all the lines in V that pass through the origin. We also have the following presentation. Given that V is finite dimensional, we can use the canonical identification $V \xrightarrow{\cong} V^*$ to associate to every hyperplane in V a unique line through the origin in V^* and $\mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}(V^*)$ tells us that $\mathbb{P}(V)$ can be also considered as the set of $n - 1$ dimensional subspaces of V .

Let us fix now basis $\mathcal{B} := \{v_1, \dots, v_n\}$ of V . This fixes coordinates over $\mathbb{P}(V)$ and we can identify

$$\mathbb{P}(V) \simeq \mathbb{P}(\mathbb{C}^n) = \mathbb{P}^{n-1}.$$

In particular, every k -dimensional subspace of $V \xrightarrow{\cong} \mathbb{C}^n$ can be identified (via projectivization) with a $k - 1$ dimensional subspaces of \mathbb{P}^{n-1} .

The preceding description of the Grassmannian in terms of matrices is very convenient to understand it as a set; unfortunately, it is not very useful for our goal of finding a structure of projective variety. Instead, it is better to employ some multilinear algebra.

Exterior Algebra 1.1.1. The exterior algebra $\bigwedge V$ is the residue class ring of the non-commutative tensor algebra

$$\mathcal{T}(V) := \bigoplus_{d \geq 0} V^{\otimes d}$$

modulo the ideal generated by all the tensor of the form $v \otimes v$, for $v \in V$. The residue class of a basic tensor $w_1 \otimes \dots \otimes w_d$ is denoted by $w_1 \wedge \dots \wedge w_k$. The exterior algebra inherits the grading of the tensor algebra

$$\bigwedge V = \bigoplus_{d \geq 0} \bigwedge^d V$$

where $\bigwedge^d V$ is the spanned by all the vectors of the forms $w_1 \wedge \dots \wedge w_d$. In particular, $\bigwedge^1 V = V$ and $\bigwedge^0 V = \mathbb{C}$. We recall for the reader that the wedge product is associative, bilinear, antisymmetric and for every permutation $\sigma \in \mathcal{S}_k$ we have

$$w_1 \wedge \dots \wedge w_k = \text{sgn}(\sigma)(w_{\sigma(1)} \wedge \dots \wedge w_{\sigma(k)}).$$

In particular, if we express w_1, \dots, w_k in terms of the basis \mathcal{B} , then using antisymmetry and bilinearity we can expand every multivector in $\bigwedge V$ in terms of

this basis. In other words

$$\begin{aligned} & \left(\sum_{i=1}^n a_{i,1} v_i \right) \wedge \cdots \wedge \left(\sum_{i=1}^n a_{i,k} v_i \right) \\ &= \sum_{1 \leq i_1 < \cdots < i_k \leq n} \det \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & \ddots & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{pmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k} \end{aligned}$$

for $k \leq n$. In particular, we see that every multivector $w_1 \wedge \cdots \wedge w_n$ in $\bigwedge^n V$ is a multiple of $v_1 \wedge \cdots \wedge v_n$, with the coefficient given by the determinant of the change of basis matrix from $\{w_1, \dots, w_n\}$ to \mathcal{B} .

Let us define a map

$$i : G(k, V) \rightarrow \mathbb{P}(\bigwedge^k V).$$

Given a subspace $W \in G(k, V)$ and a basis $\mathcal{B}_W = \{w_1, \dots, w_k\}$ of W we let $i(W)$ be the composite

$$W \rightarrow \bigwedge^k V \rightarrow \mathbb{P}(\bigwedge^k V).$$

Clearly, different choices of basis for W give different wedge products in $\bigwedge^k V$, but by we have just stated, this map is unique up to scalar multiplication, hence is well-defined on $\mathbb{P}(\bigwedge^k V)$.

Proposition 1.1.2. *i is injective.*

Proof. Let us build a left inverse $p : \mathbb{P}(\bigwedge^k V) \rightarrow G(k, V)$ of i . For every $[\omega] \in \mathbb{P}(\bigwedge^k V)$ let

$$p([\omega]) := \{v \in V \mid v \wedge \omega = 0 \in \bigwedge^{k+1} V\}.$$

It is clear that p is well-defined². We want to proof that $p \circ i(W) = W$. Let $\mathcal{B}_W := \{w_1, \dots, w_k\}$ be a basis of W in such a way that $i(W) = [w_1 \wedge \cdots \wedge w_k]$. Then for every $w \in W$ it is clear that $w \wedge w_1 \wedge \cdots \wedge w_k = 0$ and $W \subseteq p \circ i(W)$.

On the other hand, let us take $v \in p \circ i(W)$ and let us complete \mathcal{B}_W to a basis $\mathcal{B}_W \cup \{w_{k+1}, \dots, w_n\}$ of V . Then writing $v = \sum_{i=1}^n a_i w_i$, we have

$$\left(\sum_{i=1}^n a_i w_i \right) \wedge w_1 \wedge \cdots \wedge w_k = 0.$$

After distributing and using antisymmetry we see that $a_i = 0$ for all $i > k$ and therefore $v = \sum_{i=1}^k a_i w_i \in W$. \square

Remark 1.1.3. *Given that the set $\{v_{i_1} \wedge \cdots \wedge v_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}$ forms a basis for $\bigwedge^k V$, we have that $\mathbb{P}(\bigwedge^k V) \xrightarrow{\cong} \mathbb{P}^{\binom{n}{k}-1}$, and i identifies the Grassmannian with a subset of the projective space $\mathbb{P}^{\binom{n}{k}-1}$.*

²Actually, it is an easy exercise to prove that $\dim(\{v \in V \mid v \wedge \omega = 0 \in \bigwedge^{k+1} V\}) = k$

Definition 1.1.4. Let $\omega \in \bigwedge^k V$. We say that ω is totally decomposable if we can write $\omega = v_1 \wedge \cdots \wedge v_k$ where v_1, \dots, v_k are linearly independent in V .

Let us denote by D_ω the set of all $v \in V$ dividing $\omega \in \bigwedge^k V$, this is

$$D_\omega := \{v \in V \mid \exists \phi \in \bigwedge^{k-1} V \text{ such that } \omega = v \wedge \phi\} = \{v \in V \mid v \wedge \omega = 0\} \quad (1)$$

the second equality is an easy exercise.

Proposition 1.1.5. A multivector $\omega \in \bigwedge^k V$ is totally decomposable if and only if $\dim(D_\omega) = k$.

Proof. Let us first suppose that ω is totally decomposable, say $\omega = v_1 \wedge \cdots \wedge v_k$. Using the relation (1), we can see that v_1, \dots, v_k are k linearly independent elements of D_ω . Let us see that they generate D_ω . In fact, if we extend this set to a basis of V , say $\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ and we take $v = \sum_{i=1}^n a_i v_i$, then we must have

$$0 = v \wedge \omega = \left(\sum_{i=1}^n a_i v_i \right) \wedge v_1 \wedge \cdots \wedge v_k.$$

All the i -th terms $i \leq k$ will vanish after distributing, so all $a_i = 0$ for $k < i \leq n$, and v must be a linear combination of the $\{v_1, \dots, v_k\}$. This shows that $\dim(D_\omega) = k$. On the other hand, let us suppose now that $\dim(D_\omega) = k$ and let v_1, \dots, v_k be a basis. Extending this set to a basis of V we get the following equation for every $1 \leq j \leq k$

$$0 = v_j \wedge \underbrace{\left(\sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq k}} a_{\bar{i}_k} v_{\bar{i}_k} \right)}_{\in D_\omega} = \sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq k}} a_{\bar{i}_k} (v_j \wedge v_{\bar{i}_k}).$$

This equations show that if $j \notin \{\bar{i}_k\} := \{i_1, \dots, i_k\}$ for all \bar{i}_k , we must have $a_{\bar{i}_k} = 0$. In particular, all the $a_{\bar{i}_k} = 0$ for which $\{\bar{i}_k\} \neq \{1, \dots, k\}$. this shows that $\omega = a v_1 \wedge \cdots \wedge v_k$ for some scalar a , and thus, ω is decomposable. \square

Corollary 1.1.6. Let $\omega \in \bigwedge^k V$. the space D_ω has dimension $\leq k$, with equality occurring if and only if ω is totally decomposable.

Proof. Let us pick a basis v_1, \dots, v_s of D_ω and extend to a basis of V . As before,

$$\omega = \sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq k}} a_{\bar{i}_k} v_{\bar{i}_k}.$$

For $j \in \{1, \dots, n\}$ we find

$$\omega \wedge v_j = \sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq k}} a_{\bar{i}_k} v_{\bar{i}_k} \wedge v_j = \sum_{\substack{j \notin \bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq k}} a_{\bar{i}_k} v_{\bar{i}_k} \wedge v_j.$$

Now, for $j \leq s$, we have $v_j \in D_\omega$ and the equation $\omega \wedge v_j = 0$ shows that $a_{\bar{i}_k} = 0$ for every \bar{i}_k such that $j \notin \bar{i}_k$. In other words, all \bar{i}_k with $a_{\bar{i}_k} \neq 0$ must contain $\{1, \dots, s\}$, but if $s > k$, there is not such \bar{i}_k of length k , contradicting the fact that $\omega \neq 0$. Finally, if ω is totally decomposable, we are under the hypothesis of the preceding proposition. \square

In the sequel we will be considering the map

$$\begin{aligned} \varphi : \bigwedge^k V &\rightarrow \text{Hom} \left(V, \bigwedge^{k+1} V \right) \\ \omega &\mapsto \varphi_\omega : v \mapsto \varphi_\omega(v) := v \wedge \omega \end{aligned} \quad (2)$$

Lemma 1.1.7. $[\omega] \in \mathbb{P}(\bigwedge^k V)$ lies in the image of the Grassmannian under the Plücker embedding if and only if ω is totally decomposable.

Proof. If ω is totally decomposable $\omega = v_1 \wedge \dots \wedge v_k$, then the subspace of V spanned by $\{v_1, \dots, v_k\}$ is k -dimensional, hence is some $W \in G(k, V)$ an $i(W) = [\omega]$. Conversely, let us suppose $[\omega] = i(W)$ for some $W \in G(k, V)$. Choosing a basis $\{u_1, \dots, u_k\}$ for W , we know that $[\omega] = [u_1 \wedge \dots \wedge u_k]$, and so ω is totally decomposable as $\lambda u_1 \wedge \dots \wedge u_k$ for some $\lambda \in \mathbb{C}$. \square

Theorem 1.1.8. $G(k, V)$ is a projective variety.

Proof. We want to prove that $i(G(k, V)) \subseteq \mathbb{P}(\bigwedge^k V)$ is the locus of a set of homogeneous polynomials. To do that, let us consider the linear map φ . Fixing a basis of V , for every $\omega \in \bigwedge^k V$, the linear map $\varphi_\omega \in \text{Hom}(V, \bigwedge^{k+1} V)$ can be seen as an $n \times \binom{n}{k+1}$ matrix $A(\omega)$ where the entries are functions in ω , and linearity of φ shows that these functions are homogeneous of degree 1. By proposition 1.1.5 and lemma 1.1.7 a particular $[\omega']$ lies in $i(G(k, V))$ if and only if $\varphi_{\omega'}$ has rank $n - k$. In particular, all of its $(n - k + 1) \times (n - k + 1)$ minors vanish³. This implies that ω' is the zero locus of the $(n - k + 1) \times (n - k + 1)$ minors of the matrix $\varphi_{\omega'}$. \square

We will end this section by given a description of the Plücker embedding in coordinates.

As before, let $\bar{i}_k = (i_1, \dots, i_k)$ be such that $1 \leq i_1 < \dots < i_k \leq n$. Given a subspace $W \in G(k, V)$, we want to explicitly find the coordinate $\bar{i}_k(W)$. This is, the \bar{i}_k -th coordinate of the image of the Grassmannian under the Plücker embedding. To do that, let us choose a basis $\{w_1, \dots, w_k\}$ of W . Writing every $w_j = a_{1j}v_1 + \dots + a_{nj}v_n$ (the basis $\{v_1, \dots, v_n\}$ of V being fixed before) we get an $n \times k$ matrix $M_W := (a_{ij})$. The j -th column of M_W keeps the coordinates of w_j . Then

$$w_1 \wedge \dots \wedge w_k = \sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \dots < i_k \leq n}} \sum_{\sigma \in \mathcal{S}_k} \text{sign}(\sigma) a_{i_1, \sigma(1)} \dots a_{i_k, \sigma(k)} v_{i_1} \wedge \dots \wedge v_{i_k}.$$

³Here we are using the following easy result coming from linear algebra. The rank of a matrix $M \in M_{n \times m}(\mathbb{C})$ is the largest integer r such that some $r \times r$ minor does not vanish.

This can be written as follows

$$w_1 \wedge \cdots \wedge w_k = \sum_{\substack{\bar{i}_k = (i_1, \dots, i_k) \\ 1 \leq i_1 < \cdots < i_k \leq n}} \det \begin{pmatrix} a_{i_1,1} & \cdots & a_{i_1,k} \\ \vdots & \ddots & \vdots \\ a_{i_k,1} & \cdots & a_{i_k,k} \end{pmatrix} v_{i_1} \wedge \cdots \wedge v_{i_k}$$

which means that the \bar{i}_k -th coordinate for $i(W)$ is $\bar{i}_k(W) := \det(M_{W, \bar{i}_k})$, where M_{W, \bar{i}_k} is the $k \times k$ -matrix formed from the i_1, \dots, i_k -th rows of M_W .

Remark 1.1.9. *A priori, the preceding construction depends of the chosen basis for W . But again, a different basis will be mapped to a multiple of $w_1 \wedge \cdots \wedge w_k$ which are identified in $\mathbb{P}(\wedge^k V) = \mathbb{P}^{\binom{n}{k}-1}$.*

The relations between these minor correspond to equations of $G(k, V)$ in $\mathbb{P}(\wedge^k V)$, and are called **Plücker relations** [33].

Example 1.1.10. (i) The Plücker embedding of $G(1, V)$ simply maps a linear subspace $W := \text{span}(a_1 v_1 + \cdots + a_n v_n)$ to the point $(a_1 : \cdots : a_n) \in \mathbb{P}(\wedge^1 V) = \mathbb{P}(V) = \mathbb{P}^{n-1}$. Hence $G(1, n) = \mathbb{P}^{n-1}$ (as expected).

(ii) Let us suppose that $\dim(V) = 3$ and let us consider the subspace $W := \text{Gen}(v_1 + v_2, v_1 + v_3) \in G(2, V)$. Since

$$(v_1 + v_2) \wedge (v_1 + v_3) = -v_1 \wedge v_2 + v_1 \wedge v_3 + v_2 \wedge v_3$$

the Plücker coordinates of W in $\mathbb{P}(\wedge^2 V) = \mathbb{P}^{\binom{3}{2}-1} = \mathbb{P}^2$ are given by the vector $(-1 : 1 : 1)$. Alternatively, under the preceding notation we have

$$M_W = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $(-1 : 1 : 1)$ keeps the three 2×2 minors of this matrix.

(iii) ([82]) Let us suppose now that $\dim(V) = 4$. Then $\mathbb{P}(V) \xrightarrow{\cong} \mathbb{P}^3$. The Plücker coordinates are the 2×2 minors

$$p_{ij} := \det \begin{pmatrix} x_i & y_i \\ x_j & y_j \end{pmatrix} = x_i y_j - x_j y_i$$

of the matrix

$$M = \begin{pmatrix} x_0 & y_0 \\ x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{pmatrix}$$

whose columns are two distinct point in \mathbb{P}^3 . In this case, the Plücker embedding is defined by

$$\begin{array}{ccc} i : G(2, V) & \rightarrow & \mathbb{P}^5 \\ W & \mapsto & (p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}). \end{array}$$

Let us show that the Plücker coordinates of a line W (in \mathbb{P}^3) satisfies the **quadratic Plücker relation**

$$0 = p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}.$$

This is a direct computation

$$\begin{aligned} & p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12} \\ &= \begin{vmatrix} x_0 & y_0 \\ x_1 & y_1 \end{vmatrix} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_0 & y_0 \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_0 & y_0 \\ x_3 & y_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ &= p_{01} \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} - p_{02} \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + p_{03} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ &= x_0(y_1p_{23} - y_2p_{13} + y_3p_{12}) - y_0(x_1p_{23} - x_2p_{13} + x_3p_{12}) \\ &= x_0 \begin{vmatrix} x_1 & y_1 & y_1 \\ x_2 & y_2 & y_2 \\ x_3 & y_3 & y_3 \end{vmatrix} - y_0 \begin{vmatrix} x_1 & x_1 & y_1 \\ x_2 & x_2 & y_2 \\ x_3 & x_3 & y_3 \end{vmatrix} \\ &= 0. \end{aligned}$$

In particular, if we see $G(2, V)$ as a (closed) subset of \mathbb{P}^5 , we have $G(2, V) \subseteq V(p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12})$. On the other hand, Let us take $(q_{01} : q_{02} : q_{03} : q_{23} : q_{31} : q_{12}) \in \mathbb{P}^5$. Without loss of generality, we can assume that q_{01} is nonzero and the matrix

$$M = \begin{pmatrix} q_{01} & 0 \\ 0 & q_{01} \\ -q_{12} & q_{02} \\ q_{31} & q_{03} \end{pmatrix}$$

has clearly rank 2, so its columns are distinct points defining a line W . The key point now is that, when the coordinates q_{ij} satisfies the quadratic Plücker relation, they are the coordinates of W . To see this, let us first normalize q_{01} to 1. We have therefore $p_{ij} = q_{ij}$, except for

$$p_{23} = -q_{03}q_{12} - q_{02}q_{31}.$$

But, by hypothesis we know that $(q_{01} = 1) q_{23} + q_{02}q_{31} + q_{03}q_{12} = 0$ and then $p_{23} = q_{23}$. This shows that

$$G(2, V) = G(2, 4) = V(p_{01}p_{23} + p_{02}p_{31} + p_{03}p_{12}) \subseteq \mathbb{P}^5$$

is the **Plücker quadratic**.

1.2 Flag varieties

Definition 1.2.1. Let V be a finite dimensional complex vector space. A **flag** in V is a strictly increasing sequence of vector subspaces:

$$\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_l \subsetneq V.$$

The **signature** of the flag is defined to be the sequence $(\dim(V_1), \dots, \dim(V_l))$.

Now, let us take a_1, \dots, a_l, n be sequence of integers such that $0 < a_1 < \dots < a_l < n$. We put

$$\mathbb{F}(a_1, \dots, a_l; n) := \{\{0\} \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_l \subsetneq V \mid \dim(V_i) = a_i\}.$$

Remark 1.2.2. *If $l = 1$, then $\mathbb{F}(a_1; n) = G(a_1, n)$ is a projective variety.*

As in the case of the Grassmannian variety, we want to prove that $\mathbb{F}(a_1, \dots, a_l; n)$ has also the structure of a projective variety.

Proposition 1.2.3. *$\mathbb{F}(a_1, \dots, a_l; n)$ is a Zariski closed subset of $G(a_1, n) \times \dots \times G(a_l, n)$.*

Proof. We already know this for $l = 1$. For any $1 \leq i < j \leq l$, let π_{ij} be the restriction to $\mathbb{F}(a_1, \dots, a_l; n)$ of the projection $G(a_1, n) \times \dots \times G(a_l, n)$. Then

$$\mathbb{F}(a_1, \dots, a_l; n) = \bigcap_{1 \leq i < j \leq l} \pi_{ij}^{-1}(\mathbb{F}(a_i, a_j; n))$$

So, it is enough to prove the proposition for $l = 2$. To do that we will follow the same philosophy followed in 1.1. Let $r < s$ and $(U, W) \in G(r, V) \times G(s, V)$. Let us take $\{u_1, \dots, u_r\}$ and $\{w_1, \dots, w_s\}$ basis of U and W , respectively, and let us define $u := u_1 \wedge \dots \wedge u_r$ and $w = w_1 \wedge \dots \wedge w_s$. As in theorem 1.1, we can consider the map

$$\begin{aligned} \varphi \oplus \varphi := \bigwedge^r V \oplus \bigwedge^s V &\rightarrow \text{Hom}\left(V, \bigwedge^{r+1} V \oplus \bigwedge^{s+1} V\right) \\ (v_1, v_2) &\mapsto \varphi_{v_1} \oplus \varphi_{v_2} := v_1 \wedge (\bullet) \oplus v_2 \wedge (\bullet). \end{aligned}$$

The reasoning given in proposition 1.1.2 tells us that $\ker(\varphi_u \oplus \varphi_w) = U \cap W$ (this is also a consequence of the fact that u and w are totally decomposable). So $\text{rank}(\varphi_u \oplus \varphi_w) = n - \dim(U \cap W) \geq n - r$. this means

$$U \subset W \Leftrightarrow \text{rank}(\varphi_u \oplus \varphi_w) = n - r.$$

This means that the $(n - r + 1) \times (n - r + 1)$ -minors give polynomial conditions for $\mathbb{F}(r, s; n)$. \square

Example 1.2.4. *Let V be a four dimensional complex vector space with basis $\{v_1, \dots, v_4\}$. Let $(U = \text{Gen}(u), W = \text{Gen}(w_1, w_2)) \in \mathbb{F}(1, 2; 4)$. Let us suppose $u = \sum_{i=1}^4 a_i v_i$ and $w_1 \wedge w_2 = \sum_{i < j}^4 b_{ij} v_i \wedge v_j$. So*

$$u \wedge w_1 \wedge w_2 = \sum_{i < j < k}^4 (a_i b_{jk} - a_j b_{ik} + a_k b_{ij}) v_i \wedge v_j \wedge v_k$$

and therefore $U \subset W \Leftrightarrow u \wedge w_1 \wedge w_2 = 0$. So

$$\mathbb{F}(1, 2; 4) \simeq V(X_1 X_{2,3} - X_2 X_{1,3} + X_3 X_{1,2}, \dots) \subset \mathbb{P}^9.$$

Definition 1.2.5. *A flag variety of the form $\mathbb{F}(1, \dots, n-1; n)$ is called a **complete flag variety**.*

Let us consider the group of \mathbb{C} -points $GL_{n, \mathbb{C}}(\mathbb{C})$ of the algebraic group $GL_{n, \mathbb{C}}$ (definition below) and let us fix the standard basis $\{e_1, \dots, e_n\}$ of $V = \mathbb{C}^n$. We have a full flag

$$\mathcal{F} := 0 \subsetneq \mathbb{C}e_1 \subsetneq \dots \subsetneq \mathbb{C}e_1 \oplus \dots \oplus \mathbb{C}e_{n-1} \subsetneq \mathbb{C}^n \in \mathbb{F}(1, \dots, n-1; n) \quad (3)$$

Proposition 1.2.6. $\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})$ acts transitively on complete flag varieties.

Proof. Let $\{U_i\}$ and $\{W_i\}$ in $\mathbb{F}(1, \dots, n-1; n)$. Choosing basis $\{u_i\}$ and $\{w_i\}$ such that for every $1 \leq j \leq n$ the sets $\{u_i\}_{i \leq j}$ and $\{w_i\}_{i \leq j}$ define basis of U_j and W_j , respectively, we can therefore define $A \in \mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})$ by $A \cdot u_i = w_i$. \square

From the preceding proposition, we can conclude that any flag $\mathcal{G} \in \mathbb{F}(1, \dots, n-1; n)$ has the form

$$\mathcal{G} = g \cdot \mathcal{F} := 0 \subsetneq \mathbb{C}g \cdot e_1 \subsetneq \dots \subsetneq \mathbb{C}g \cdot e_1 \oplus \dots \oplus \mathbb{C}g \cdot e_{n-1} \subsetneq \mathbb{C}^n$$

for some $g \in \mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})$. Under this action, the stabilizer $B(\mathbb{C}) := \text{Stab}_{\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})}(\mathcal{F})$ is the (Borel) subgroup of upper triangular matrices in $\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})$. This reasoning tells us that the surjective orbit map

$$\begin{aligned} o_{\mathcal{F}} : \mathbb{G}L_{n,\mathbb{C}}(\mathbb{C}) &\rightarrow \mathbb{F}(1, \dots, n-1; n) \\ g &\mapsto g \cdot \mathcal{F} \end{aligned}$$

descends to the orbit space $\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})/B(\mathbb{C})$ and gives a (geometric) bijection

$$\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})/B(\mathbb{C}) = \mathbb{F}(1, \dots, n-1; n).$$

In particular for $n = 2$, we have

$$\mathbb{G}L_{2,\mathbb{C}}(\mathbb{C}) / \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} = \mathbb{F}(1; 2) = G(1, 2) = \mathbb{P}^1(\mathbb{C}). \quad (4)$$

In what follows we propose to give a sense to the quotient $\mathbb{G}L_{n,\mathbb{C}}(\mathbb{C})/B(\mathbb{C})$ and to prove that this has the structure of a projective variety.

1.3 The functor of points

Let $\mathbf{Sch}_{\mathbb{C}}$ be the category of schemes over \mathbb{C} . We start by recalling to the reader that we can apply the Yoneda embedding [47, III, section 2, first lemma]

$$\mathbf{Sch}_{\mathbb{C}}^{\text{opp}} \rightarrow \mathbf{Fun}(\mathbf{Sch}_{\mathbb{C}}, \mathbf{Set})$$

to regard a \mathbb{C} -scheme X as its functor of points

$$h_X : \mathbb{C}\text{-alg} \rightarrow \mathbf{Set}.$$

We want to determine how we can use this information to think about quotients of algebraic groups. A first essay in how to define quotients (under this perspective) is to define the following functor

$$\begin{aligned} \mathbb{G}L_{n,\mathbb{C}}/\mathbb{B} : \mathbb{C}\text{-Alg} &\rightarrow \mathbf{Set} \\ R &\mapsto \mathbb{G}L_{n,\mathbb{C}}(R)/\mathbb{B}(R), \end{aligned}$$

where \mathbb{B} is the subgroup of $\mathbb{G}L_{n,\mathbb{C}}$ consisting in upper triangular matrices. Let us remark that this (*naive*) definition is independent of \mathbb{C} and we can consider the same objects based over \mathbb{Z} . Under this approach, it is natural to expect (exactly as in (4)) that for every \mathbb{Z} -algebra R we will have

$$\mathbb{G}L_{2,\mathbb{Z}}(R)/\mathbb{B}(R) \xrightarrow{\cong} \mathbb{P}^1(R).$$

Unfortunately, this is not the case. For example, the reader can take $n = 2$ and $R = \mathbb{Z}[\sqrt{-5}]$ to realize that

$$\mathrm{GL}_{n,\mathbb{Z}}(\mathbb{Z}[\sqrt{-5}])/\mathbb{B}(\mathbb{Z}[\sqrt{-5}]) \rightarrow \mathbb{P}^1(\mathbb{Z}[\sqrt{-5}]) \quad (5)$$

is not surjective. This example shows that our naive definition needs some refinement in order to get the expected scheme (i.e. \mathbb{P}^1 regarded as a functor of points).

Definition 1.3.1. *Let V be a complex vector space of finite dimension n . For each sequence of integers $\underline{a} := a_1 < \dots < a_l \leq n$, we define the associated flag variety to be the functor*

$$\begin{aligned} \mathbb{F}(\underline{a}) : \mathbf{C}\text{-Alg} &\rightarrow \mathbf{Set} \\ R &\mapsto \mathbb{F}(\underline{a})(R) \end{aligned}$$

where

$$\begin{aligned} \mathbb{F}(\underline{a})(R) \\ := \{R\text{-submods } V_1 \subsetneq \dots \subsetneq V_l = V \otimes_{\mathbb{C}} R, \text{ each } V_i \text{ an } R\text{-submod. of rank } a_i\}. \end{aligned}$$

If $\underline{a} = \underline{n} = (1, \dots, n-1, n)$ we will say that $\mathbb{F}(\underline{n})$ is the **full flag variety**. We can observe that

$$\begin{aligned} \mathrm{GL}_{n,\mathbb{Z}}(\mathbb{C})/\mathbb{B}(\mathbb{C}) &\rightarrow \mathbb{F}(\underline{n})(\mathbb{C}) \\ g &\mapsto g \cdot \mathcal{F} \end{aligned}$$

is an isomorphism. However, based on our previous example, we can see that this does not hold for general R . The problem arrives from a functorial perspective and is that the naive quotient presheaf

$$\begin{aligned} \mathbf{AffSch}_{\mathbb{C}}^{\mathrm{opp}} &\rightarrow \mathbf{Set} \\ R &\mapsto \mathrm{GL}_{n,\mathbb{Z}}(R)/\mathbb{B}(R) \end{aligned}$$

is not a sheaf.

1.4 Briefly categorical introduction to the sheaf theory

Let us fix for a moment a topological space X . A *presheaf* on X is a functor

$$\mathcal{P} : (\mathbf{Open}_X)^{\mathrm{opp}} \rightarrow \mathbf{Set}$$

and a **sheaf** is a presheaf \mathcal{P} that satisfies a glueing condition. This is, if $\{U_i \rightarrow U\}_{i \in I}$ is covering in \mathbf{Open}_X then the natural map $\mathcal{P}(U) \rightarrow \prod_{i \in I} \mathcal{P}(U_i)$ represents the equalizer of the following diagram

$$\prod_{i \in I} \mathcal{P}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{P}(U_i \times_U U_j).$$

The maps given by the canonical projections $U_i \times_U U_j \rightarrow U_i$ and $U_i \times_U U_j \rightarrow U_j$.

We can define a similar notion of sheaf on the category of topological spaces, \mathbf{Top} . A **presheaf** on \mathbf{Top} is functor

$$\mathcal{P} : \mathbf{Top}^{\mathrm{opp}} \rightarrow \mathbf{Set}$$

and a **sheaf** on \mathbf{Top} is a presheaf \mathcal{P} satisfying glueing conditions with respect to coverings families of open immersions.

Example 1.4.1. For all objects $X \in \mathbf{Top}$, the presheaf

$$h_X : \mathbf{Top}^{\text{opp}} \rightarrow \mathbf{Set} \\ Y \mapsto \text{Hom}(Y, X)$$

is a sheaf. This is called the **representable sheaf**.⁴

If we replace the category \mathbf{Top} with the category $\mathbf{AffSch}_{\mathbb{Z}}$, in this setting a **presheaf** is a functor

$$\mathbf{AffSch}_{\mathbb{Z}} \rightarrow \mathbf{Set}.$$

Example 1.4.2. The functor

$$\mathbf{AffSch}_{\mathbb{Z}} \rightarrow \mathbf{Set} \\ R \mapsto \mathbb{G}L_{n, \mathbb{Z}}(R)/\mathbb{B}(R) \quad (6)$$

is a presheaf.

The key point to replay the definitions on \mathbf{Top} is to decide what kind of *covers* and *glueing conditions* we should impose to get a notion of a sheaf. There is all a gamma of options to do this. For instance, étale topology, fpqc topology, Zariski topology or fppf topology.⁵ We will work with the last one.

In this setting, **covers** are jointly surjective families

$$\bigcup_{i=1}^d \text{Spec}(R_i) \rightarrow \text{Spec}(R)$$

with each $R \rightarrow R_i$ a finitely presented algebra and such that $R_1 \times \cdots \times R_d$ is a faithfully flat R -module (equivalently, and using the hypothesis that the family is already jointly surjective, each R_i is a flat R -module).

Definition 1.4.3. A *sheaf for the fppf topology*⁶ is a presheaf

$$\mathcal{F} : \mathbf{AffSch}_{\mathbb{Z}}^{\text{opp}} \rightarrow \mathbf{Set}$$

satisfying the following properties:

- (i) (*Local condition*) $\mathcal{F}(\prod_{i=1}^d R_i) = \prod_{i=1}^d \mathcal{F}(R_i)$,
- (ii) (*Glueing condition*) For all full faithful flat morphism $R \rightarrow R'$, the canonical map $\mathcal{F}(R') \rightarrow \mathcal{F}(R)$ represents the equalizer in \mathbf{Set} of the diagram

$$\mathcal{F}(R') \rightrightarrows \mathcal{F}(R' \otimes_R R').$$

given that the map (5) is not surjective the presheaf (6) is not a sheaf for the fppf topology. This example suggest that the correct notion of quotients of algebraic groups in our categorical setting is that $\mathbb{G}L_{n, \mathbb{Z}}/\mathbb{B}$ should be the *sheafification* of the presheaf (6). We will show at the end of the section 2 that this approach coincides with the *classical definition* of a quotient space (definition 2.3.2).

⁴We warn to the reader that for a general site \mathcal{C} (in the sense of [76, Sites, definition 6.2]) the contravariant functor h_X is not necessarily a sheaf.

⁵" Fidèlement plat de présentation finie ".

⁶This is called a *faisceau* in [39].

2 Linear algebraic groups

Through this subsection k will always denote an algebraically closed field (the reader can always suppose $k = \mathbb{C}$). In this section we propose to introduce (linear) algebraic groups together with their main properties. References for this section are [14, 35, 75].

2.1 Initial definitions and basic properties

Definition 2.1.1. A **linear algebraic group** is an affine k -variety \mathbb{G} equipped with a group structure such that the group operations

$$\begin{aligned} \mu : \mathbb{G} \times_{\text{Spec}(k)} \mathbb{G} &\rightarrow \mathbb{G} & \iota : \mathbb{G} &\rightarrow \mathbb{G} \\ (g, h) &\mapsto (gh), & g &\mapsto g^{-1} \end{aligned}$$

are morphisms of varieties.

Let us give some examples.

Example 2.1.2. (i) The **additive group** $\mathbb{G}_a = (k, +)$. This is the locus of the zero polynomial and we have $k[\mathbb{G}_a] = k[T]$.

(ii) The **multiplicative group** $\mathbb{G}_m = (k^*, \times)$. As an algebraic variety, we can identify it with

$$\{(x, y) \in k^2 \mid xy = 1\}$$

which is clearly a closed subset of k^2 , being the zero set of the polynomial $T_1 T_2 - 1$. Moreover $k[\mathbb{G}_m] = k[T_1, T_2]/(T_1 T_2 - 1) = k[T, T^{-1}]$ and the componentwise multiplication and the inversion are given by polynomials, so \mathbb{G}_m is a linear algebraic group.

(iii) The **general linear algebraic group** \mathbb{GL}_n is also a linear algebraic group. one way to check this is identifying $M_{n \times n}(k)$ with k^{n^2} , then we can identify \mathbb{GL}_n with

$$\{(A, y) \in k^{n^2} \times k \mid \det(A) \cdot y = 1\} \quad (7)$$

via $A \mapsto (A, \det(A^{-1}))$. Using Laplace expansion and the fact that \det is in particular a polynomial expression, we have that \mathbb{GL}_n is a closed subset of k^{n^2+1} and an algebraic variety. Finally, the expression (7) helps us to calculate the ring of regular functions as follows:

$$\begin{aligned} k[\mathbb{GL}_n] &= k[T_{ij}, Y \mid 1 \leq i, j \leq n]/(\det(T_{ij}) \cdot Y - 1) \\ &= k[T_{ij}, Y \mid 1 \leq i, j \leq n][\det(T_{ij})^{-1}]. \end{aligned}$$

(iv) The **special linear group** $\mathbb{SL}_n := \{A \in \mathbb{GL}_n \mid \det(A) = 1\}$ is a closed subgroup of \mathbb{GL}_n and it is therefore a linear algebraic group. By definition, its ring of regular functions is

$$k[\mathbb{SL}_n] = k[T_{ij}, Y \mid 1 \leq i, j \leq n]/(\det(T_{ij}) - 1).$$

(v) The following are also closed subgroups of \mathbb{GL}_n and therefore linear algebraic groups.

(a) The group of invertible upper triangular matrices

$$\mathbb{B} = \{(a_{ij}) \in \mathbb{GL}_n \mid a_{ij} = 0 \text{ for } i > j\}.$$

(b) The group of upper triangular matrices with 1's on the diagonal

$$\mathbb{U} = \{(a_{ij}) \in \mathbb{GL}_n \mid a_{ij} = 0 \text{ for } i > j \text{ and } a_{ii} = 1\}.$$

(c) The group of invertible diagonal matrix

$$\mathbb{D} = \{(a_{ij}) \in \mathbb{GL}_n \mid a_{ij} = 0 \text{ for } i \neq j\}.$$

In this particular case, we have

$$\begin{aligned} k[\mathbb{D}] &= k[T_1^{\pm 1}, \dots, T_n^{\pm 1}] = k[T_1^{\pm 1}] \otimes_k \dots \otimes_k k[T_n^{\pm 1}] \\ &= k[\mathbb{G}_m] \times_{\text{Spec}(k)} \dots \times_{\text{Spec}(k)} k[\mathbb{G}_m]. \end{aligned}$$

Now, in order to define a category, let us consider the maps between them. To do that, we need to preserve both the geometrical structure and the algebraic structure (or group structure of algebraic groups).

Definition 2.1.3. *Let \mathbb{G}_1 and \mathbb{G}_2 be linear algebraic groups. A map $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ is a morphism of algebraic groups if it is a group homomorphism and a morphism of affine varieties.*

In the category that we just introduce, which we will denote by \mathbf{AlgGps}_k , we would like to recover some important properties of the algebraic structure of the objects. In particular we want this category closed under images and kernels. We need the following results.

Lemma 2.1.4. *Let U_1, U_2 be two dense opens subsets of an algebraic group \mathbb{G} . Then $\mathbb{G} = U \cdot V$.*

Proof. Let us take $x \in \mathbb{G}$. By definition and hypothesis, we know that $\iota^{-1}(U_1) = U_1^{-1}$ and $\mu(x, \iota(U_1)) = xU_1^{-1}$ are also open dense ⁷ subsets of \mathbb{G} . Therefore, given that U_2 is also dense, we must have $xU_1^{-1} \cap U_2 \neq \emptyset$ and $x \in U_1 \cdot U_2$. \square

We recall for the reader that a subset U of a topological space X is called **locally closed** if U is open in \bar{U} . We also recall that U is called **constructible** if it is a finite union of locally closed subsets. Every morphism of algebraic varieties maps constructible sets to constructible sets [76, Morphism of Schemes, section 22, theorem 22.3]. Moreover, if Y is a constructible set of a variety X , then it contains an open dense subset of \bar{X} [76, Topology, constructible sets, lemma 15.15].

Proposition 2.1.5. *Let \mathbb{H} be a subgroup of \mathbb{G} . Then:*

(i) $\overline{\mathbb{H}}$ is a subgroup of \mathbb{G} .

⁷It is clear that $\mu(x, \bullet)$ (multiplication by x) is an homeomorphism.

(ii) If \mathbb{H} is constructible, then $\mathbb{H} = \overline{\mathbb{H}}$.

Proof. Let us start with (i). Topologically speaking, we have the following equalities $\overline{\mathbb{H}^{-1}} = \overline{\mathbb{H}}^{-1} = \overline{\mathbb{H}}$, and (as in the proof of the preceding proposition) for $x \in \mathbb{H}$, we have $x\overline{\mathbb{H}} = \overline{x\mathbb{H}} = \overline{\mathbb{H}}$. Therefore $\mathbb{H} \cdot \overline{\mathbb{H}} \subseteq \overline{\mathbb{H}}$. Moreover, for $x \in \mathbb{H}$, we have $\mathbb{H}x \subseteq \overline{\mathbb{H}}$, and so $\overline{\mathbb{H}x} \subseteq \overline{\mathbb{H}}$. We can conclude that $\overline{\mathbb{H}}$ is closed under inverses and multiplication, so $\overline{\mathbb{H}}$ is a subgroup of \mathbb{G} .

Let us proof now (ii). If \mathbb{H} is constructible, then it contains an open dense subset U of $\overline{\mathbb{H}}$. After (i), we know that $\overline{\mathbb{H}}$ is a linear algebraic group, so by the preceding lemma, we can conclude that $\overline{\mathbb{H}} = U \cdot U \subset \mathbb{H} \cdot \mathbb{H} = \mathbb{H}$. \square

Corollary 2.1.6. *Let $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a morphism of algebraic groups. Then $\ker(\varphi)$ and $\varphi(\mathbb{G}_1)$ are closed and therefore linear algebraic groups.*

Proof. $\ker(\varphi) = \varphi^{-1}(e)$ (identity element of the trivial subgroup of \mathbb{G}_2 which is in particular closed) and $\varphi(\mathbb{G}_1)$ is a constructible subgroup of \mathbb{G}_2 , so it must be closed by the preceding proposition. \square

Let see now how the algebraic structure of linear algebraic groups allows us to study its geometrical structure. More exactly, let us see what the algebraic structure can say about the components (connected and irreducible) of a linear algebraic group. We recall for the reader that an affine variety X can always be written as $\cup_{i=1}^d X_i$, where X_i are maximal irreducible subsets, called **irreducible components** of X [29, Part I, proposition 1.5]. Here irreducible means that X_i can not be written as a union of non-empty closed subsets. It is clear that an irreducible variety is connected and it is a topological fact that a morphism of varieties maps irreducible subsets to irreducible subsets, and if X and Y are irreducible then $X \times Y$ is irreducible.

Proposition 2.1.7. *Let \mathbb{G} be a linear algebraic group.*

- (i) *The irreducible components of \mathbb{G} are pairwise disjoint, and so are the connected components of \mathbb{G} .*
- (ii) *The irreducible component \mathbb{G}° containing the identity is a closed normal subgroup of finite index.*
- (iii) *Any closed subgroup of \mathbb{G} of finite index contains \mathbb{G}° .*

Proof. Let X_1 and X_2 be irreducible components of \mathbb{G} . Let us suppose that $X_1 \cap X_2 \neq \emptyset$ and pick $g \in X_1 \cap X_2$. Given that multiplication by g^{-1} is an isomorphism of varieties, we know that $g^{-1}X_i$ are also irreducible components, and we have $1 \in g^{-1}X_1 \cap g^{-1}X_2$.

Now, given that $X_1 \times X_2$ is irreducible in $\mathbb{G} \times \mathbb{G}$, it follows that $\mu(X_1 \times X_2) = X_1 \cdot X_2$ is irreducible in \mathbb{G} . Moreover, we have $X_1 \subseteq X_1 \times X_2$ since $1 \in X_2$ and therefore, by maximality, we get $X_1 = X_1 \cdot X_2$. Similarly $X_2 = X_1 \cdot X_2 = X_2$.

On the other hand, we already know that $(\mathbb{G}^\circ)^{-1}$ is an irreducible component of \mathbb{G} . As it contains 1, it must be \mathbb{G}° by (i). Similarly we can show that for every $h \in \mathbb{G}^\circ$, we have $h\mathbb{G}^\circ = \mathbb{G}^\circ$. In particular, if $g, h \in \mathbb{G}^\circ$, then $gh \in \mathbb{G}^\circ$ and therefore \mathbb{G}° is a subgroup. Moreover, for every $g \in \mathbb{G}^\circ$, conjugation by g is an isomorphism of varieties, so $g\mathbb{G}g^{-1}$ is again an irreducible component

containing 1, and so it equals \mathbb{G}° . Thus \mathbb{G}° is a normal subgroup. Let us prove $[\mathbb{G}, \mathbb{G}^\circ] < \infty$. To do that, let us consider an irreducible component X of \mathbb{G} and let us pick $g \in X$. As before, we can show that $X = g\mathbb{G}^\circ$ and so all the irreducible components of \mathbb{G} are cosets of \mathbb{G}° , since there are only finitely many irreducible components of \mathbb{G} , \mathbb{G}° must have finite index.

Let us finally prove the third part. Let $\mathbb{H} \subseteq \mathbb{G}$ be a closed subgroup of finite index. Then $\mathbb{H}^\circ \subseteq \mathbb{G}^\circ \subseteq \mathbb{G}$ and we have $[\mathbb{G}, \mathbb{H}^\circ] = [\mathbb{G} : \mathbb{H}][\mathbb{H}, \mathbb{H}^\circ]$, which is finite by (ii). Hence, we can write $\mathbb{G}^\circ = \sqcup g\mathbb{H}^\circ$, a finite disjunction union of cosets of \mathbb{H}° . Since \mathbb{G}° is connected, it follows that $\mathbb{G}^\circ = \mathbb{H}^\circ \leq \mathbb{H}$. \square

Corollary 2.1.8. *Let $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a morphism of linear algebraic groups, then $\phi(\mathbb{G}_1^\circ) = \phi(\mathbb{G}_1)^\circ$.*

Proof. By corollary 2.1.6, we know that $\varphi(\mathbb{G}^\circ)$ is closed, connected, contains 1 and has finite index in $\varphi(\mathbb{G})$. The result follows by (iii) in the previous theorem. \square

It is an easy exercise to see that an affine variety X is irreducible if and only if its ring of regular functions is an integral domain.

Example 2.1.9. The preceding discussion tells us that \mathbb{G}_a , \mathbb{G}_m and in general \mathbb{GL}_n are connected linear algebraic groups. Moreover, \mathbb{D} is connected since it is a direct product of connected algebraic groups (n -copies of \mathbb{G}_m).

It is also true that \mathbb{B} and \mathbb{U} are connected. To show this, we need the following result whose proof is founded in [35, Proposition 7.5].

Proposition 2.1.10. *Let \mathbb{G} be a linear algebraic group and $f_i : Y_i \rightarrow \mathbb{G}$, $i \in I$, a family of morphisms from irreducible varieties, such that $1 \in X_i := \text{im}(f_i)$ for all $i \in I$. Then $H = \langle X_i \mid i \in I \rangle$ is a closed, connected subgroup of \mathbb{G} . In particular, if \mathbb{G} is generated by a family of closed connected subgroups, then it is connected.*

From linear algebra, we know that \mathbf{SL}_n is generated by subgroups

$$H_{ij} = \{(a_{kl}) \in \mathbf{GL}_n \mid a_{kk} = 1 \text{ and } a_{kl} = 0 \text{ for } (k, l) \neq (i, j)\} \quad (i \neq j).$$

Similarly, \mathbb{U} is generated by the subgroups H_{ij} with $i < j$. These subgroups are all isomorphic to \mathbb{G}_a , which is connected. Therefore, the preceding proposition gives us that \mathbf{SL}_n and \mathbb{U} are connected. Similarly, one can show that \mathbb{B} is connected.

Proposition 2.1.11. *Let \mathbb{H}, K be subgroups of a linear algebraic group \mathbb{G} , with K closed and connected. Then $[\mathbb{H}, K]$ is closed and connected.*

Proof. For $h \in \mathbb{H}$, define $\varphi_h : K \rightarrow \mathbb{H}$ by $g \mapsto [h, g]$. This defines a morphism being a composition of multiplication and inversion. Also, for all $h \in \mathbb{H}$, we have $\varphi_h(1) = 1$. Hence

$$[\mathbb{H}, K] = \langle \varphi_h(K) \mid h \in \mathbb{H} \rangle$$

is closed and connected by the preceding proposition. \square

The preceding proposition tells us that closed subgroups are well-behaved under commutators. In particular, if \mathbb{G} is a connected linear algebraic group, then its derived subgroup $\mathbb{G}^{(1)} = [\mathbb{G}, \mathbb{G}]$ is a closed connected subgroup. Inductively, we see that its n -th derived subgroup is a closed and connected subgroup.

Definition 2.1.12. Let \mathbb{G} be a linear algebraic group. We define $\mathbb{G}^{(0)} = \mathbb{G}$ and $\mathbb{G}^{(i)} = [\mathbb{G}^{(i-1)}, \mathbb{G}^{(i-1)}]$ for $i \geq 1$. We obtain the derived series of \mathbb{G} :

$$\mathbb{G} = \mathbb{G}^{(0)} \geq \mathbb{G}^{(1)} \geq \dots$$

We say that \mathbb{G} is **solvable** if $\mathbb{G}^{(d)} = 1$ for some d . The smallest such d is called the **derived length** of \mathbb{G} .

Similarly, we can define $\mathcal{C}^{(0)}\mathbb{G} = \mathbb{G}$ and $\mathcal{C}^i\mathbb{G} = [\mathcal{C}^{i-1}\mathbb{G}, \mathbb{G}]$ for $i \geq 1$. We then define \mathbb{G} to be **nilpotent** if $\mathcal{C}^n\mathbb{G} = 1$ for some n .

Example 2.1.13. Our first example of solvable groups are \mathbb{G}_a , \mathbb{G}_m and \mathbb{D} because they are all abelian. Actually we have $\mathbb{B}^{(1)} = \mathbb{U}$ and

$$\mathcal{C}^{(m)}\mathbb{U} = \{(a_{ij}) \in \mathbb{U} \mid a_{ij} = 0 \text{ for } 0 < i - j \leq m\}$$

and so \mathbb{U} is nilpotent, and thus solvable.⁸

Definition 2.1.14. For an irreducible variety X , we define $\dim(X)$ to be the maximal length of a chain of prime ideals in $k[X]$. In general for a reducible variety X , we define

$$\dim(X) := \max\{\dim(X_i) \mid 1 \leq i \leq d\},$$

where the X_i are the irreducible components of X .

Example 2.1.15. For a linear algebraic group \mathbb{G} , we have $\dim(\mathbb{G}) = \dim(\mathbb{G}^\circ)$. Since the components of \mathbb{G} are the cosets of \mathbb{G}° , which are all isomorphic to \mathbb{G}° .

Proposition 2.1.16. Let $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a morphism of linear algebraic groups. Then

$$\dim \varphi(\mathbb{G}_1) + \dim \ker(\varphi) = \dim(\mathbb{G}_1)$$

Proof. First of all, every fiber $\varphi^{-1}(g)$ is a coset of $\ker(\varphi)$, and thus has the same dimension ($= \dim \ker(\varphi)$). On the other hand, we have

$$\dim(\mathbb{G}_2) = \dim(\mathbb{G}_2)^\circ = \dim(\varphi(\mathbb{G}_1)^\circ) = \dim(\varphi(\mathbb{G}_1^\circ)).$$

All in all, the identity $\dim \varphi^{-1}(g) = \dim(\mathbb{G}_1^\circ) - \dim(\varphi(\mathbb{G}_1^\circ))$ [29, Part III, proposition 9.5] gives us the result. \square

Let us use the previous proposition to calculate the dimension of some algebraic groups.

⁸Clearly $\mathbb{G}^{(i)} \subseteq \mathcal{C}^i\mathbb{G}$, so *nilpotent* implies *solvable*. The opposite is not true! For example, we can consider the group of *nilpotent* matrix

$$\mathbb{N} = \{(a_{ij}) \mid a_{ij} = 0 \text{ for } i \geq j\}.$$

Then $\mathbb{B}^{(n)} = 0$ and $\mathcal{C}^i\mathbb{B} = \mathbb{N}$ for all $i \geq 0$. In particular, \mathbb{B} is solvable but not nilpotent.

Example 2.1.17. (i) $\dim(\mathbb{G}_a) = \dim(\mathbb{G}_m) = 1$, because $k[\mathbb{G}_a] = k[T]$ and $k[\mathbb{G}_m] = k[T][T^{-1}]$. By induction we have $\dim(\mathbb{D}) = n$ for all n .

(ii) Let R be an affine k -algebra.⁹ By [5, Theorem 5.6.7] we have

$$\text{Krull dim}(R) = \text{tr deg}_k \text{Frac}(R).$$

Since the field of fractions of

$$k[T_{ij} \mid 1 \leq i, j \leq n][\det(T_{ij})^{-1}] = k(T_{ij} \mid 1 \leq i, j \leq n)$$

we have

$$\dim(\mathbb{G}L_n) = \text{Krull dim}(k[\mathbb{G}L_n]) = \text{tr deg}_k(k[\mathbb{G}L_n]) = n^2.$$

(iii) Applying the previous examples and corollary 2.1.16 to the surjection

$$\det : \mathbb{G}L_n \rightarrow \mathbb{G}_m.$$

we can see that $\dim(\mathbf{S}L_n) = n^2 - 1$.

2.2 Tori, unipotent and connected solvable groups

Let V be a finite dimensional k -vector space. We call an endomorphism $x \in \text{End}(V)$ **semi-simple** if the roots of its minimal polynomial are all distinct. As k is by assumption algebraically closed, then the preceding definition is equivalently of saying that x is **semi-simple** if and only if x is diagonalizable. An endomorphism x is called **nilpotent** if there exists a positive integer n such that $x^n = 0$. We have the following well known fact called the **additive Jordan decomposition**. The reader can find its proof in [34, Chapter II, 4.3] (cf. proposition 3.8.8 below).

Theorem 2.2.1. *If $\alpha \in \text{End}(V)$, then there exist $x_s, x_n \in \text{End}(V)$ such that x_s is semi-simple, x_n is nilpotent and $\alpha = x_s + x_n$. Moreover, $x_s x_n = x_n x_s$. In particular, x_s and x_n are both polynomials in α with constant coefficient equals to zero.*

Definition 2.2.2. *An endomorphism $u \in \text{End}(V)$ is **unipotent** if $u - 1$ is nilpotent.*

We also have a *multiplicative version* of the Jordan decomposition.

Proposition 2.2.3. *For $g \in GL(V)$, there exists $x_s, x_u \in GL(V)$ such that $g = x_s x_u = x_u x_s$, where x_u is unipotent and x_s semisimple.*

Proof. From the additive decomposition, we can write $g = x_s + x_n$. Since g is invertible, so is x_s and we may define $x_u = 1 + x_s^{-1} x_n$. As x_n is nilpotent and $x_s x_n = x_n x_s$, we have $x_u - 1 = x_s^{-1} x_n$ is nilpotent. Therefore x_u is unipotent and $x_s x_u = x_s + x_n = g$. \square

We can transfer this definition to an arbitrary linear algebraic group [35, Theorem 15.3].

⁹An integral domain which is also a finite-dimensional algebra over k .

Theorem 2.2.4. *Let \mathbb{G} be a linear algebraic group.*

- (i) *For any embedding ι of \mathbb{G} in to some $\mathbb{G}L(V)$ and for any $g \in \mathbb{G}$, there exist unique $g_s, g_u \in \mathbb{G}$ such that $g = g_u g_s = g_s g_u$, where $\iota(g_s)$ is semisimple and $\iota(g_u)$ is unipotent.*
- (ii) *The decomposition $g = g_u g_s$ is independent of the chosen embedding.*
- (iii) *Let $\varphi : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a morphism of linear algebraic groups. Then $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$.*

Given a linear algebraic group \mathbb{G} , each $h \in \mathbb{G}$ gives rise to a morphism $\mathbb{G} \rightarrow \mathbb{G}$ given by $g \mapsto gh$, which induces a $k[\mathbb{G}]$ -algebra homomorphism

$$\begin{aligned} \rho_h : k[\mathbb{G}] &\rightarrow k[\mathbb{G}] \\ f &\mapsto \rho_h(f)(g) := f(gh). \end{aligned} \quad (8)$$

This defined a \mathbb{G} -action on $k[\mathbb{G}]$.

Proposition 2.2.5. *Let \mathbb{G} be a linear algebraic group and V a finite dimensional subspace of $k[\mathbb{G}]$. There exists a finite dimensional \mathbb{G} -invariant subspace X containing V . In particular, $k[\mathbb{G}]$ is a union of finite dimensional subspaces.*

Proof. We can suppose $V = k \cdot v$ is a one-dimensional subspace. By definition, the multiplication morphism $\mu : \mathbb{G} \times_{\text{Spec}(k)} \mathbb{G} \rightarrow \mathbb{G}$ gives rise to a co-morphism $\mu^* = k[\mathbb{G}] \rightarrow k[\mathbb{G} \times_{\text{Spec}(k)} \mathbb{G}] = k[\mathbb{G}] \otimes_k k[\mathbb{G}]$. We can write

$$\mu^*(v) = \sum_{i \in I} f_i \otimes g_i$$

so $\rho_h(v) = \sum_{i \in I} g_i(h) f_i$. Hence the finite dimensional subspace generated by $\{f_i \mid i \in I\}$ contains $\rho_h(v)$ for all $h \in \mathbb{G}$. It follows that the subspace X generated by $\{\rho_h(v) \mid h \in \mathbb{G}\}$ is contained in it and so is finite dimensional. It is clearly \mathbb{G} -invariant and it contains V . \square

Let \mathbb{G} be a linear algebraic group. We write $\mathbb{G}_u := \{g \in \mathbb{G} \mid g \text{ is unipotent}\}$ and $\mathbb{G}_s := \{g \in \mathbb{G} \mid g \text{ is semisimple}\}$ for the subsets of unipotent and semisimple elements of \mathbb{G} . If $\mathbb{G} = \mathbb{G}_u$ we then say that \mathbb{G} is a *unipotent group*.

Remark 2.2.6. \mathbb{G}_u is a closed subset since the set of unipotent polynomials elements of $\mathbb{G}L_n$ is closed, given by the polynomial $(T - 1)^n$.

Example 2.2.7. By definition, we have \mathbb{U} is unipotent and more generally subgroups of \mathbb{U} turn out to be unipotent subgroups of $\mathbb{G}L_n$. Moreover, for $\mathbb{G} = \mathbb{B}$ we have $\mathbb{G}_u = \mathbb{U}$. We can also note that $\mathbb{G}_a \simeq \mathbb{U} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ is a unipotent group. Of course, the unicity in the Jordan decomposition tells us that $\mathbb{G} = \mathbb{G}_s$ for $\mathbb{G} = \mathbb{G}_m$ or more generally for \mathbb{D} .

Theorem 2.2.8. *Let \mathbb{G} be a unipotent subgroup of $\mathbb{G}L(V)$ for some non-zero finite dimensional vector space V . Then \mathbb{G} has a common eigenvector in V .*

Proof. Let us identify V with k^n where $n = \dim(V)$. We use induction on n . The result is clear if $n = 1$, so let us assume $\dim(V) > 1$. Suppose that V has a proper non-zero subspace W stable under \mathbb{G} . Then by choosing appropriate bases, we may assume that

$$\mathbb{G} \leq \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

More specifically, every element $g \in \mathbb{G}$ can be written in the form

$$\begin{pmatrix} \phi(g) & * \\ 0 & \psi(g) \end{pmatrix}$$

where $\phi : \mathbb{G} \rightarrow \mathbb{G}L(W)$ is the canonical, and $\psi(g) : \mathbb{G} \rightarrow \mathbb{G}L(V/W)$. Now $\psi(\mathbb{G})$ is also unipotent, so by induction hypothesis there exists a common eigenvector $v \in W \subset V$ for \mathbb{G} . Therefore we may assume that V is an irreducible \mathbb{G} -module. We need the following theorem of Burnside: if R is a subalgebra of $\text{End}(V)$ which acts irreducibly on V , then $R = \text{End}(V)$. Now, the assumption that \mathbb{G} is unipotent implies $\text{Tr}(x) = \text{Tr}(1) = \dim(V)$ for all $x \in \mathbb{G}$. Writing x as $1 + n$ with n nilpotent, we have for all $y \in \mathbb{G}$

$$\text{Tr}(y) = \text{Tr}(xy) = \text{Tr}(y + ny) = \text{Tr}(y) + \text{Tr}(ny).$$

Therefore $\text{Tr}(ny) = 0$. Now, the k -linear combinations of the elements of \mathbb{G} must also satisfy this. These form a subalgebra R of $\text{End}(V)$, which acts irreducibly on V since \mathbb{G} does. Burnside's theorem then implies that for all $y \in \text{End}(V)$ and for all $x = 1 + n \in \mathbb{G}$, $\text{Tr}(ny) = 0$. Taking y to be the standard unit matrices E_{ij} , we see that we must have $n = 0$. Hence $\mathbb{G} = 1$ and since V is irreducible, $\dim(V) = 1$, a contradiction. \square

Corollary 2.2.9. *If $\mathbb{G} \leq \mathbb{G}L_n$ is a unipotent subgroup, then \mathbb{G} is conjugate to a subgroup \mathbb{U} . Since \mathbb{U} is nilpotent, it follows that \mathbb{G} is nilpotent and therefore soluble.*

Proof. The preceding theorem gives us a common eigenvector $v \in V = k^n$. Let $V_1 = k \cdot v$. Then \mathbb{G} acts on V/V_1 and the image of \mathbb{G} in $\mathbb{G}L(V/V_1)$ is clearly unipotent. Induction on $\dim(V)$ allows us to construct a basis V with respect to which elements of \mathbb{G} are represented by upper triangular matrices. Since they are also unipotent, it follows that these matrices are in \mathbb{U} . \square

2.2.1 Connected solvable groups

We wish to establish a similar result to theorem 2.2.8 for connected solvable groups. This is analogue of Lie's theorem for Lie algebras, except that Lie's theorem only holds in characteristic zero while we don't make any assumption on $\text{char}(k)$.

Lemma 2.2.10. *Let $M \subset \mathbb{G}L_n$ be a commuting set of matrices. Then M is triangonalisable¹⁰.*

Proof. The reader can find the proof of the lemma in [35, Proposition 15.4]. \square

¹⁰We can find a basis with respect to which all elements of M are represented by upper triangular matrices.

Theorem 2.2.11 (Lie-Kolchin). *Let \mathbb{G} be a connected solvable group of $\mathbb{G}L(V)$, with $V \neq 0$ finite dimensional. Then \mathbb{G} has a common eigenvector in V , i.e. V has a \mathbb{G} -stable one-dimensional subspace.*

Proof. Let us start by remarking that if \mathbb{G} is solvable then its closure $\overline{\mathbb{G}}$ is also solvable. So we can suppose that \mathbb{G} is closed. We will argue by induction on $\dim(V)$ and on the derived length d of \mathbb{G} .

If $n = 1$ the result is trivial. Let us suppose $n > 1$ and $d = 1$. By definition \mathbb{G} is commutative and the result follows from the preceding lemma. Let us assume $d > 1$ and let us suppose that there is a proper subspace $W \subsetneq V$ which is stabilised by \mathbb{G} . By choosing an appropriate basis, we may write any $g \in \mathbb{G}$ of the form

$$\begin{pmatrix} \varphi(g) & * \\ 0 & \psi(g) \end{pmatrix}$$

where $\varphi : \mathbb{G} \rightarrow \mathbb{G}L(W)$ is the canonical restriction morphism and $\psi : \mathbb{G} \rightarrow \mathbb{G}L(V/W)$. Now, $\varphi(\mathbb{G})$ is connected and solvable and acts on W . Given that $\dim(W) < \dim(V)$, hypothesis induction gives us $v \in W$ such that v is a common eigenvector of \mathbb{G} . Now, we only need to consider the case when \mathbb{G} acts irreducibly on V .

Let $\mathbb{G}^{(1)} = [\mathbb{G}, \mathbb{G}]$. This is closed and connected by proposition 2.1.10 and obviously solvable with derived length $d - 1$. By induction hypothesis there is a common eigenvector $v \in V$ for $\mathbb{G}^{(1)}$. Since $\mathbb{G}^{(1)}$ is a normal subgroup of \mathbb{G} , we have that gv is also a common eigenvector of $\mathbb{G}^{(1)}$. Let W be the non-zero subspace of V spanned by the common eigenvectors of $\mathbb{G}^{(1)}$. By the above W is \mathbb{G} -invariant. Since \mathbb{G} acts irreducibly on V , it follows that $W = V$. Hence V has a basis consisting of common eigenvectors of $\mathbb{G}^{(1)}$. In particular, the elements of $\mathbb{G}^{(1)}$ are diagonal matrices with respect to this basis, and so $\mathbb{G}^{(1)}$ is commutative.

Now, for fixed $h \in \mathbb{G}^{(1)}$, all conjugates ghg^{-1} , for $g \in \mathbb{G}$ are in $\mathbb{G}^{(1)}$, and hence diagonal with the same eigenvalues as h . This implies that we only have finitely many possibilities for ghg^{-1} . In particular, the image of the morphism $g \mapsto ghg^{-1}$ is finite by the above discussion, and connected since \mathbb{G} is connected since \mathbb{G} is connected. We must have $\mathbb{G}h\mathbb{G}^{-1} = h$, i.e. $h \in Z(\mathbb{G})$. So $\mathbb{G}^{(1)} \subseteq Z(\mathbb{G})$. Now, every element of $Z(\mathbb{G})$ commutes with \mathbb{G} in this action on V . By Schur's lemma they are represented by scalar multiples of the identity. This implies that elements of $\mathbb{G}^{(1)}$ have determinant 1. Therefore there are only many possibilities for the element of $\mathbb{G}^{(1)}$. Given that $\mathbb{G}^{(1)}$ is connected it follows that $\mathbb{G}^{(1)} = 1$ and so \mathbb{G} is commutative, contradicting $d > 1$. \square

An analogous reasoning to the one given in the proof of corollary 2.2.9, but using the preceding theorem, gives us the

Corollary 2.2.12. *Let \mathbb{G} be a connected, solvable subgroup of $\mathbb{G}L_n$. Then \mathbb{G} is conjugated to a subgroup of \mathbb{B} .*

2.2.2 Borel subgroups

Let us study now connected solvable subgroups of a linear algebraic group \mathbb{G} .

Definition 2.2.13. A subgroup $\mathbb{B} \subseteq \mathbb{G}$ is called a **Borel subgroup** if it is a maximal closed, connected, solvable subgroup.

Example 2.2.14. From corollary 2.2.12, we see that a connected solvable subgroup of $\mathbb{G}L_n$ is conjugated to a subgroup of \mathbb{B} . Along this section we have showed that \mathbb{B} is a closed connected solvable subgroup of $\mathbb{G}L_n$. In particular, this implies that \mathbb{B} is a Borel subgroup of $\mathbb{G}L_n$. Moreover, if \mathbb{B} is a Borel subgroup of $\mathbb{G}L_n$, then it is conjugated to a subgroup of \mathbb{B} . By maximality, this conjugate must be equal to the whole \mathbb{B} . Thus all Borel subgroups of $\mathbb{G}L_n$ are conjugated.

2.2.3 Actions of algebraic groups

Definition 2.2.15. Let \mathbb{G} be a linear algebraic group, and let X be a variety. We say that X is a \mathbb{G} -space if there is a group action

$$\mathbb{G} \times_{\text{Spec}(k)} X \rightarrow X$$

of \mathbb{G} on X which is also a morphism of varieties. If the action of \mathbb{G} on X is transitive, X is said to be **homogeneous**.

Example 2.2.16. (i) \mathbb{G} is itself a \mathbb{G} -space via conjugation.

(ii) Let V be a finite dimensional k -vector space. A **rational representation** of \mathbb{G} is a morphism $\varphi : \mathbb{G} \rightarrow \mathbb{G}L(V)$. In this situation V is a \mathbb{G} -space via the action

$$\begin{aligned} \mathbb{G} \times_{\text{Spec}(k)} V = \mathbb{A}_k^{\dim(V)} &\rightarrow \mathbb{A}_k^{\dim(V)} \\ (g, v) &\mapsto \varphi(g)v \end{aligned}$$

(iii) The associated projective space $\mathbb{P}(V)$ is also a \mathbb{G} -space via the action

$$\begin{aligned} \mathbb{G} \times_{\text{Spec}(k)} \mathbb{P}(V) &\rightarrow \mathbb{P}(V) \\ (g, \langle v \rangle) &\mapsto \langle \varphi(g)v \rangle \end{aligned}$$

Proposition 2.2.17. Let X be a \mathbb{G} -space. For every $x \in X$ the stabiliser

$$\mathbb{G}_x := \{g \in \mathbb{G} \mid g \cdot x = x\}$$

is a closed subgroup of \mathbb{G} .

Proof. Be definition, the map

$$\begin{aligned} \varphi_x : \mathbb{G} &\rightarrow X \\ g &\mapsto g \cdot x \end{aligned}$$

is a morphism of varieties and so $\mathbb{G}_x = \varphi_x^{-1}(\{x\})$ is closed. \square

Proposition 2.2.18. Let X be a \mathbb{G} -space.

(i) Every orbit $\mathbb{G} \cdot x$ is open in its closure.

(ii) *Orbits of minimal dimension are closed.*

Proof. By definition, the orbit map

$$\begin{aligned} o_x : \mathbb{G} &\rightarrow X \\ g &\mapsto g \cdot x \end{aligned}$$

is a morphism of varieties with image $\mathbb{G} \cdot x$, which is constructible and therefore, it contains an open dense subset Y of $\overline{\mathbb{G} \cdot x}$. Given that

$$\mathbb{G} \cdot x = \bigcup_{g \in \mathbb{G}} g \cdot Y$$

the result (i) follows.

On the other hand, for every $x \in X$ and $g \in \mathbb{G}$ we have that $g \cdot \overline{\mathbb{G} \cdot x}$ is closed and contains $\mathbb{G} \cdot x$. Therefore $\overline{\mathbb{G} \cdot x} \subseteq g \cdot \overline{\mathbb{G} \cdot x}$. Similarly $\overline{\mathbb{G} \cdot x} \subseteq g^{-1} \cdot \overline{\mathbb{G} \cdot x}$. Multiplying by g , we see that $\overline{\mathbb{G} \cdot x} = g \cdot \overline{\mathbb{G} \cdot x}$. As g was arbitrary, we can conclude that

$$\overline{\mathbb{G} \cdot x} = \bigcup_{g \in \mathbb{G}} g \cdot \overline{\mathbb{G} \cdot x}.$$

Let us take $x \in X$, such that $\dim(\mathbb{G} \cdot x)$ is minimal and let us suppose that $\mathbb{G} \cdot x$ is not closed. Then

$$\overline{\mathbb{G} \cdot x} \setminus \mathbb{G} \cdot x = \bigcup_{h \in \overline{\mathbb{G} \cdot x} \setminus \mathbb{G} \cdot x} \mathbb{G} \cdot h$$

and for every $h \in \overline{\mathbb{G} \cdot x} \setminus \mathbb{G} \cdot x$, we know by (i) that $\mathbb{G} \cdot h$ is closed in $\overline{\mathbb{G} \cdot x}$. But, if $Y \subseteq \overline{\mathbb{G} \cdot x}$ is an irreducible component intersecting $\mathbb{G} \cdot x$, then

$$(\overline{\mathbb{G} \cdot x} \setminus \mathbb{G} \cdot x) \cap Y$$

is a proper closed subset of Y , thus of strictly smaller dimension. The result follows from the definition of dimension¹¹. \square

Theorem 2.2.19 (Chevalley). *Let \mathbb{H} be a closed subgroup. There exists a rational representation $\varphi : \mathbb{G} \rightarrow \mathbb{G}L(V)$ and a one-dimensional subspace $W \subseteq V$ such that*

$$H = \{g \in \mathbb{G} \mid \varphi(g)W = W\}.$$

¹¹**Auxiliary Proposition:** Let X be a variety, U a dense open subset and $Z = X \setminus U$. Then $\dim(Z) < \dim(X)$.

Proof. Let X_1, \dots, X_r be the irreducible components of X and let Z_1, \dots, Z_s be the irreducible component of Z . Since U is dense in X , it has non-empty intersection with each X_i . Moreover, every Z_i is contained in one of the maximal irreducible subset of X , say X_j . We have $Z_i \subsetneq X_j$ (because $U \cap X_j \neq \emptyset$). Given that a proper, closed, irreducible subset of an irreducible variety must be of strictly lower dimension, we can conclude that every component Z_1, \dots, Z_s has strictly less dimension than that of some irreducible component of X . \square

Now to end the proof of the proposition we may take $X = \overline{\mathbb{G} \cdot x}$ and $U = \mathbb{G} \cdot x$.

Proof. Let $I \subseteq k[\mathbb{G}]$ be the ideal of \mathbb{H} . Given that $k[\mathbb{G}]$ is Noetherian, this ideal is finitely generated, say $I = (F_1 \cdots, F_s)$. By proposition 2.2.5, there exists a finite dimensional subspace F of $k[\mathbb{G}]$ containing the F_j and a corresponding morphism

$$\rho : \mathbb{G} \rightarrow \mathbb{G}L(F)$$

and $E := F \cap I$ is \mathbb{H} -invariant.

Now, if $x \in \mathbb{G}$ is such that $\rho_x(E) = E$, then we have

$$\rho(I) = \rho_x(E)\rho_x(k[\mathbb{G}]) = Ek[\mathbb{G}] = I.$$

In particular, all the functions in I vanish at x , and thus $x \in \mathbb{H}$. Therefore we have

$$\mathbb{H} = \{x \in \mathbb{G} \mid \rho_x(E) = E\}.$$

Finally, If $d = \dim(E)$, we set $V := \bigwedge^d F$, with $\varphi : \mathbb{G} \rightarrow \mathbb{G}L(V)$ the rational representation induced by the natural \mathbb{G} -action. The one-dimensional subspace $W = \bigwedge^d E$ is $\varphi(\mathbb{H})$ -invariant by construction. Now, assume that $\varphi(g)W = W$ for some $g \in \mathbb{G}$. Let w_1, \dots, w_d be a basis for E , and v_1, \dots, v_d be a basis for $\rho_g(E)$. By assumption, we have $\varphi(g)(w_1 \wedge \cdots \wedge w_d) \in W$, but by hypothesis it must be a multiple of $v_1 \wedge \cdots \wedge v_d$. It follows that each $v_i \in E$ and so $\rho_g(E) = E$. \square

Corollary 2.2.20. *Any linear algebraic group can be embedded as a closed subgroup of $\mathbb{G}L_n$ for some n .*

Proof. Let us take $\mathbb{H} = \{1\}$. By the preceding theorem, we get a rational representation $\rho : \mathbb{G} \rightarrow \mathbb{G}L(V) \simeq \mathbb{G}L_{\dim(V)}$. Since $\ker(\rho) \subseteq \mathbb{H}$, we can conclude that ρ is injective. \square

2.3 Topological properties of homogeneous spaces

In section 3 we will give detailed definitions of the tangent space. In this part of the notes we will give quickly review the definition and some important properties. The reader can accept for a moment the facts presented here or he can take a look to the section cited (cf. [75, Chapter 4].)

Quotient Space 2.3.1. Now let us give the structure of variety to a set of cosets \mathbb{G}/\mathbb{H} . Let $\mathbb{H} \subset \mathbb{G}$ be a closed subgroup and V as in the previous theorem. Let $\bar{v} \in \mathbb{P}(V)$ be the point corresponding to the line $\langle v \rangle \subseteq V$ stabilised by \mathbb{H} . Set

$$X := \mathbb{G} \cdot \bar{v} \subseteq \mathbb{P}(V),$$

then X is clearly a homogeneous \mathbb{G} -space endowed with a canonical surjective map

$$\begin{aligned} \varphi : \mathbb{G} &\rightarrow X \\ g &\mapsto g \cdot \bar{v} \end{aligned} \tag{9}$$

with fibers the coset of ¹² \mathbb{H} . This induces a bijection

$$\bar{\varphi} : \mathbb{G}/\mathbb{H} \rightarrow X.$$

Using this bijection we can endow \mathbb{G}/\mathbb{H} with a structure of variety. Indeed, given the that $\bar{X} \subseteq \mathbb{P}(V)$ is a closed subset, it is a projective variety. Also, as X is an orbit in a \mathbb{G} -space, then it is open in its closure and therefore a quasi-projective variety.

By construction, the natural map $\pi : \mathbb{G} \rightarrow \mathbb{G}/\mathbb{H}$ is a morphism of varieties.

We have the following notions from [14, Section 6]. Given X and Y k -algebraic varieties, we will say that a morphism (of k -algebraic varieties) $\psi : X \rightarrow Y$ is a **quotient morphism** if

- (i) ψ is surjective and open.
- (ii) If $U \subseteq X$ is open, then the co-morphism ψ^\sharp induces an isomorphism from $\mathcal{O}_Y(\psi(U))$ onto the set of $f \in \mathcal{O}_X(U)$ which are constant on the fibers of $\psi|_U$.

Let us suppose that X is a \mathbb{G} -space in the sense of the definition 2.2.15.

Definition 2.3.2. We say that a morphism of k -algebraic varieties $\psi : X \rightarrow Y$ is an **orbit map** if ψ is a surjective morphism such that the fibers of ψ are the orbits of \mathbb{G} in X . A **quotient** of X by \mathbb{G} is an orbit map which is a quotient morphism.

Universal mapping property 2.3.3. If $\pi : X \rightarrow Z$ is any morphism of k -algebraic varieties which is constant on the fibers of \mathbb{G} , then there is a unique morphism $\pi' : Y \rightarrow Z$ making commutative the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & Y \\ \downarrow \pi & \swarrow \pi' & \\ Z & & \end{array} \quad (10)$$

Remark 2.3.4. It is possible to show that the morphism φ in (9) is affine and that the topology on \mathbb{G}/\mathbb{H} is the quotient topology obtained from φ [35, Theorem 5.5.5]. Moreover, if $\pi : \mathbb{G} \rightarrow Z$ is a morphism of algebraic varieties which is constant on the \mathbb{H} orbits. Then $\pi'(g\mathbb{H}) = \pi(g)$ is well-defined and makes commutative the diagram (10). By [14, Proposition 6.7], to prove that X is the quotient of \mathbb{G} by \mathbb{H} we need to prove that φ is separable (in the sense of [14, Section 8]). This is a property that in our case depends of the Lie algebras ¹³ of \mathbb{G} and \mathbb{H} , so we will end this discussion in section 3.

¹²Let $x = \tilde{g} \cdot \bar{v} \in X$. Then

$$g \in \varphi^{-1}(x) \Leftrightarrow (\tilde{g}^{-1}g) \cdot \bar{v} = \bar{v} \Leftrightarrow (\tilde{g}^{-1}g) \cdot \langle v \rangle = \langle v \rangle \Leftrightarrow \tilde{g}^{-1}g \in \mathbb{H} \Leftrightarrow g \in \tilde{g}\mathbb{H}$$

This proves that $\varphi^{-1}(x) = \tilde{g}\mathbb{H}$.

¹³If \mathfrak{g} and \mathfrak{h} denote the Lie algebras of \mathbb{G} and \mathbb{H} , respectively. Then the *tangent map* $T_e(\varphi)$ will be the natural surjection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$. This surjectivity property implies that X is the quotient of \mathbb{G} by \mathbb{H} [14, Proposition 6.7 (ii)].

We end this section with the following discussion (cf. [21]). At the end of subsection 1.3 we have defined the **geometric quotient** $(\mathbb{G}/\mathbb{H})^\dagger$ as the fppf-sheafification of the presheaf

$$\begin{aligned} (\mathbf{Sch}/\mathrm{Spec}(k))_{\mathrm{fppf}} &\rightarrow \mathbf{Set} \\ S &\mapsto \mathbb{G}(S)/\mathbb{H}(S). \end{aligned}$$

By theorem A.2.8, we also know that $(\mathbb{G}/\mathbb{H})^\dagger$ is the algebraic space associated to the étale equivalence relation

$$\begin{aligned} R := \mathbb{G} \times_{\mathrm{Spec}(k)} \mathbb{H} &\rightarrow \mathbb{G} \times_{\mathrm{Spec}(k)} \mathbb{G} \\ (g, h) &\mapsto (g, gh). \end{aligned} \tag{11}$$

In this case, we have a presentation $\pi : h_{\mathbb{G}} \rightarrow (\mathbb{G}/\mathbb{H})^\dagger$. From a **schematic** point of view, the **quotient** (if it exists!) will represent the functor $(\mathbb{G}/\mathbb{H})^\dagger$. In other words, the geometric quotient lies in the image of the Yoneda's embedding¹⁴

$$\begin{aligned} (\mathbf{Sch}/\mathrm{Spec}(k))_{\mathrm{fppf}} &\rightarrow \mathcal{AS}/\mathrm{Spec}(k) \\ \mathbb{G} \rightarrow Q &\mapsto h_{\mathbb{G}} \rightarrow (\mathbb{G}/\mathbb{H})^\dagger. \end{aligned}$$

We want to prove that the **classical quotient** in remark 2.3.4 represents the sheafification of the presheaf (11), as well, and via the Yoneda's embedding, this will be the **(geometric)** quotient in the sense of [24, Definition 3.1.2].

In section 3 we will show that the tangent map at the identity $T_e\varphi$ of the map (9) is surjective. Via $\mathbb{G}(k)$ -translation we see that $T_g\varphi$ is surjective for all $g \in \mathbb{G}(k)$ and therefore by [17, Chapter 2, section 2, proposition 8] φ is a smooth morphism, in particular φ is an fppf-morphism and therefore q represents the (geometric) quotient of \mathbb{G} module the flat equivalence relation (cf. [76, Part 2: Groupoids schemes, commentary just before lemma 39.20.3]).

$$R' := \mathbb{G} \times_{\mathbb{G}/\mathbb{H}} \mathbb{G} \rightarrow \mathbb{G} \times \mathbb{G}$$

In order to match with the scheme-theoretic notion, we need to prove that $R = R'$ as subschemes of $\mathbb{G} \times \mathbb{G}$. By smoothness it suffices to prove that $R(k) = R'(k)$. Since $\mathbb{G}(k)/\mathbb{H}(k) \rightarrow (\mathbb{G}/\mathbb{H})^\dagger(k)$ is a bijection [23, Chapter 3, section 1, 1.15], the desired equality follows.

A Algebraic spaces

As we have seen in sections 1 and 2, there are two possibilities to construct algebraic quotients in the category of k -schemes \mathbf{Sch}_k , and at the end of section 2 we have proved that they are actually the same by using the fact that the *algebraic space* defined by an étale equivalence relation is representable. In this appendix we propose to give an overview of the theory of algebraic spaces. We essentially follow word by word the results exhibited in [76, Part 4: Algebraic spaces].

¹⁴ $(\mathbf{Sch}/\mathrm{Spec}(k))_{\mathrm{fppf}}$ and $\mathcal{AS}/\mathrm{Spec}(k)$ are the categories of k -schemes with fppf-coverings and of algebraic spaces, respectively. Notations are given in the appendix of this section.

A.1 General properties

Let S be an affine scheme. In what follows, we will always denote by $(\mathbf{Sch}/S)_{\text{fppf}}$ the *big fppf site* of S in the sense of [76, Topologies on schemes, definition 7.8]. The underlying category have fibre products and final object S .

Remark A.1.1. *If $T \in \text{Ob}(\mathbf{Sch}/S)_{\text{fppf}}$, then the representable presheaf*

$$h_T : (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \mathbf{Set}$$

is a sheaf ([76, Algebraic spaces, remark before section 3]).

Let us suppose now that $F, G : (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \mathbf{Set}$ are functors and $\Phi : F \rightarrow G$ is a transformation of functors. We will say that Φ is **representable** if for every object $T \in (\mathbf{Sch}/S)_{\text{fppf}}$ and any $a \in G(T)$ the fiber product $h_T \times_{a,G} F$ (in the sense of [76, Chapter 4: categories, section 8]) is representable. In this case, if V_a represents the previous fiber product, i.e. there exists an isomorphism $h_{V_a} \rightarrow h_T \times_{a,G} F$, then by Yoneda's embedding, the projection $h_T \rightarrow h_T \times_{a,G} F \rightarrow h_{V_a}$ is represented by a unique morphism of schemes

$$\xi_a : T \rightarrow V_a. \tag{12}$$

A.2 Algebraic spaces and quotient sheaves

Definition A.2.1. *An algebraic space over S is presheaf*

$$\mathfrak{X} : (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \mathbf{Set}$$

with the following properties

- (i) *The presheaf \mathfrak{X} is a sheaf.*
- (ii) *The diagonal morphism $\mathfrak{X} \rightarrow \mathfrak{X} \times \mathfrak{X}$ is representable.*
- (iii) *There exists a scheme $T \in \text{Ob}(\mathbf{Sch}/S)_{\text{fppf}}$ and a map $h_T \rightarrow \mathfrak{X}$ which is surjective, and *étale* (definition?).*

As the reader probably expects, our first example of an algebraic space is the representable sheaf h_T for $T \in \text{Ob}((\mathbf{Sch}/S)_{\text{fppf}})$. In fact, given that $(\mathbf{Sch}/S)_{\text{fppf}}$ has fiber products, the diagonal morphism $h_T \rightarrow h_T \times_T h_T$ is clearly representable, and the identity map $h_T \rightarrow h_T$ is surjective étale.

Definition A.2.2. *We will denote by \mathcal{AS}/S the category of algebraic spaces. A morphism in \mathcal{AS}/S is a transformation of functors.*

Remark A.2.3. *As a consequence of the Yoneda's embedding we have that the category $(\mathbf{Sch}_S)_{\text{fppf}}$ is contained in the category of algebraic spaces \mathcal{AS}/S . The embedding is defined via representable sheaves $T \mapsto h_T$.*

Let us review now how to glue algebraic spaces (cf. [76, Part 4: Algebraic spaces, lemma 8.3]). We will denote by $\mathcal{S}(\mathbf{Sch}/S)_{\text{fppf}}$ the category of sheaves in the sense of 1.4. Moreover, if $\Phi : F \rightarrow G$ is a representable morphism of sheaves and \mathcal{P} is a property of morphism of functors, such that \mathcal{P} is stable under base change

and it is fppf local on the base,¹⁵ then we will say that Φ **has the property** \mathcal{P} if the morphism (1.4) has the property \mathcal{P} , for all $T \in \text{Ob}(\mathbf{Sch}/S)_{\text{fppf}}$ and $a \in G(T)$. Finally, we recall for the reader that the *coproduct* $F \sqcup G$ is the sheafification of the coproduct sheaf (cf. [76, Part I: Sites, Lemma 10.13]).

Lemma A.2.4. *Let $\mathfrak{X} : (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \mathbf{Set}$ be a presheaf. We assume*

- (1) $\mathfrak{X} \in \mathcal{S}(\mathbf{Sch}/S)_{\text{fppf}}$.
- (2) *There exists an index set I and subfunctors $\mathfrak{X}_i \subset \mathfrak{X}$ such that*
 - (a) *each $\mathfrak{X}_i \in \mathcal{AS}/S$,*
 - (b) *each $\mathfrak{X}_i \rightarrow \mathfrak{X}$ is representable,*
 - (c) *each $\mathfrak{X}_i \rightarrow \mathfrak{X}$ is an open immersion,*
 - (d) *the map $\mathfrak{X}_i \rightarrow \mathfrak{X}$ is surjective in $\mathcal{S}(\mathbf{Sch}/S)_{\text{fppf}}$, and*
 - (e) $\sqcup \mathfrak{X}_i$ *is an algebraic space.*

Then \mathfrak{X} is an algebraic space.

Let us find now a **presentation** of an algebraic space. Let $T, R \in \mathbf{Sch}/S$. We will say that a morphism of schemes $j : R \rightarrow T \times_S T$ is an **étale equivalence relation** if j is an equivalence relation and $s, t : R \rightarrow T \times_S T \rightrightarrows T$ are étale morphisms of schemes. We warn to the reader that in what follows we will abuse of the notation and we will say that R is an (étale) equivalence relation.

Definition A.2.5. *Let \mathfrak{X} be an algebraic space. A **presentation** of \mathfrak{X} is given by a scheme $T \in \mathbf{Sch}/S$ and an étale equivalence relation R on T , and a surjective étale morphism $h_T \rightarrow \mathfrak{X}$ such that $h_R \simeq h_T \times_{\mathfrak{X}} h_T$.*

As we have remarked, the representable presheaves $h_T, h_R : (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} \rightarrow \mathbf{Set}$ are sheaves, and the morphism j induces a morphism of sheaves $h_j : h_R \rightarrow h_T \times h_T$. In particular, for every object $U \in \text{Ob}((\mathbf{Sch}/S)_{\text{fppf}})$ we have an equivalence relation $(\bullet)h_R(U)(\bullet)$ induced by $h_j(U) : h_R(U) \rightarrow h_T(U) \times h_T(U)$. We get a presheaf

$$\begin{array}{ccc} (\mathbf{Sch}/S)_{\text{fppf}}^{\text{opp}} & \rightarrow & \mathbf{Set} \\ U & \mapsto & h_T(U)/(\bullet)h_R(U)(\bullet). \end{array} \quad (13)$$

The étale sheafification of (13) will be called the **quotient sheaf** and denoted by T/R .

Remark A.2.6. *Let k be an algebraic closed field and $S = \text{Spec}(k)$. Let X be a smooth algebraic variety and \mathbb{G} a linear algebraic group acting on the left on X . By smoothness, the morphism*

$$\begin{array}{ccc} (s, t) : \mathbb{G} \times_{\text{Spec}(k)} X & \rightarrow & X \times_{\text{Spec}(k)} X \\ (g, x) & \mapsto & (x, g \cdot x) \end{array}$$

defines an étale equivalence relation. The quotient sheaf will be denoted by X/\mathbb{G} .

¹⁵Let $f : X \rightarrow Y$ be a morphism of S -schemes and $\{Y_i \rightarrow Y\}$ an étale covering of Y . We say that \mathcal{P} is **fppf local on the base** if we have: $f \in \mathcal{P}$ if, and only if, $Y_i \times_X Y \rightarrow Y_i \in \mathcal{P}$ (cf. [76, Part 2: Descent, definition 35.19.1]).

We will show that if R is an étale equivalence relation, then T/R is an algebraic space. The following lemma tells us that this is always the case if T is affine (cf. [76, Part 4: Algebraic space, lemma 10.4]). In the general case, we might construct a covering of the sheaf T/R by quotient sheaves induced by affine schemes, and then to apply lemma A.2.4 (cf. theorem A.2.8 below).

Let $(\mathbf{Aff}/S)_{\text{fppf}}$ be the category of affine S -schemes whose coverings are fppf-coverings.

Lemma A.2.7. *Let $T \in (\mathbf{Aff}/S)_{\text{fppf}}$ and $j : R \rightarrow T \times_S T$ be an étale equivalence relation on T . Then the quotient $\mathfrak{X} := T/R$ is an algebraic space.*

Theorem A.2.8. *Let $T \in (\mathbf{Sch}/S)_{\text{fppf}}$ and $j : R \rightarrow T \times_S T$ be an étale equivalence relation. Then the quotient T/R is an algebraic space, and $h_T \rightarrow T/R$ is a presentation of T/R .*

Proof. Reduction to the category $(\mathbf{Aff}/S)_{\text{fppf}}$. We start with the following observation: let $T' \rightarrow T$ be a surjective étale morphism. In particular, $\{T' \rightarrow T\}$ is an fppf-covering, and the restriction R' of R to T' (cf. [76, Part II: Grupoids Schemes, definition 3.3]) induces an isomorphism $T'/R' \simeq T/R$ (cf. [76, Part II: Grupoids Schemes, lemma 20.6]). By [76, Part 4: Algebraic spaces, lemma 10.1] R' is an étale relation on T' , and we may replace T by T' . Following this remark we may assume that $U = \bigcup T_i$, with $T_i \in (\mathbf{Aff}/S)_{\text{fppf}}$.

T/R is an algebraic space. Let us start by considering the restriction R_i of R to T_i . As we have remarked, the relation R_i is an étale relation on T_i . Moreover, if $\mathfrak{X}_i := T_i/R_i$, then $\sqcup \mathfrak{X}_i \rightarrow T/R$ is surjective. By lemma A.2.7 we know that \mathfrak{X}_i is an algebraic space and by [76, Part 4: Algebraic space, lemma 10.3] the map $\mathfrak{X}_i \rightarrow T/R$ is étale and surjective. From lemma [76, Part 4: Algebraic spaces, lemma 8.4] we see that $\sqcup \mathfrak{X}_i$ is an algebraic space, and therefore lemma A.2.4 tells us that T/R is an algebraic space.

$h_T \rightarrow T/R$ is a presentation. This is [76, Part 4: Algebraic spaces, lemma 10.4]. \square

3 The Lie algebra of an algebraic group

In the first part of this section we will introduce the Lie algebra of a (linear) algebraic group. Following the arguments given in [51] we will give several interpretations and we will show that they are all equivalent. In the second part we will follow [64] to discuss the most important properties of (complex) semisimple Lie algebras, and [19] to introduce Weyl groups and root systems. This part starts to prepare the "algebraic side" of the Beilinson-Berstein correspondence. At the end of this section, we sketch the fact that category of semisimple algebraic groups in characteristic zero is visible already in the theory of semisimple Lie algebras.

Through this section k will denote an algebraically closed field of characteristic zero.

3.1 Terminology and basic properties

As we will explain in this section, the Lie algebra of an algebraic group can be considered as a linear approximation of the group, and therefore it is endowed of an algebraic nature. This suggests that the study of the Lie algebras is much more natural than the study of the group.

Definition 3.1.1. A **Lie algebra** over a field k is a k -vector space \mathfrak{g} together with a k -bilinear map

$$[\bullet, \bullet] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

called the **bracket** such that

- (i) $[x, x] = 0$, for all $x \in \mathfrak{g}$,
- (ii) $[x, [y, z]] + [y, [x, z]] + [z, [x, y]] = 0$, for all $x, y, z \in \mathfrak{g}$.

Condition (ii) is called **Jacobi's identity**.

We will denote by $\mathcal{L}ie_k$ the category of k -Lie algebras, whose morphisms are k -linear maps respecting the respective brackets.

A **Lie subalgebra** of a Lie algebra \mathfrak{g} is a k -vectors subspace \mathfrak{h} such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. A Lie algebra \mathfrak{g} is said to be **abelian** if $[\mathfrak{g}, \mathfrak{g}] = 0$. Finally, an **ideal** in a Lie algebra \mathfrak{g} is a k -vector subspace \mathfrak{a} such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$.

Remark 3.1.2. The preceding definition suggests that \mathfrak{a} is a left ideal, but it is clear that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$ if and only if $[\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$. In particular, all left (resp. right) ideals are two-sided ideals.

Even if the following notion will be used later, we want to introduce it here for further references.

Definition 3.1.3. Let \mathfrak{g} be a finite dimensional Lie algebra.

- (i) A **representation** of \mathfrak{g} is a vector space V together with a function $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ such that

$$\rho([x, y]) = [\rho(x), \rho(y)] = \rho(x)\rho(y) - \rho(y)\rho(x).$$

- (ii) A representation is said to be **irreducible** if there is no non-trivial, proper subspace W such that $\rho(x)W \subset W$ for all $x \in \mathfrak{g}$.

Example 3.1.4. One of the most important representations of a given (abstract) Lie algebra $(\mathfrak{g}, [\bullet, \bullet])$ is the **adjoint representation**

$$\begin{aligned} \text{ad} : \mathfrak{g} &\rightarrow \text{End}(\mathfrak{g}) \\ x &\mapsto \text{ad}(x) := [x, \bullet]. \end{aligned}$$

We have the following examples of Lie algebras (cf. subsection 3.6).

\mathfrak{gl}_n : as a k -vector space $\mathfrak{gl}_n := M_n(k)$ the space of squared matrices of size n . The bracket is the usual one defined by $[A, B] := AB - BA$.

\mathfrak{sl}_n : as a k -vector space, it is the subspace of \mathfrak{gl}_n consisting of matrices whose trace is equal to zero. In order to see that this is a Lie subalgebra of \mathfrak{gl}_n we need to show that \mathfrak{sl}_n is stable under the bracket. This follows from the identity $\text{Tr}(AB) = \text{Tr}(BA)$.

\mathfrak{b} and \mathfrak{n} : given that the product of two (strictly) upper diagonal matrices is an (strictly) upper triangular matrix, the spaces $\mathfrak{b} := \{(a_{ij}) \mid a_{ij} = 0 \text{ for all } i > j\}$ and $\mathfrak{n} := \{(a_{ij}) \mid a_{ij} = 0 \text{ for all } i \geq j\}$ are in fact Lie subalgebras of \mathfrak{gl}_n .

\mathfrak{t} : the vector space of diagonal matrices is clearly a Lie algebra.

3.2 Functor of points

Let R be a k -algebra and let us consider $R[\varepsilon] := R[x]/(x^2)$ the ring of **dual numbers**. It is clear that $R[\varepsilon] = R \oplus R\varepsilon$ as an R -module, and $\varepsilon^2 = 0$. We have the following sequence

$$R \xrightarrow{i} R[\varepsilon] \xrightarrow{\pi} R$$

with $i(a) = a + \varepsilon 0$ and $\pi(a + \varepsilon b) = a$. We recall for the reader that via the Yoneda's embedding we are considering our algebraic groups as functors

$$k\text{-Alg} \rightarrow \mathbf{Gps}.$$

So, for an algebraic k -group \mathbb{G} , the preceding sequence gives us a sequence of group homomorphisms¹⁶

$$\mathbb{G}(R) \xrightarrow{i} \mathbb{G}(R[\varepsilon]) \xrightarrow{\pi} \mathbb{G}(R). \quad (14)$$

We let

$$\mathfrak{g}(R) := \ker(\mathbb{G}(R[\varepsilon]) \xrightarrow{\pi} \mathbb{G}(R))$$

Example 3.2.1. Let $\mathbb{G} = \mathbb{G}L_n$. For each matrix $A \in M_n(R)$, $I_n + \varepsilon A$ is an element of $M_n(R[\varepsilon])$, and $(I_n + \varepsilon A)(I_n - \varepsilon A) = I_n$; therefore $I_n + \varepsilon A \in \mathfrak{g}(R)$. Given that every element of $\mathfrak{g}(R)$ has this form, we can conclude that the map

$$\begin{array}{ccc} M_n(R) & \rightarrow & \mathfrak{g}(R) \\ A & \mapsto & E(A) \stackrel{\text{def}}{=} I_n + \varepsilon A. \end{array}$$

is a bijection. Therefore $\mathfrak{g}(R) = \{I_n + \varepsilon A \mid A \in M_n(R)\}$.

Example 3.2.2. Let $\mathbb{G} = \mathbb{G}L_V$ where V is a finite-dimensional vector space over k . Every element of $V(\varepsilon) \stackrel{\text{def}}{=} k[\varepsilon] \otimes_k V$ can be written uniquely in the form $x + \varepsilon y$ with $x, y \in V$, i.e., $V(\varepsilon) = V \oplus \varepsilon V$. For k -linear endomorphism α and β of V , define $\alpha + \varepsilon\beta$ to be the map $V(\varepsilon) \rightarrow V(\varepsilon)$ such that

$$(\alpha + \varepsilon\beta)(x + \varepsilon y) = \alpha(x) + \varepsilon(\alpha(y) + \beta(x)); \quad (15)$$

by definition $\alpha + \varepsilon\beta$ is a $k[\varepsilon]$ -linear map. Now, if we consider

$$V(\varepsilon) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid x, y \in V \right\},$$

¹⁶We abuse of the notation y we note i and π the group morphisms $\mathbb{G}(i)$ and $\mathbb{G}(\pi)$, respectively. Under this convention we have, by functoriality, $\pi \circ i = id_{\mathbb{G}(R)}$.

then it is clear that the set of k -linear endomorphism of $V(\varepsilon)$ is equal to the set of 2×2 matrices of k -linear endomorphisms of V (multiplication on the left), and ε acts as $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. This means that the $k[\varepsilon]$ -linear endomorphisms of $V(\varepsilon)$ are in correspondence with those matrices that commute with $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, in other words, those matrices of the form $\begin{pmatrix} \alpha & 0 \\ \beta & \alpha \end{pmatrix}$, which corresponds to the morphisms (15). This shows that

$$\mathbb{G}L_V(k[\varepsilon]) = \{\alpha + \varepsilon\beta \mid \alpha \text{ invertible}\}$$

and following the example 3.2.1 we can conclude that

$$\mathfrak{g}(k) = \{\text{id}_V + \varepsilon\alpha \mid \alpha \in \text{End}(V)\}. \quad (16)$$

3.3 Description in terms of derivations

Definition 3.3.1. *Let A be a k -algebra and M an A -module. A k -linear map $D : A \rightarrow M$ is a k -derivation of A into M if D satisfies the **Leibniz rule***

$$D(fg) = f \cdot D(g) + g \cdot D(f).$$

The Leibniz rule implies that $D(1) = 0$. By k -linearity, this implies that $D(c) = 0$, for all $c \in k$. Conversely, every additive map $A \rightarrow M$ satisfying the Leibniz rule and zero on k is a k -derivation.

Let $\alpha : A \rightarrow R[\varepsilon]$ be a k -linear algebra. Let us write

$$\alpha(f) = \alpha_0(f) + \varepsilon\alpha_1(f), \quad \alpha_0(f), \alpha_1(f) \in R$$

Then $\alpha(fg) = \alpha(f)\alpha(g)$ if and only if

$$\alpha_0(fg) = \alpha_0(f)\alpha_0(g), \quad \alpha_1(fg) = \alpha_0(f)\alpha_1(g) + \alpha_0(g)\alpha_1(f).$$

The first condition say that α_0 is a k -algebra homomorphism $A \rightarrow R$. When we use α_0 to make R into an A -module, the second condition say that α_1 is a k -derivation $A \rightarrow R$.

Let \mathbb{G} be an algebraic group and let us consider $\epsilon : k[\mathbb{G}] \rightarrow k$ be the identity element of $\mathbb{G}(k)$. By definition, the elements of $\mathfrak{g}(R)$ are the k -algebra homomorphisms $k[\mathbb{G}] \xrightarrow{\alpha} R[\varepsilon]$ such that the composite

$$k[\mathbb{G}] \xrightarrow{\alpha} R[\varepsilon] \xrightarrow{\varepsilon \mapsto 0} R$$

is ε , in other words such that $\alpha_0 = \epsilon$. According to the preceding discussion, we can conclude that

$$\mathfrak{g}(R) = \{\epsilon + \varepsilon D \mid D \text{ is a derivation}\}. \quad (17)$$

Let $\text{Der}_{k,\varepsilon}(k[\mathbb{G}], R)$ be the the set of k -derivations $k[\mathbb{G}] \rightarrow R$ with R regarded as an $k[\mathbb{G}]$ -module though $k[\mathbb{G}] \xrightarrow{\epsilon} k \rightarrow R$. Let us also consider I the augmentation ideal of $k[\mathbb{G}]$, defined by the exact sequence

$$0 \rightarrow I \rightarrow k[\mathbb{G}] \xrightarrow{\epsilon} k \rightarrow 0 \quad (18)$$

Proposition 3.3.2. *There are natural one-to-one correspondences*

$$\mathfrak{g}(R) \leftrightarrow \text{Der}_{k,\varepsilon}(k[\mathbb{G}], R) \leftrightarrow \text{Hom}_{k\text{-linear}}(I/I^2, R).$$

Proof. The first correspondence is by (17). On the other hand, by definition, the Leibnitz rule in this case is

$$D(fg) = \varepsilon(f) \cdot D(g) + \varepsilon(g) \cdot D(f). \quad (19)$$

In particular, $D(fg) = 0$ if $f, g \in I$. Moreover, given that $\varepsilon(c) = c$ for $c \in k$, the sequence (18) splits via the canonical decomposition

$$\begin{aligned} k[\mathbb{G}] &= k \oplus I \\ f &\leftrightarrow (\varepsilon(f), f - \varepsilon(f)) \end{aligned}$$

of $k[\mathbb{G}]$ as a k -vector space. Under this presentation and given that any derivation $k[\mathbb{G}] \rightarrow R$ is zero on k , then it is determined by its restriction to I , which is therefore any k -linear map $I \rightarrow R$ that is zero on I^2 . \square

Let us now define the **adjoint map** $\text{Ad}: \mathbb{G}(R) \rightarrow \text{Aut}(\mathfrak{g})$. To do that, we recall for the reader that we have two maps

$$i: \mathbb{G}(R) \rightarrow \mathbb{G}(R[\varepsilon]) \quad \text{and} \quad \pi: \mathbb{G}(R[\varepsilon]) \rightarrow \mathbb{G}(R)$$

such that $\pi \circ i = \text{id}_{\mathbb{G}(R)}$. We define

$$\begin{aligned} \text{Ad}: \mathbb{G}(R) &\rightarrow \text{Aut}(\mathfrak{g}(R)) \\ g &\mapsto \text{Ad}(g)(x) := i(g) \cdot x \cdot i(g)^{-1}. \end{aligned}$$

Here $g \in \mathbb{G}(R)$ and $x \in \mathfrak{g}(R) \subset \mathbb{G}(R[\varepsilon])$. We have the following relations: first of all, we have

$$\text{Ad}(g)(x + x') = \text{Ad}(g)x + \text{Ad}(g)x'$$

for $g \in \mathbb{G}(R)$ and $x, x' \in \mathfrak{g}(R)$. Furthermore, the R -module structure of $\mathbb{G}(R)$ can be defined as follows: every element $r \in R$ defines a homomorphism of R -algebras

$$\begin{aligned} R[\varepsilon] &\xrightarrow{u_r} R[\varepsilon] \\ a + \varepsilon b &\mapsto a + \varepsilon r b \end{aligned}$$

such that $\pi \circ u_r = \pi$ and $u_r \circ i = i$. From the first relation we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}(R[\varepsilon]) & \xrightarrow{\mathbb{G}(u_r)} & \mathbb{G}(R[\varepsilon]) \\ \downarrow \mathbb{G}(\pi) & & \downarrow \mathbb{G}(\pi) \\ \mathbb{G}(R) & \xrightarrow{\text{id}} & \mathbb{G}(R) \end{array}$$

and $\mathbb{G}(u_r)$ defines a morphism

$$\begin{aligned} r \cdot (\bullet): \mathfrak{g}(R) &\rightarrow \mathfrak{g}(R) \\ \eta &\mapsto r \cdot \eta := \mathbb{G}(u_r)(\eta) \end{aligned} \quad (20)$$

By using this action and the relation $u_r \circ i = i$, we have that

$$\begin{aligned} r \cdot \text{Ad}(g)(x) &= r \cdot (i(g) \cdot x \cdot i(g)^{-1}) = \mathbb{G}(u_r)(i(g)) \cdot \mathbb{G}(u_r)(x) \cdot \mathbb{G}(u_r)(i(g)^{-1}) \\ &= i(g) \cdot (r \cdot x) \cdot i(g)^{-1} \\ &= \text{Ad}(g)(r \cdot x). \end{aligned}$$

Therefore Ad maps into $\text{Aut}_{R\text{-linear}}(\mathfrak{g}(R))$.

Finally, let $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ be a morphism of algebraic groups over k . By functoriality, we get the following commutative diagram

$$\begin{array}{ccc} \mathbb{G}_1(R[\varepsilon]) & \xrightarrow{\pi} & \mathbb{G}_1(R) \\ \downarrow f & & \downarrow f \\ \mathbb{G}_2(R[\varepsilon]) & \xrightarrow{\pi} & \mathbb{G}_2(R) \end{array}$$

and so f induces a homomorphism

$$df : \mathfrak{g}_1(R) \rightarrow \mathfrak{g}_2(R), \quad (21)$$

which is natural in R .

3.4 Definition of the functor Lie

Let Lie be the functor

$$\begin{array}{ccc} \text{Lie} : & \mathbf{AlgGps}_k & \rightarrow & \mathbf{Vect}_k \\ & \mathbb{G} & \mapsto & \mathfrak{g}(k) := \ker(\mathbb{G}(k[\varepsilon]) \rightarrow \mathbb{G}(k)) \end{array} \quad (22)$$

(the k -structure as in (20)). Using the identification $\mathfrak{g}(R) = \mathfrak{g} \otimes_k R$ we can define the **adjoint representation** of \mathbb{G} on the vector space $\mathfrak{g}(k)$, by using the map Ad introduced in the previous subsection

$$\text{Ad} : \mathbb{G} \rightarrow \mathbb{GL}_{\mathfrak{g}(k)}. \quad (23)$$

By applying the functor Lie to (23) and using the relation (16) in example 3.2.2, we get a k -linear map

$$\text{ad} : \mathfrak{g}(k) \rightarrow \text{End}(\mathfrak{g}(k)).$$

For $x, y \in \mathfrak{g}(k)$ we set

$$[x, y] := \text{ad}(x)(y). \quad (24)$$

Lemma 3.4.1. *For \mathbb{GL}_n the construction gives*

$$[A, X] = AX - XA.$$

Proof. Let us recall that from the example 3.2.1 we know that

$$\text{Lie}(\mathbb{GL}_n) := \mathfrak{gl}_n(k) := \{I_n + \varepsilon A \mid A \in M_n(k)\}.$$

Now, in our context the sequence (14) translates into the sequence

$$\begin{array}{ccccc} \mathbb{GL}_n(k) & \xrightarrow{i} & \mathbb{GL}_n(K[\varepsilon]) & \xrightarrow{\pi} & \mathbb{GL}_n(k) \\ A & \mapsto & I_n + \varepsilon A & & \\ & & X + \varepsilon Y & \mapsto & X \end{array}$$

and an element $i(A) = I_n + \varepsilon A \in \mathbb{G}L_n(K[\varepsilon])$ acts on $M_n(k[\varepsilon])$ as

$$\begin{aligned} i(A)(X + \varepsilon Y)i(A)^{-1} &= (I_n + \varepsilon A)(X + \varepsilon Y)(I_n - \varepsilon A) \\ &= X + \varepsilon Y + \varepsilon(AX - XA). \end{aligned}$$

Therefore, if in the example 3.2.2 we take $V = M_n(k)$ we can conclude that $\text{ad}(A) = \text{id} + \varepsilon\varphi$, where $\varphi(X) = AX - XA$. \square

Directly from the definition the reader can check that if $f : \mathbb{G} \rightarrow \mathbb{H}$ is a morphism of algebraic groups, then we have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{g}(k) \times \mathfrak{g}(k) & \xrightarrow{\text{ad}} & \mathfrak{g}(k) \\ \downarrow df \times df & & \downarrow df \\ \mathfrak{h}(k) \times \mathfrak{h}(k) & \xrightarrow{\text{ad}} & \mathfrak{h}(k). \end{array} \quad (25)$$

Theorem 3.4.2. [51, Theorem 3.8] *There exists a unique functor Lie*

$$\text{Lie} : \mathbf{AlgGps}_k \rightarrow \mathbf{Lie}_k$$

from the category of algebraic groups over k to the category of Lie algebras such that:

- (i) $\text{Lie}(\mathbb{G}) = \mathfrak{g}(k)$ as a k -vector space, and
- (ii) the bracket on $\text{Lie}(\mathbb{G}L_n) = \mathfrak{gl}_n$ is $[X, Y] = XY - YX$.

Proof. The second part of the theorem follows from lemma 3.4.1. The statement (i) is just (22), so we only need to prove that the bracket defined in (24) endows $\text{Lie}(\mathbb{G})$ with a structure of Lie algebra. To do that, we remark for the reader that the diagram (25) tells us that the differential map $df : \text{Lie}(\mathbb{G}) \rightarrow \text{Lie}(\mathbb{H})$, defined by a morphism $f : \mathbb{G} \rightarrow \mathbb{H}$ of algebraic groups, is compatible with the two brackets. Since the bracket on $\text{Lie}(\mathbb{G}L_n)$ makes it into a Lie algebra, and by corollary 2.2.20 every algebraic group can be embedded in $\mathbb{G}L_n$, the bracket on $\text{Lie}(\mathbb{G})$ makes into a Lie algebra. This also proves uniqueness in the theorem. \square

3.5 Tangent spaces

Let us describe now the Lie algebra of an algebraic group as *tangent space* to the identity element. We will follow the arguments given in [75].

Let us suppose for a while that X is an affine algebraic variety. Let us denote by $k[X]$ the **algebra of regular functions** on X and I the ideal of functions vanishing on X . For every $x \in X$, we will also denote by \mathfrak{m}_x the maximal ideal in $k[X]$ of functions vanishing at x . Given that k was supposed to be algebraically closed, we have a canonical identification $k[X]/\mathfrak{m}_x = k$. In particular, we can view k as a $k[X]$ -module, denoted by k_x , via the *evaluation* morphism $f \mapsto f(x)$.

Definition 3.5.1. *Let X be an affine algebraic variety. For every $x \in X$, we define the **tangent space** $T_x X$ of X at x to be the k -vector space $\text{Der}_k(k[X], k_x)$.*

if $\psi : X \rightarrow Y$ is a morphism of affine algebraic varieties, then the comorphism map $\psi^\sharp : k[Y] \rightarrow k[X]$ induces a linear map between the tangent spaces

$$d\psi_x : \begin{array}{ccc} T_x X & \rightarrow & T_{\psi(x)} Y \\ \nu & \mapsto & \nu \circ \psi^\sharp \end{array} \quad (26)$$

the **differential** of ψ at x . It is clear that if $\varphi : Y \rightarrow Z$ is a second morphism of affine algebraic varieties, then we have the *chain rule*

$$d(\varphi \circ \psi)_x = d\varphi_{\psi(x)} \circ d\psi_x. \quad (27)$$

In particular, if ψ is an isomorphism then so is $d\psi_x$.

Let us see that the information about the tangent space can be read locally. To do that, let us denote by \mathcal{O}_x the local ring of regular function in x . We recall for the reader that its maximal ideal \mathfrak{n}_x consists of the functions vanishing in x . As before, we have $\mathcal{O}_x/\mathfrak{n}_x = k$, and we see k as an \mathcal{O}_x -module.

Lemma 3.5.2. *We have an isomorphism of k -vector spaces $\alpha : \text{Der}_k(\mathcal{O}_x, k) \xrightarrow{\cong} \text{Der}_k(k[X], k_x)$.*

Proof. We have a canonical algebra homomorphism $k[X] \rightarrow \mathcal{O}_x$, this maps induces the morphism α . The inverse is defined by differentiating a quotient (cf. [75, Lemma 4.1.5]). \square

The results that allows us to consider general algebraic varieties is the following.

Proposition 3.5.3. *Let $x \in X$ and U be an affine open subvariety of X . Then $T_x U \simeq T_x X$.*

Proof. By hypothesis, we have $\mathcal{O}_{U,x} \simeq \mathcal{O}_{X,x}$ and the proposition follows by lemma 3.5.2. \square

Definition 3.5.4. *Let X be an algebraic variety and $x \in X$. We define the **tangent space** of X at x by*

$$T_x X := \varprojlim_{\substack{U \\ x \in U}} T_x X,$$

where the projective limit is relative to the set of affine neighbourhoods of x , ordered by inclusion.

By proposition 3.5.3 we have $T_x X \simeq T_x U$ for every affine open neighbourhood U containing x , and the transition maps in definition 3.5.4 are all isomorphisms. In particular, when dealing with questions concerning tangent spaces we can consider them locally.

Definition 3.5.5. *We say that X is **smooth** in x if $\dim_k(T_x X) = \dim(X)$.*

Let us try now to reinterpret the preceding definition in terms of differentials. This will allow us to end the statements in remark 2.3.4.

Let A be a k -algebra and $m : A \otimes_k A \rightarrow A$ the (co)-product morphism. Let $I := \text{Ker}(m) \subseteq A \otimes_k A$. This ideal is generated by the elements $a \otimes 1 - 1 \otimes a$, and the quotient algebra $A \otimes_k A/I$ is isomorphic to A . We define the **module of differentials** $\Omega_{A/k}$ of the k -algebra A by

$$\Omega_{A/k} := I/I^2.$$

As we have remarked, this can be considered as an A -module. Moreover, the module of differentials comes equipped with a k -derivation

$$\begin{aligned} d_{A/k} : A &\rightarrow \Omega_{A/k} \\ a &\mapsto d_{A/k}(a) := (a \otimes 1 - 1 \otimes a) + I^2. \end{aligned}$$

The couple $(\Omega_{A/k}, d_{A/k})$ is universal for k -derivations of A [75, Theorem 4.2.2]. This universality tells us that if X is an irreducible affine variety and $x \in X$, then the tangent space $T_x X$ is isomorphic to $\text{Hom}(\Omega_{k[X]/k}, k_x)$. More exactly, if we put $\Omega_X := \Omega_{k[X]/k}$, then

$$\begin{aligned} \text{Hom}(\Omega_X, k_x) &\rightarrow T_x X := \text{Der}_k(k[X], k_x) \\ \varphi &\mapsto \varphi \circ d_{k[X]/k} \end{aligned}$$

is an isomorphism. Moreover, following our *local* description in 3.5.2, if $\Omega_{X,x}$ denotes the localisation at x (the stalk at x of the associated sheaf $\widetilde{\Omega}_X$), then we have

$$T_x X \simeq \text{Hom}_k(\Omega_{X,x}, k) \tag{28}$$

We have the following basic result about smooth points [17, Chapter 2, section 2, proposition 11].

Proposition 3.5.6. *Let X be an irreducible variety of dimension d . If x is a smooth point of X , there exists an affine open neighbourhood U of x such that Ω_U is a free $k[U]$ -module with a basis $\{dg_1, \dots, dg_d\}$, for suitable $g_i \in k[U]$.*

As a consequence of the preceding discussion, we see that if X is an affine algebraic variety such that Ω_X is a free $k[X]$ -module, then by (28) we can conclude that X is smooth.

Let us suppose now that X, Y are irreducible varieties. A morphism $\varphi : X \rightarrow Y$ is called **dominant** if $\varphi(X)$ is dense in Y . This property implies that we have an injective map of quotient fields $k(Y) \rightarrow k(X)$. We say that φ is **separable** if this extension is separably generated.¹⁷ The main result to keep in mind is the following [75, Chapter 4, theorem 4.3.6].

Proposition 3.5.7. *Let $\varphi : X \rightarrow Y$ be a morphism of irreducible varieties.*

- (i) *If x is a smooth point of X such that $\varphi(x)$ is a smooth point of Y and $d\varphi_x$ is surjective, then φ is dominant and separable.*

¹⁷Let $F := k(Y)(x_1, \dots, x_l)$. The **transcendence degree** $\text{trdeg}_{k(Y)} F$ of F over $k(Y)$ is the maximal of the number of x_i that are algebraically independent over $k(Y)$. If $\text{trdeg}_{k(Y)} F = l$ we say that F is **purely transcendental**. We say that $k(X)$ is **separably generated** over $k(Y)$ if there exists a purely transcendental extension F over $k(Y)$, contained in $k(X)$, such that $k(X)$ is separably algebraic over F .

(ii) If φ is dominant and separable, then the points $x \in X$ satisfying (i) is a non-empty open subset of X .

The preceding proposition implies the following important properties for homogeneous spaces (definition 2.2.15).

Theorem 3.5.8. *Let \mathbb{G} be a connected algebraic group.*

- (i) *Let X be a homogeneous space for \mathbb{G} . Then X is irreducible and smooth.*
- (ii) *Let $\varphi : X \rightarrow Y$ be a \mathbb{G} -equivariant morphism of homogeneous spaces. Then φ is separable if, and only if, the differential $d\varphi_x$ is surjective for some $x \in X$.*
- (iii) *Let $\varphi : \mathbb{G} \rightarrow \mathbb{G}'$ be a surjective morphism of algebraic groups. Then φ is separable if, and only if, the differential $d\varphi_e$ is surjective.*

Proof. We follow word by word the proof given in [75, Chapter 4, theorem 4.3.7]. Let us prove (i). To do that, we consider X a homogeneous space and $x \in X$. By hypothesis, the orbit map $o_x : \mathbb{G} \rightarrow X$ is surjective, and therefore as \mathbb{G} is in particular irreducible (proposition 2.1.7), we can conclude that X is irreducible. On the other hand, for every $g \in \mathbb{G}$ the map

$$\begin{aligned} \mu_g : X &\rightarrow X \\ x &\mapsto \mu_g(x) := g \cdot x \end{aligned}$$

is clearly an isomorphism. By functoriality, (27), the differential $(d\mu_g)_x$ is an isomorphism. The first part of the theorem follows from proposition 3.5.6 and transitivity of the action. The second part of the theorem clearly follows from the proposition 3.5.7 because φ is surjective being a \mathbb{G} -equivariant morphism between homogeneous spaces. The final part of the theorem is an easy consequence of (ii) (cf. remark below). \square

Remark 3.5.9. *Without say it, we have used the following easy observation. If $d\varphi_x$ is surjective for some x , then $d\varphi_x$ is surjective for all $x \in X$.*

Let us recall now that in proposition 2.2.5 we have showed that \mathbb{G} acts (on the left) on $k[\mathbb{G}]$ locally finitely via the morphisms ρ_h in (8). The same argument proves that the right \mathbb{G} -action

$$\begin{aligned} \lambda_h : k[\mathbb{G}] &\rightarrow k[\mathbb{G}] \\ f &\mapsto (\lambda_h f)(g) := f(h^{-1}g). \end{aligned}$$

We consider the space of **left invariant derivations** of \mathbb{G} , which is defined by

$$\mathcal{L}(\mathbb{G}) := \{D \in \text{Der}_k(k[\mathbb{G}], k[\mathbb{G}]) \mid D \circ \lambda_h = \lambda_h \circ D, \text{ for all } h \in \mathbb{G}\}.$$

This is a Lie algebra under the bracket operation $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$. In a suggestive way we let

$$\mathfrak{g} := T_e \mathbb{G}$$

be the tangent space to \mathbb{G} at the identity e . We recall for the reader that \mathfrak{g} can be considered as the space of point derivations $\mathfrak{g} = \text{Der}_k(\mathcal{O}_{\mathbb{G}, e}, k)$. Given that every derivation D defined on $k[\mathbb{G}]$ can be extended to a derivation of the local ring $\mathcal{O}_{\mathbb{G}, e}$ via the quotient rule, we can define a k -linear map

$$\begin{aligned} \mathcal{L}(\mathbb{G}) &\rightarrow \mathfrak{g} \\ D &\mapsto f \mapsto (Df)(e) \end{aligned} \tag{29}$$

Theorem 3.5.10. *The map (29) is an isomorphism of k -vector spaces. In particular $\dim_k(\mathcal{L}(\mathbb{G})) = \dim(\mathbb{G})$.*

Proof. We let to reader to verify that the **convolution**

$$\begin{aligned} \mathfrak{g} &\rightarrow \mathcal{L}(\mathbb{G}) \\ \delta &\mapsto (f * \delta)(h) := \delta(\lambda_{h^{-1}} f) \end{aligned}$$

is well-defined. This means that $f \mapsto f * \delta$ is a left-invariant derivation. Let us verify that this is the inverse of the map (29). To do that, we take $D \in \mathcal{L}(\mathbb{G})$ and we consider δ the derivation $f \mapsto (Df)(e)$. We have

$$(f * \delta)(h) = \delta(\lambda_{h^{-1}} f) = (D\lambda_{h^{-1}} f)(e) = (\lambda_{h^{-1}} Df)(e) = (Df)(x).$$

On the other hand, if δ is a point derivation at e . Then $(f * \delta)(e) = \delta(\lambda_e f) = \delta(f)$ and the derivation $f \mapsto (f * \delta)(e)$ is just δ . \square

Via the preceding isomorphism, we can endow the tangent space $\mathfrak{g} := T_e \mathbb{G}$ with a structure of Lie algebra. We have the following relations

$$\mathfrak{g}(k) = \text{Der}_{k,\epsilon}(k[\mathbb{G}], k) = T_e \mathbb{G} = \mathcal{L}(\mathbb{G}) \quad (30)$$

By uniqueness and functoriality, the Lie algebra structures on $\mathfrak{g}(k)$, (24), and $\mathcal{L}(\mathbb{G})$ coincide. This gives us several interpretations of the Lie algebra of an algebraic group, all of them equivalent.

Before getting some examples let us end the discussion about homogeneous spaces started in the previous section. We recall for the reader that we want to prove that (9) is a quotient map in the sense of the definition 2.3.2. To do that we rephrase theorem 3.5.8 as follows ([14, II, proposition 6.7]). Let X be a \mathbb{G} -space.

Proposition 3.5.11. *Let $x \in X$ and let us consider the orbit map $o_x : \mathbb{G} \rightarrow \mathbb{G} \cdot x$. Then $\mathbb{G} \cdot x$ is a smooth variety defined over k and locally closed in X . Moreover o_x is an orbit map for the action of $\mathbb{G} \cdot x$ on \mathbb{G} . The following conditions are equivalent:*

- (i) o_x is a quotient of \mathbb{G} by \mathbb{G}_x .
- (ii) o_x is separable, i.e. the **tangent map** $d_e o_x : \mathfrak{g} \rightarrow T_x(\mathbb{G} \cdot x)$ is surjective.

Remark 3.5.12. *In theorem 3.5.8 we use connectedness of \mathbb{G} to prove that X is irreducible. We do not have this hypothesis in the preceding proposition.*

At the moment, we have proved that $X := \mathbb{G}/\mathbb{H}$ is a smooth quasi-projective variety (theorem 3.5.8 and 2.3.1). The canonical map (9) $\varphi : \mathbb{G} \rightarrow \mathbb{G}/\mathbb{H}$ is by construction a morphism of algebraic varieties. Moreover, we recall for the reader that X has been also defined as a \mathbb{G} -orbit space $\mathbb{G} \cdot \bar{v}$, where $\bar{v} \in \mathbb{P}(V)$ is the line stabilised by \mathbb{H} , and V is the vector space in theorem 2.2.19. Under this regard, the canonical map φ coincides with the orbit map $o_{\bar{v}}$ and $\mathbb{G}_{\bar{v}} = \mathbb{H}$. So, to prove that φ is a quotient map, we need to prove that $d_e \varphi$ is surjective. To do that, let us bring back again the notation coming from the proof of theorem

2.2.19. By using proposition 2.2.5 we could find a rational representation $\rho : \mathbb{G} \rightarrow \mathrm{GL}(F)$ which contains an \mathbb{H} -invariant subspace E , such that

$$\mathbb{H} = \left\{ g \in \mathbb{G} \mid g \cdot \bigwedge^{\dim(E)} E = \bigwedge^{\dim(E)} E \right\}$$

Let us denote by $W := \bigwedge^{\dim(E)} E$ and $V := \bigwedge^{\dim(E)} F$. Let $\psi : \mathbb{G} \rightarrow \mathrm{GL}(V)$ be the rational representations induced by the \mathbb{G} -action on F . Using (21) and the same reasoning given at the end of proposition 2.2.5, we see that if $\mathfrak{h} := \mathrm{Lie}(\mathbb{H})$, then¹⁸

$$\mathfrak{h} = \{ \zeta \in \mathfrak{g} \mid d\psi(\zeta)(W) \subseteq W \}$$

All in all, let $o_{\bar{v}} : \mathbb{G} \rightarrow \mathbb{G} \cdot \bar{v}$ (here $\bar{v} := [W] \in \mathbb{P}(V)$) be the orbit map, and let us consider the differential $d_e o_{\bar{v}}$. Let $vi \in W$, and let us remark that if $\gamma : V \setminus \{0\} \rightarrow \mathbb{P}(V)$ is the projection, and $o_v : \mathbb{G} \rightarrow \mathbb{G} \cdot v$ is defined by $o_v(g) := \varphi(g)v$, then $\varphi = \gamma \circ o_v$. In particular, by functoriality we have $d_e \varphi = d_v \gamma \circ d_e o_v$. If we identify $T_v V$ with V , the map $d_v \gamma : T_v V \rightarrow T_{\gamma(v)} \mathbb{P}(V)$ is the projection from V with respect to $k \cdot v$, this tells us that $\ker(d_e \varphi) = (d_e o_v)^{-1}(k \cdot v) = \mathrm{Stab}_{\mathfrak{g}}(W) = \mathfrak{h}$. Let us see that this implies that $d_e \varphi$ is surjective. On the one hand we have

$$\dim(\mathfrak{h}) = \dim(\mathbb{H}) = \dim(\mathbb{G}) - \dim(\mathbb{G} \cdot \bar{v}) = \dim(\mathbb{G}) - \dim(T_{\bar{v}} \mathbb{G} \cdot \bar{v}), \quad (31)$$

and on the other hand

$$\dim(\ker(d_e \varphi)) = \dim(\mathbb{G}) - \dim(\mathrm{im}(d_e \varphi)). \quad (32)$$

From (31), (32), and the fact that $\ker(d_e \varphi) = \mathfrak{h}$, we can conclude that $d_e \varphi$ is surjective.

3.6 Examples

Let us use the relations (30) to compute the Lie algebras of the (linear) algebraic group studied in section 2. We recall for the reader that every connected, linear algebraic group is smooth by theorem 3.5.8. Using again the relation (30) we have that

$$\dim(\mathbb{G}) = \dim_k(\mathfrak{g}),$$

and therefore, we can use the Lie algebra to compute the dimension of the group.

GL_n: By example 3.2.1 we know that as k -vector space, the Lie algebra \mathfrak{gl}_n of GL_n equals the k -vectors space $M_n(k)$ of squared matrices of size n . By lemma 3.4.1, the structure of Lie algebra of \mathfrak{gl}_n is given by the usual bracket $[A, B] = AB - BA$.

SL_n: By definition

$$\mathrm{Lie}(\mathbf{SL}_n) = \{ I + \varepsilon M_n(k[\varepsilon]) \mid \det(I + \varepsilon A) = 1 \}$$

Expanding $\det(I + \varepsilon A)$ as a sum of polynomials in the variable ε , we easily figure out that the only non-zero term is

$$\prod_{i=1}^n (1 + \varepsilon a_{ii}) = 1 + \varepsilon \sum_{i=1}^n a_{ii}$$

¹⁸We have $d\psi : \mathfrak{g} \rightarrow \mathrm{End}(V)$. We abuse of the notation denoting again $d\psi$ the canonical extension $\mathfrak{g} \rightarrow \mathrm{End}(V)$.

(terms including at least two off-diagonal entries are zero because $\varepsilon^2 = 0$). Hence

$$\det(I + \varepsilon A) = 1 + \varepsilon \operatorname{Tr}(A)$$

and therefore $\operatorname{Lie}(\mathbf{SL}_n) = \mathfrak{sl}_n = \{A \in \mathfrak{gl}_n \mid \operatorname{Tr}(A) = 0\}$.

The group of invertible upper triangular matrices \mathbb{B} : By definition

$$\operatorname{Lie}(\mathbb{B}) = \{I + \varepsilon A \in M_n(k[\varepsilon]) \mid a_{ij} = 0 \text{ for all } i > j\}.$$

So $\operatorname{Lie}(\mathbb{B}) = \mathfrak{b}$ is the Lie algebra of upper triangular matrices.

The unipotent group \mathbb{U} : This is the group of upper triangular matrices having all the diagonal terms equal to 1. So, by definition we have

$$\operatorname{Lie}(\mathbb{U}) = \{I + \varepsilon A \in M_n(k[\varepsilon]) \mid a_{ij} = 0 \text{ for all } i > j, \text{ and } 1 + \varepsilon a_{ii} = 1\}.$$

This implies that $\operatorname{Lie}(\mathbb{U}) = \mathfrak{n}$ the Lie algebra of strictly upper triangular matrices.

The group of invertible diagonal matrices \mathbb{T} : being sure that at this point the reader has understood the philosophy behind the computations, we can say that it is clear that $\operatorname{Lie}(\mathbb{T}) = \mathfrak{t}$, the Lie algebra of diagonal matrices.

3.7 Solvable and nilpotent Lie algebras

In this subsection we will review the classical theory of solvable and Lie algebras. The main references are [63, 64]. We start with the following notions.

Let \mathfrak{g} be a k -Lie algebra. We put $\mathcal{C}(\mathfrak{g})_1 := [\mathfrak{g}, \mathfrak{g}]$, and inductively we define $\mathcal{C}(\mathfrak{g})_n := [\mathfrak{g}, \mathcal{C}(\mathfrak{g})_{n-1}]$ to be the **lower central series** of \mathfrak{g} . On the other hand, we can also put $\mathcal{C}(\mathfrak{g})^1 := [\mathfrak{g}, \mathfrak{g}]$, and inductively define $\mathcal{C}(\mathfrak{g})^n := [\mathcal{C}(\mathfrak{g})^{n-1}, \mathcal{C}(\mathfrak{g})^{n-1}]$ the **upper central series**.

Definition 3.7.1. *We say that \mathfrak{g} is **nilpotent** if the lower central series vanishes, i.e. there exists a positive integer n such that $\mathcal{C}(\mathfrak{g})_n = 0$. We say that \mathfrak{g} is **solvable** if the upper central series vanishes.*

Let us recall that in subsection 3.4 we have introduced the relation $[x, y] = \operatorname{ad}(x)(y)$, with $\operatorname{ad} : \mathfrak{g} \rightarrow \operatorname{End}(\mathfrak{g})$ is the adjoint representation.¹⁹ The reader can easily verify that the following conditions are equivalent:

- (i) There exists $n \in \mathbb{Z}_{>0}$, such that $\mathcal{C}(\mathfrak{g})_{n+1} = 0$.
- (ii) For all $x_0, \dots, x_n \in \mathfrak{g}$, we have

$$[x_0, [x_1, [\dots, x_n] \dots]] = \operatorname{ad}(x_0)\operatorname{ad}(x_1) \dots \operatorname{ad}(x_{n-1})(x_n) = 0.$$

- (iii) There exists a descending series of ideals

$$\mathfrak{g} \supseteq \mathfrak{a}_0 \supseteq \dots \supseteq \mathfrak{a}_n = 0$$

such that $[\mathfrak{g}, \mathfrak{a}_i] \subseteq \mathfrak{a}_{i+1}$, for all $0 \leq i \leq n-1$.

¹⁹If $\mathfrak{g} := \operatorname{Lie}(\mathbb{G})$ is the Lie algebra of an algebraic group, then this relation is established in (24).

Example 3.7.2. Let V be an n dimensional k -vector space, and let

$$\mathcal{F} := 0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_{n-1} \subsetneq V$$

be a full flag. Let $\mathfrak{n}(\mathcal{F})$ be the subset of the Lie algebra $\mathfrak{g} := \text{End}(V)$ consisting of those endomorphisms $x \in \mathfrak{g}$, such that $x \cdot V_i \subset V_{i-1}$. It is clear that $\mathfrak{n}(\mathcal{F})$ is stable under the bracket of \mathfrak{g} and therefore is a Lie subalgebra. Moreover, using (ii) in the preceding discussion we see that $\mathfrak{n}(\mathcal{F})$ is a nilpotent Lie algebra with $\mathcal{C}(\mathfrak{n}(\mathcal{F}))_{n-1} = 0$

We intend to prove the following theorem (the reader will easily realize that the necessary conditions follows immediately from (ii) in the previous discussion).

Theorem 3.7.3 (Engel's theorem). *A Lie algebra \mathfrak{g} is nilpotent if and only if $\text{ad}(x)$ is nilpotent for each $x \in \mathfrak{g}$.*

We will first prove an $\text{End}(V)$ -version of the preceding theorem²⁰ and then we will use canonical isomorphism theorems to extend the proof to the general case. We prepare the route to prove the sufficient condition with the following discussion.

Let us fix a vector space V and a Lie algebra \mathfrak{g} such that $\mathfrak{g} \subseteq \text{End}(V)$.

Eigenvalues of nilpotent elements. Let V be a finite dimensional k -vector space. Let us prove that zero is the only eigenvalue of each nilpotent element $\phi \in \text{End}(V)$. To do that we start by remarking that zero is an eigenvalue of ϕ because is not invertible. On the other hand, if λ is an eigenvalue of ϕ with eigenvector $v \in V$, then $0 = \phi^k(v) = \lambda^k v$, so $\lambda = 0$ because $\text{char}(k) = 0$.

ad-nilpotency. Let us suppose that \mathfrak{g} consists of nilpotent endomorphisms of V , and let us prove that for all $x \in \mathfrak{g}$, the endomorphism $\text{ad}(x) \in \text{End}(\mathfrak{g})$ is also nilpotent. In fact, if $n \in \mathbb{N}$ is such that $x^n = 0$, then for all $y \in \mathfrak{g}$ we have

$$\text{ad}^{2n}(x)(y) = \sum_{i=0}^{2n} r_i x^i y x^{2n-i}.$$

Now all what we have to remark is that in each term in the sum there are at least n factors of x on one of the sides of y , so the whole sum is equal to zero.

Proposition 3.7.4. *Let $\mathfrak{g} \subseteq \text{End}(V)$ be a Lie algebra of nilpotent endomorphisms of V . There exists a vector $v \in V$ such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$.*

Proof. We will proceed by induction on the dimension of \mathfrak{g} . If $\dim(\mathfrak{g}) = 1$, then \mathfrak{g} is clearly abelian, but also it consists of scalar multiples of a nilpotent endomorphism $\phi \in \text{End}(V)$. Given that zero is the only eigenvalue of ϕ , there exists a non-zero vector $v \in V$ such that $\phi(v) = 0$. Let us suppose now that $\dim(\mathfrak{g}) > 1$, and let us consider the following situation.

²⁰By **Ado's theorem**: every finite-dimensional abstract Lie algebra is isomorphic to a **concrete** Lie algebra over a finite dimensional vector space (vector spaces of linear transformations). We can see that it suffices to prove Engel's theorem for the $\text{End}(V)$ -version. We prefer to prove Engel's theorem with out using Ado's theorem because, except for some special cases, this theorem is surprisingly tricky to prove. We will discuss the semi-simple case in the appendix of this section by using Tannaka duality, but a concrete prove can be founded in [78].

Maximal proper subalgebras of nilpotent algebras. Before continuing the proof, we want to construct an ideal $\mathfrak{a} \subset \mathfrak{g}$ of co-dimension 1. In fact, we show that we can take any maximal proper sub-algebra. Let \mathfrak{a} be such an algebra. So $\text{ad}(\mathfrak{a})$ preserves \mathfrak{a} and $\text{ad}\mathfrak{a}$ induces an \mathfrak{a} -action by nilpotent endomorphisms on the quotient vector space $\mathfrak{g}/\mathfrak{a}$. By inductive hypothesis, we can find a vector $\bar{y} \in \mathfrak{g}/\mathfrak{a}$ such that $\text{ad}(x)(\bar{y}) = 0 \in \mathfrak{g}/\mathfrak{a}$ for every $x \in \mathfrak{a}$. If $y \in \mathfrak{g}$ is a pre-image of \bar{y} , then $\text{ad}(x)(y) \in \mathfrak{a}$ for all $x \in \mathfrak{a}$. This clearly implies that \mathfrak{a} is an ideal of the algebra $\text{Span}(\mathfrak{a}, y)$ of co-dimension 1. However, as we declared \mathfrak{a} to be a maximal proper sub-algebra, we necessarily have $\text{Span}(\mathfrak{a}, y) = \mathfrak{g}$. We can go back to the proof of the proposition.

Let us take an ideal $\mathfrak{a} \subset \mathfrak{g}$ with co-dimension 1. Applying the inductive hypothesis to \mathfrak{a} , we can find $v \in V$ such that $x \cdot v = 0$ for all $x \in \mathfrak{a}$. Let W be the subspace of all vectors with this property and $y \in \mathfrak{g} \setminus \mathfrak{a}$. Let us also take $w \in W$, $x \in \mathfrak{a}$, and let us analyse the following relation:

$$xy \cdot w = yx \cdot w + [x, y] \cdot w.$$

First of all, we know that $yx \cdot w = 0$ by definition of W . Moreover, given that \mathfrak{a} is an ideal we have $[x, y] \in \mathfrak{a}$ and therefore $[x, y] \cdot w = 0$. We have $xy \cdot w = 0$ for any $x \in \mathfrak{a}$ and we can conclude that $y \cdot w \in W$. This tells us that y preserves W . However, we know that y acts nilpotently and we can find $v \in W$ such that $y \cdot v = 0$. As we have remarked $\text{Span}(\mathfrak{a}, y) = \mathfrak{g}$ and we have that $x \cdot v = 0$ for any $x \in \mathfrak{g}$. \square

Corollary 3.7.5 (Engel's theorem $\text{End}(V)$ -version). *Let $\mathfrak{g} \subset \text{End}(V)$ be a concrete Lie algebra of nilpotent endomorphisms of V . There exists a basis of V such that each element is strictly upper triangular.*

Proof. We proceed by induction on $\dim(V)$. If $\dim(V) = 1$, then $\dim(\mathfrak{g})$ is either 0 or 1. In both cases the matrices with respect to any basis are all zero because they are nilpotent 1×1 -matrices. Now, let us assume that the corollary is true for any vector space of dimension less or equal than $n - 1$, and let us suppose that $\dim(V) = n$. By theorem 3.7.4, there exists a vector $v \in V$ such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$. Applying inductive hypothesis to the $n - 1$ -dimensional space $V/\text{Span}(v)$, we can find a basis $\bar{v}_1, \dots, \bar{v}_{n-1}$ for $V/\text{Span}(v)$ such that all x are strictly upper triangular. Taking the pre-images of those vectors, together with v , we get a basis for V , such that all x are strictly upper triangular. \square

Before giving the proof of the theorem 3.7.3, let us give one remark more that will be used in the proof of the theorem. Let

$$z(\mathfrak{g}) := \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}$$

be the **center** of the Lie algebra \mathfrak{g} . This is clearly an abelian ideal of \mathfrak{g} . We claim that if $\mathfrak{g}/z(\mathfrak{g})$ is nilpotent, then so in \mathfrak{g} . The bracket is defined by

$$\begin{aligned} [\bullet, \bullet] &: \mathfrak{g}/z(\mathfrak{g}) \times \mathfrak{g}/z(\mathfrak{g}) \rightarrow \mathfrak{g}/z(\mathfrak{g}) \\ (x + z(\mathfrak{g}), y + z(\mathfrak{g})) &\mapsto [x, y] + z(\mathfrak{g}). \end{aligned}$$

By hypothesis, there exists $n \in \mathbb{N}$ such that $\mathcal{C}(\mathfrak{g}/z(\mathfrak{g}))_n = 0$. We have a well-defined surjection $\mathcal{C}(\mathfrak{g})_n \rightarrow \mathcal{C}(\mathfrak{g}/z(\mathfrak{g}))_n$ which tells us that $\mathcal{C}(\mathfrak{g})_n \subseteq z(\mathfrak{g})$. Then $\mathcal{C}(\mathfrak{g})_{n+1} = [\mathfrak{g}, \mathcal{C}(\mathfrak{g})_n] \subseteq [z(\mathfrak{g}), \mathfrak{g}] = 0$.

We can finally give the proof of Engel's theorem.

Proof of theorem 3.7.3. We start the proof of the theorem by recalling for the reader that by **ad-nilpotency** the Lie algebra $\kappa := \text{ad}(\mathfrak{g})$ is a nilpotent Lie subalgebra of $\text{End}(\mathfrak{g})$. By corollary 3.7.5 we can conclude that κ is nilpotent. Moreover, we have $\ker(\text{ad}) = z(\mathfrak{g})$ and therefore

$$\begin{aligned} \mathfrak{g}/z(\mathfrak{g}) &\rightarrow \kappa \\ x + z(\mathfrak{g}) &\mapsto \text{ad}(x) \end{aligned}$$

is an isomorphism of Lie algebras. We can conclude that $\mathfrak{g}/z(\mathfrak{g})$ is nilpotent and in the light of the previous discussion we can conclude that \mathfrak{g} is nilpotent. \square

Example 3.7.6. The Lie algebra $\mathfrak{n} = \text{Lie}(\mathbb{U})$ is nilpotent.

In the example 3.7.2 we have seen that every fully flag \mathcal{F} in a vector space V defines a nilpotent Lie sub-algebra $\mathfrak{n}(\mathcal{F})$ of $\text{End}(V)$. Let us prove that every nilpotent Lie sub-algebra of $\text{End}(V)$ is contained in a nilpotent Lie algebra $\mathfrak{n}(\mathcal{F})$.

Corollary 3.7.7. *Let V be a finite dimensional vector space and \mathfrak{g} a Lie sub-algebra of $\text{End}(V)$ consisting of nilpotent endomorphisms. Then there is a fully flag \mathcal{F} such that $\mathfrak{g} \subseteq \mathfrak{n}(\mathcal{F})$.*

Proof. We will proceed by induction on the flag. To define $V_1 \supseteq \{0\}$ we use proposition 3.7.4 to obtain $v \in V$ such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$. We take $V_1 := \text{Span}(v)$. Let us suppose that we have defined $0 \subsetneq V_1 \subsetneq \dots \subsetneq V_{n-2}$ such that $\dim(V_i) = i$ and $x \cdot V_i \subset V_{i-1}$ for all $1 \leq i \leq n-2$. By construction, we will have a morphism of Lie algebras $\iota : \mathfrak{g} \rightarrow \text{End}(V/V_{n-1})$ and $\mathfrak{g}/\ker(\iota)$ can be considered as nilpotent Lie subalgebra of $\text{End}(V/V_{n-1})$. By proposition 3.7.4 we can find $\bar{v}_{n-1} \in V/V_{n-2}$ such that $\bar{x} \cdot \bar{v}_{n-1} = 0$ in V/V_{n-2} for all $\bar{x} \in \mathfrak{g}/\ker(\iota)$. Taking a lift $v_{n-1} \in V$ of \bar{v}_{n-1} , making $V_{n-1} := \text{Span}(V_{n-2}, v_{n-1})$ and considering the full flag $\mathcal{F} = (V_i)$, we end the proof of the corollary because $\mathfrak{g} \subseteq \mathfrak{n}(\mathcal{F})$. \square

Example 3.7.8. If \mathcal{F} denotes the canonical flag variety (3) in $V = k^n$. Then $\text{Lie}(\mathbb{U}) = \mathfrak{n} = \mathfrak{n}(\mathcal{F})$.

We want to derive now similar result to the corollary 3.7.5 for **solvable** Lie algebras. We start with the following examples.

Example 3.7.9. (i) For every $n \in \mathbb{Z}_{>0}$ we have $\mathcal{C}(\mathfrak{g})^n \subset \mathcal{C}(\mathfrak{g})_n$. Therefore, every nilpotent algebra is solvable.

(ii) It follows immediately from the definitions that every subalgebra and every quotient of a solvable Lie algebra is a solvable Lie algebra. In particular, every extension of solvable algebras is solvable. In fact, if

$$0 \rightarrow \mathfrak{h} \rightarrow \mathfrak{e} \xrightarrow{p} \mathfrak{g} \rightarrow 0$$

is an extension of \mathfrak{e} , then it is easy to see that $p(\mathcal{C}(\mathfrak{e})^n) = \mathcal{C}(\mathfrak{g})^n = 0$. Therefore, if $\mathcal{C}(\mathfrak{g})^n = 0$ and $\mathcal{C}(\mathfrak{h})^l = 0$, by exactness we have $\mathcal{C}(\mathfrak{e})^n \subseteq \mathfrak{h}$ and $\mathcal{C}(\mathfrak{e})^{n+l} = \mathcal{C}(\mathfrak{e})^n \subseteq \mathfrak{h} \subseteq \mathcal{C}(\mathfrak{h})^l = 0$.

(iii) Let $\mathcal{F} = (V_i)$ be a full flag of a vector space V , and let $\mathfrak{b}(\mathcal{F})$ be the subalgebra of $\text{End}(V)$ consisting of terms $x \in \text{End}(V)$ such that $x \cdot V_i \subset V_i$ for all i . The algebra $\mathfrak{b}(\mathcal{F})$ is solvable. In fact, it is possible to construct a basis of V such that $\mathfrak{b}(\mathcal{F})$ consists of upper triangular matrices. We also remark for the reader that the Engel's theorem implies that $\mathfrak{b}(\mathcal{F})$ is not nilpotent.

Theorem 3.7.10. *Let $\mathfrak{g} \subset \text{End}(V)$ be a concrete solvable Lie algebra. There is a vector $v \in V$ such that v is an eigenvector for all $x \in \mathfrak{g}$.*

Proof. We will proceed by induction on \mathfrak{g} as we have done in proposition 3.7.4. If $\dim(\mathfrak{g}) = 1$, then \mathfrak{g} consists of scalar multiples of a non-zero single element y . Given that k is algebraically closed, it contains a root of the characteristic polynomial of y and therefore we can take an eigenvector $v \in V$ which will be an eigenvector for every $x \in \mathfrak{g}$. Let us suppose now that $\dim(\mathfrak{g}) \geq 2$ and that the theorem is true for every solvable concrete Lie algebra of dimension at least $\dim(\mathfrak{g}) - 1$.

Construction of an ideal of co-dimension 1. Let us consider $\mathcal{C}(\mathfrak{g})^1 := [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Given that \mathfrak{g} is solvable $\mathcal{C}(\mathfrak{g})^1$ is a proper ideal and therefore we can consider the quotient Lie algebra $\bar{\mathfrak{g}} := \mathfrak{g}/\mathcal{C}(\mathfrak{g})^1$. Given that $\bar{\mathfrak{g}}$ is an abelian algebra (in particular nilpotent) the reasoning given in proposition 3.7.4 gives us an ideal $\bar{\mathfrak{a}}$ with co-dimension 1. The pre-image \mathfrak{a} of $\bar{\mathfrak{a}}$ in \mathfrak{g} will also be an ideal of co-dimension 1. Let us go back to the proof of the theorem.

By inductive hypothesis, there exists a vector $v \in V$ which is an eigenvector of every element of \mathfrak{a} . Given $x \in \mathfrak{a}$ we will denote by λ_x the eigenvalue of v . We have the weight space

$$V_{\lambda_x} := \{v \in V \mid x \cdot v = \lambda_x v \text{ for all } x \in \mathfrak{a}\}.$$

As in the proof of proposition 3.7.4 we will show that for some $y \in \mathfrak{g} \setminus \mathfrak{a}$, the weight space V_{λ_x} . In other words, $y \cdot V_{\lambda_x} \subset V_{\lambda_x}$. Let $x \in \mathfrak{a}$, $w \in V_{\lambda_x}$ and $y \in \mathfrak{g} \setminus \mathfrak{a}$. As before, we want to analyse the expression

$$xy \cdot w = y \cdot x + [x, y] \cdot w = \lambda_x y \cdot w + \lambda_{[x, y]} w. \quad (33)$$

If the second term in the sum on the right were zero then it would be immediately that $y \cdot w \in V_{\lambda_x}$. This is less obvious than in 3.7.4. We will verify this by proving first that $\lambda_{[x, y]} = 0$. Let us consider the subspace

$$U := \text{Span}(w, y \cdot w, \dots, y^k \cdot w, \dots).$$

It is clear that $y \cdot U \subset U$. We want to see, via induction, that $\mathfrak{a} \cdot U \subset U$.

\mathfrak{a} -invariance of U . By definition $x \cdot w \in U$ and by (33) we see that $xy \cdot w \in \text{Span}(y \cdot w, w) \subset U$. Let us suppose that $xy^{k-1} \cdot w \in U$ (for any $x \in \mathfrak{a}$). We have

$$xy^k \cdot w = yxy^{k-1} \cdot w + [x, y]y^{k-1} \cdot w.$$

We have assumed that $xy^{k-1} \cdot w \in U$ and given that \mathfrak{a} is an ideal, we can conclude, by inductive hypothesis, that the sum on the right lies in U .

\mathfrak{a} acts upper triangular on U . Let us see that the action of $x_i \mathfrak{a}$ is upper triangular on U with respect to the basis $w, y \cdot w, \dots$. To do this, we will use induction to prove that $xy^k \cdot w$ is a linear combination of $y^i \cdot w$ for $i \leq k$. First of all, by definition $x \cdot w = \lambda_x w$. Let us suppose that $xy^{k-1} \cdot w = \sum_{i=1}^{k-1} a_i^{(x)} y^i \cdot w$. We have

$$\begin{aligned} xy^k \cdot w &= yxy^{k-1} \cdot w + [x, y]y^{k-1} \cdot w \\ &= \left(\sum_{i=0}^{k-1} a_i^{(x)} y^{i+1} \cdot w \right) + \left(\sum_{i=0}^{k-1} a_i^{([x, y])} y^i \cdot w \right) \\ &= a_0^{([x, y])} w + \sum_{i=1}^k (a_{i-1}^{(x)} + a_i^{([x, y])}) y^i \cdot w, \end{aligned}$$

and we can see that $xy^k \cdot w \in \text{Span}(w, \dots, y^k \cdot w)$. We can finally end the proof of the theorem.

We have just proved that $\text{Tr}(x|_U) = \lambda_x \dim(U)$. Moreover, as \mathfrak{a} is an ideal, we know that $[x, y] \in \mathfrak{a}$ (we recall for the reader that $x \in \mathfrak{a}$ and $y \in \mathfrak{g} \setminus \mathfrak{a}$), which acts on U as a commutator and therefore has trace zero. But given that $\dim(U) \neq 0$, we necessarily have $\lambda_{[x, y]} = 0$. This completes the proof of the theorem. \square

Corollary 3.7.11 (Lie's theorem). *Let $\mathfrak{g} \subset \text{End}V$ be a concrete solvable Lie algebra. There is a basis of V , such that the matrix of the action of every element of \mathfrak{g} , with respect to this basis, is an upper triangular matrix.*

We let the proof of the preceding corollary as an easy exercise to the reader. The idea is to follow the same lines of reasoning that in corollary 3.7.5, together with the following fact that extensions of solvable Lie algebras are solvable (example 3.7.9 (ii)).

Example 3.7.12. It is clear that $\mathcal{C}(\mathfrak{b})^1 = [\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. So $\mathcal{C}(\mathfrak{b})^1$ is nilpotent and therefore solvable. This immediately implies that \mathfrak{b} is solvable.²¹

Exactly as we have reasoned in corollary 3.7.7 we can use Lie's theorem to prove

Corollary 3.7.13. *Let $\mathfrak{g} \subset \text{End}(V)$ be concrete solvable Lie algebra. There is a flag \mathcal{F} of V such that $\mathfrak{g} \subset \mathfrak{b}(\mathcal{F})$.*

Example 3.7.14. If \mathcal{F} denotes the canonical flag (3) in $V = k^n$, then $\text{Lie}(\mathbb{B}) = \mathfrak{b} = \mathfrak{b}(\mathcal{F})$.

We end this subsection with the following important result.

Definition 3.7.15. *Let V be a finite-dimensional vector space, and let us take $x, y \in \text{End}(V)$. We define the **Killing form** B_V to be*

$$B_V(\bullet, \bullet) : \begin{array}{ccc} \text{End}(V) \times \text{End}(V) & \rightarrow & k \\ (x, y) & \mapsto & B_V(x, y) := \text{Tr}(xy). \end{array}$$

²¹Using the examples 3.7.14 and 3.7.9 (iii) we see that \mathfrak{b} is not nilpotent. We can also see this directly from $\mathcal{C}(\mathfrak{b})_n = [\mathfrak{b}, \mathcal{C}(\mathfrak{b})_{n-1}] = [\mathfrak{b}, \mathfrak{n}] = \mathfrak{n}$.

If $\mathfrak{g} \subseteq \text{End}(V)$ is a concrete Lie algebra, we will omit the subscript \mathfrak{g} when referring to the bilinear form

$$B(x, y) := B_{\mathfrak{g}}(\text{ad}(x), \text{ad}(y)).$$

In this case we will denote by \mathfrak{g}^{\perp} the **orthogonal** complement of \mathfrak{g} taken with respect to $B(\bullet, \bullet)$.

We intend to prove a criterion of solvability involving the killing form. We will need the following technical lemma whose proof can be found in [34, Chapter II, lemma 4.3].

Lemma 3.7.16. *Let V be a finite dimensional vector space, and U and W two subspaces of $\text{End}(V)$. Let us consider the subset L of $\text{End}(V)$ defined by*

$$L := \{l \in \text{End}(V) \mid [l, U] \subset W\},$$

and let us suppose that $x \in L$ is such that for every $l \in T$, we have $\text{Tr}(xl) = 0$. Then x is nilpotent.

Theorem 3.7.17 (Cartan's criterion). *Let $\mathfrak{g} \subset \text{End}(V)$ be a Lie algebra. Then \mathfrak{g} is solvable if and only if $\mathcal{C}(\mathfrak{g})^1 \subset \mathfrak{g}^{\perp}$.*

Proof. By corollary 3.7.13 there exists a flag \mathcal{F} such that $\mathfrak{g} \subseteq \mathfrak{b}(\mathcal{F})$. This implies $\mathcal{C}(\mathfrak{g})^1 \subset \mathcal{C}(\mathfrak{b}(\mathcal{F}))^1 = \mathfrak{n}(\mathcal{F})$. In particular, for $x \in \mathfrak{g}$ and $y \in \mathcal{C}(\mathfrak{g})^1$ the product $xy \in \mathfrak{n}(\mathcal{F})$ and therefore $B(x, y) = \text{Tr}(xy) = 0$, i.e. $\mathcal{C}(\mathfrak{g})^1 \subset \mathfrak{g}^{\perp}$.

Let us suppose now that $\mathcal{C}(\mathfrak{g}) \subset \mathfrak{g}^{\perp}$, and let us consider the set L in the previous lemma with $U = \mathfrak{g}$ and $W = \mathcal{C}(\mathfrak{g})^1$. Let $l \in L$ and $x = \sum_{i=0} [y_i, z_i] \in \mathcal{C}(\mathfrak{g})^1$. We have

$$\text{Tr}(lx) = \sum_i \text{Tr}(l[y_i, z_i]) = \sum_i \text{Tr}([l, y_i], z_i) = 0 \quad (34)$$

because $[l, y_i] \in \mathcal{C}(\mathfrak{g})^1$ and by hypothesis $\mathcal{C}(\mathfrak{g})^1 \subset \mathfrak{g}^{\perp}$. So x is nilpotent. By corollary 3.7.5 we can conclude that $\mathcal{C}(\mathfrak{g})^1$ is nilpotent and therefore \mathfrak{g} is solvable. \square

Corollary 3.7.18. *Let \mathfrak{g} be a Lie algebra such that for the Killing form $B(\bullet, \bullet)$ we have $\mathcal{C}(\mathfrak{g})^1 \subset \mathfrak{g}^{\perp}$. Then \mathfrak{g} is solvable.*

Proof. Let us consider the **adjoint representation** $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$. By theorem 3.7.17 we know that $\text{ad}(\mathfrak{g})$ is solvable. Moreover, given that $\ker(\text{ad}) = \mathfrak{z}(\mathfrak{g})$, then $\ker(\text{ad})$ is abelian and therefore solvable. This implies that \mathfrak{g} is solvable being an extension of $\text{ad}(\mathfrak{g})$ by $\ker(\text{ad})$. \square

3.8 Semi-simple Lie algebras

The Lie algebra \mathfrak{sl}_2 is an important example of **semi-simple** Lie algebras (and even stronger, of **simple** Lie algebras). The study of its representations form the basis of the theory of representations of finite dimensional Lie algebras. In this subsection we intend to introduce the most general facts about semi-simple Lie algebras and we will work out \mathfrak{sl}_2 as an example. In fact, at the end of this subsection we will classify the irreducible representations of \mathfrak{sl}_2 .²² We will follow [79, 63].

²²Later, we will classify this representations by using **twisted differential operators**.

Definition 3.8.1. Let \mathfrak{g} be a finite dimensional non-abelian Lie algebra.

- (i) We say that \mathfrak{g} is **simple** if it has no non-trivial proper ideals.
- (ii) We say that \mathfrak{g} is **semi-simple** if it has no non-trivial abelian ideals.

Let us recall that

$$\mathfrak{sl}_2 = \{A \in M_{2 \times 2}(k) \mid \text{Tr}(A) = 0\}.$$

A k -basis is given by the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The reader can easily check the following relations

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h. \quad (35)$$

Lemma 3.8.2. The Lie algebra \mathfrak{sl}_2 is a simple Lie algebra.

Proof. Let us take \mathfrak{a} a non-zero ideal of \mathfrak{sl}_2 . Let $z = \gamma_1 e + \gamma_2 f + \gamma_3 h$ be a non-zero element of \mathfrak{a} . Using the relation (35) we see that $x := [h, [h, z]] = 4\gamma_1 e + 4\gamma_2 f$. As $z \in \mathfrak{a}$ and \mathfrak{a} is an ideal, we must have $x \in \mathfrak{a}$. Therefore $z - \frac{1}{4}x = \gamma_3 h \in \mathfrak{a}$. We analyse two cases.

$\gamma_3 \neq 0$ We immediately have $h \in \mathfrak{a}$ and the relations (35) implies that $\frac{1}{2}[h, e] = e \in \mathfrak{a}$. On the same fashion $f \in \mathfrak{a}$. We have proved $\mathfrak{a} = \mathfrak{sl}_2$.

$\gamma_3 = 0$ We have $[e, z] = \gamma_2 h \in \mathfrak{a}$ and $[f, z] = \gamma_1 h \in \mathfrak{a}$. Given that $z \neq 0$, we can conclude that $h \in \mathfrak{a}$ and we can proceed as in the previous case.

We have proved that $\mathfrak{a} = \mathfrak{sl}_2$. □

Lemma 3.8.3. A Lie algebra \mathfrak{g} is semi-simple if and only if \mathfrak{g} has no non-trivial solvable ideals.

Proof. Let us start assuming that \mathfrak{g} is semi-simple, and let us take \mathfrak{a} a non-trivial solvable ideal. By definition, the upper central series of \mathfrak{a} vanishes, i.e, there exists $n \in \mathbb{Z}_{>0}$ such that $\mathcal{C}(\mathfrak{a})^n = [\mathcal{C}(\mathfrak{a})^{n-1}, \mathcal{C}(\mathfrak{a})^{n-1}] = 0$. This implies that $\mathcal{C}(\mathfrak{a})^{n-1}$ is a non-trivial ideal, which is a contradiction. The converse is immediate. □

If \mathfrak{a}_1 and \mathfrak{a}_2 are solvable ideals of \mathfrak{g} , then the canonical isomorphism of Lie algebras (this is a easy exercise)

$$\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{\mathfrak{a}_1} \simeq \frac{\mathfrak{a}_2}{\mathfrak{a}_1 \cap \mathfrak{a}_2}$$

tells us that $\mathfrak{a}_1 + \mathfrak{a}_2$ is also solvable. Therefore, there is a largest solvable ideal \mathfrak{r} of \mathfrak{g} . We will call this ideal the **radical** of \mathfrak{g} . In light of the preceding lemma we can say that \mathfrak{g} is semi-simple if its radical \mathfrak{r} is zero.

We have the following result from Cartan's Criterion (theorem 3.7.17).

Proposition 3.8.4. A Lie algebra \mathfrak{g} is semi-simple if and only if the Killing form $B(\bullet, \bullet)$ is non-degenerate.

Proof. The same reasoning given in (3.7) tells us that

$$\mathfrak{s} := \{x \in \mathfrak{g} \mid B(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$$

is an ideal. The assertions of the theorem lie on \mathfrak{s} . Let us suppose that \mathfrak{g} is semi-simple. Given that $\mathcal{C}(\mathfrak{s})^\perp \subset \mathfrak{s}^\perp$ (by definition) corollary 3.7.18 we can conclude that \mathfrak{s} is solvable in \mathfrak{g} , but given that \mathfrak{g} is semi-simple then \mathfrak{s} must be trivial by lemma 3.8.3.

Let us suppose now that $\mathfrak{s} = 0$ and let us take an abelian ideal \mathfrak{a} of \mathfrak{g} . Let $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. We set $\phi := \text{ad}(x) \circ \text{ad}(y)$. Given that \mathfrak{a} is an abelian ideal $\phi(\mathfrak{g}) \subset \mathfrak{a}$ and $\phi(\mathfrak{a}) = 0$. This means that $\phi^2(\mathfrak{a}) = 0$ and ϕ is nilpotent. Since $\text{cher}(k) = 0$, we can conclude that $\text{Tr}(\phi) = \text{Tr}(\text{ad}(x) \circ \text{ad}(y)) = B(x, y) = 0$. This tells us that $x \in \mathfrak{s}$, so $\mathfrak{s} = 0$. This completes the proof of the proposition. \square

Let us organise the preceding results as follows. The following assertions are equivalent:

- (i) The Lie algebra \mathfrak{g} is semi-simple.
- (ii) The radical \mathfrak{r} vanishes.
- (iii) The Killing form $B(\bullet, \bullet)$ is non-degenerated.

In what follows we will use the notation $\mathcal{C}(\mathfrak{g}) := \mathcal{C}(\mathfrak{g})^\perp := [\mathfrak{g}, \mathfrak{g}]$.

Corollary 3.8.5. *If \mathfrak{g} is semi-simple, then $\mathcal{C}(\mathfrak{g}) = \mathfrak{g}$.*

Proof. We have $\mathfrak{g} = \mathcal{C}(\mathfrak{g}) \oplus \mathcal{C}(\mathfrak{g})^\perp$. Let $x \in \mathcal{C}(\mathfrak{g})$ and let $y, z \in \mathfrak{g}$. By (3.7), we know that $B([x, y], z) = B(x, [y, z]) = 0$. Given that in our case the Killing form is non-degenerate we have $[x, y] = 0$. This means that $y \in z(\mathfrak{g})$, which is a solvable ideal and therefore, by semi-simplicity, $z(\mathfrak{g}) = 0$. We can conclude that $\mathcal{C}(\mathfrak{g})^\perp = 0$ and $\mathcal{C}(\mathfrak{g}) = \mathfrak{g}$. \square

As we have stated in the introduction of this subsection, we aim to study representations of semi-simple Lie algebras. The following digression about **generalized eigenspaces and generalized weight spaces** will be particularly important. We will always suppose that V is a finite-dimensional k -vector space.

Definition 3.8.6. *Let $\vartheta \in \text{End}(V)$ and $\lambda \in k$. The subspace*

$$V_{[\lambda]}(\vartheta) := \{v \in V \mid (\vartheta - \lambda I)^N v = 0 \text{ for some integer } N = N(v)\}$$

*is called a **generalized eigenspace** of ϑ with eigenvalue λ .*

It is clear that the usual eigenspace $V_\lambda(\vartheta) := (\ker(\vartheta - \lambda I))$ of ϑ with eigenvalue λ is a subspace of $V_{[\lambda]}$.

Example 3.8.7. By Engel's theorem (3.7.5) or the discussion about eigenvalues of nilpotent endomorphisms (page 43) we see that if ϑ is nilpotent, then $V = V_{[0]}$.

We recall for the reader that we have assumed that k is algebraically closed of zero characteristic.

Proposition 3.8.8. Let $\vartheta \in \text{End}(V)$. Let $\lambda_1, \dots, \lambda_s$ be all the eigenvalues of ϑ and n_1, \dots, n_s be their multiplicities. We have the **generalized eigenspace decomposition**:

$$V = \bigoplus_{i=1}^s V_{[\lambda_i]}(\vartheta)$$

where $\dim(V_{[\lambda_i]}(\vartheta)) = n_i$.

Proof. Using Jordan canonical form in some basis e_1, \dots, e_n , we have that the matrix of ϑ is of the form

$$\vartheta = \begin{pmatrix} J_{\lambda_1} & & & \\ & J_{\lambda_2} & & \\ & & \ddots & \\ & & & J_{\lambda_n} \end{pmatrix},$$

where J_{λ_i} is an $n_i \times n_i$ matrix with λ_i on the diagonal, 0 or 1 in each entry just above the diagonal, and zero elsewhere. Let

$$V_{[\lambda_i]}(\vartheta) = \text{Span}(e_1, \dots, e_{n_1}), V_{[\lambda_2]}(\vartheta) = \text{Span}(e_{n_1+1}, \dots, e_{n_1+n_2}), \dots,$$

so that J_{λ_i} acts on $V_{[\lambda_i]}(\vartheta)$, in other words $V_{[\lambda_i]}(\vartheta)$ are ϑ -invariant and $\vartheta|_{V_{[\lambda_i]}(\vartheta)} = \lambda_i I_{n_i} + \mathfrak{n}_i$, with \mathfrak{n}_i nilpotent. \square

Definition 3.8.9. Let \mathfrak{g} be a Lie algebra and $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ a (finite-dimensional) representation, and $\lambda \in \mathfrak{g}^* := \text{Hom}(\mathfrak{g}, k)$ be a linear functional on \mathfrak{g} . The **generalized weight space** of \mathfrak{g} in V attached to λ is

$$\begin{aligned} V_\lambda^\mathfrak{g} &:= \{v \in V \mid (\rho(x) - \lambda(x)I)^N v = 0 \text{ for some } N = N(x), \text{ for all } x \in \mathfrak{g}\} \\ &= \bigcap_{x \in \mathfrak{g}} V_{[\lambda(x)]}(\rho(x)) \end{aligned}$$

Let us fix $x \in \mathfrak{g}$. From the preceding theorem we have the following decompositions

$$V = \bigoplus_{\gamma \in k} V_{[\gamma]}(\rho(x)),$$

For instance, if we consider the adjoint representation, we have

$$\mathfrak{g} = \bigoplus_{\gamma \in k} \mathfrak{g}_{[\gamma]}(\rho(x)),$$

where

$$\mathfrak{g}_{[\gamma]}(\rho(x)) := \{y \in \mathfrak{g} \mid (\text{ad}(x) - \gamma I)^N y = 0 \text{ for some } N \in \mathbb{N}\}.$$

We have the following important result [79, Lemma 19.3.8].

Proposition 3.8.10. Let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of the Lie algebra \mathfrak{g} . For every $x \in \mathfrak{g}$, we have the relation $\rho(\mathfrak{g}_{[\gamma_1]}(\rho(x)))V_{[\gamma_2]}(\rho(x)) \subset V_{[\gamma_1+\gamma_2]}(\rho(x))$.

Theorem 3.8.11. *Let \mathfrak{g} be a Lie algebra and $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ a representation. Let \mathfrak{h} be a nilpotent subalgebra of \mathfrak{g} . Then the following equalities hold*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}} \quad (36a)$$

$$\rho(\mathfrak{g}_\alpha^{\mathfrak{h}})V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}}. \quad (36b)$$

Proof. We analyse the following cases.

Case 1. For each $t \in \mathfrak{h}$, $\rho(t)$ has only one eigenvalue. Given that \mathfrak{h} is nilpotent, and therefore solvable, we can use Lie's theorem (theorem 3.7.10) to find define a **weight** $\lambda \in \mathfrak{h}^*$ such that $\lambda(t)$ will be the eigenvalue of $\rho(t)$ and $V = V_\lambda^{\mathfrak{g}}$.

Case 2. For some $t_0 \in \mathfrak{h}$, $\rho(t_0)$ has at least two distinct eigenvalue. Given that \mathfrak{h} is nilpotent, we have seen that $\text{ad}(t)$ is also nilpotent for all $t \in \mathfrak{h}$. Thus $\mathfrak{h} \subset \mathfrak{g}_{[\rho(t)]}$, for any $t \in \mathfrak{h}$. By the preceding proposition $\rho(\mathfrak{h})V_{[\rho(x)]} \subset V_{[\rho(x)]}$. Proposition 3.8.8 tells us that V can be written as a direct sum of the generalized spaces of t_0 and we have seen that each $V_{[\rho(t_0)]}$ is invariant under the action of \mathfrak{h} . In other words, $V_{[\rho(t_0)]}$ is also a representation of \mathfrak{h} such that $\dim(V_{[\rho(t_0)]}) < \dim(V)$. We may apply an inductive argument on $\dim(V)$ to establish the first equality of the theorem.

Let us establish the inclusion $\rho(\mathfrak{g}_\alpha^{\mathfrak{h}})V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}}$. To do that, let us take $\alpha, \lambda \in \mathfrak{h}^*$, and $x \in \mathfrak{g}_\alpha^{\mathfrak{h}}$, i.e. we are considering $x \in \mathfrak{g}_{[\alpha(t)]}(\rho(t))$ for all $t \in \mathfrak{h}$. By the preceding proposition $\rho(x)V_{\alpha(t)}(\rho(t)) \subset V_{[\lambda(t)+\alpha(t)]}(\rho(t))$ for all $t \in \mathfrak{h}$. We can conclude that if we take $v \in \bigcap_{t \in \mathfrak{h}} V_{[\lambda(t)]}(\rho(t))$, then $\rho(x)v \in \bigcap_{t \in \mathfrak{h}} V_{[\lambda(t)+\alpha(t)]}(\rho(x))$. Given that $\bigcap_{t \in \mathfrak{h}} V_{[\lambda(t)]}(\rho(t)) = V_\lambda^{\mathfrak{h}}$, by definition of a generalized weight space, we have the inclusion. \square

3.9 Cartan subalgebras

We start this subsection with the following definition which is a consequence of Jacobi identity.

Definition 3.9.1. *Let \mathfrak{h} be a Lie subalgebra of a Lie algebra \mathfrak{g} . Then*

$$N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$$

*is a subalgebra of \mathfrak{g} , called the **normalizer** of \mathfrak{h} .*

Remark 3.9.2. *By definition $\mathfrak{h} \subset N_{\mathfrak{g}}(\mathfrak{h})$ and the normalizer of \mathfrak{h} is the maximal subalgebra containing \mathfrak{h} .*

Lemma 3.9.3. *Let us suppose that \mathfrak{g} is a nilpotent Lie algebra. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Lie subalgebra, such that $\mathfrak{h} \subsetneq \mathfrak{g}$. Then $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$.*

Proof. Let us start by considering the lower central series

$$\mathfrak{g} \supset \mathcal{C}(\mathfrak{g})_1 \supset \cdots \supset \mathcal{C}(\mathfrak{g})_n = 0.$$

The last equation follow by nilpotency. Let us take $k \in \mathbb{N}$ to be the maximal positive integer such that $\mathcal{C}(\mathfrak{g})_k \not\subseteq \mathfrak{h}$. Clearly $1 < k < n$ and, by the choice made on k , we have $[\mathcal{C}(\mathfrak{g})_k, \mathfrak{h}] \subset \mathcal{C}(\mathfrak{g})_{k+1} \subset \mathfrak{h}$. In other words, $\mathcal{C}(\mathfrak{g})_{k+1} \subset N_{\mathfrak{g}}(\mathfrak{h})$ and therefore $\mathfrak{h} \subsetneq N_{\mathfrak{g}}(\mathfrak{h})$. \square

Definition 3.9.4. Let \mathfrak{g} be a Lie algebra. We say that a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a **Cartan subalgebra** if it satisfies the following two conditions:

(i) \mathfrak{h} is a nilpotent algebra.

(ii) \mathfrak{h} is itself normalizing. In other words $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Remark 3.9.5. Lemma 3.9.3 tells us that any Cartan subalgebra of \mathfrak{g} is in fact a maximal nilpotent subalgebra. In particular \mathfrak{g} is a Cartan subalgebra in itself if and only if \mathfrak{g} is a nilpotent Lie algebra.

Let us recall that we have denoted by \mathfrak{t} the abelian Lie subalgebra of $\mathfrak{gl}_n = M_n(k)$ consisting of diagonal matrices. An easy calculation shows that $N_{\mathfrak{gl}_n}(\mathfrak{t}) = \mathfrak{t}$, and therefore \mathfrak{t} is Cartan subalgebra of \mathfrak{gl}_n . In fact, we have the following general fact.

Proposition 3.9.6. Let $\mathfrak{g} \subset M_n(k)$ be a concrete Lie algebra which contains a diagonal matrix $d = (d_1, \dots, d_n)$ with $d_i \neq d_j$, for all $i \neq j$. Let $\mathfrak{h} := \mathfrak{g} \cap \mathfrak{t}$. Then \mathfrak{h} is a Cartan subalgebra.

Proof. Let us check that \mathfrak{h} satisfies the conditions in definition 3.9.4. We remark for the reader that $\mathfrak{h} \subset \mathfrak{t}$ is abelian and therefore nilpotent. Let us take now $y \in \mathfrak{g}$ such that $[y, \mathfrak{h}] \subset \mathfrak{h}$. In particular $[y, t] \in \mathfrak{h}$ for all $t \in \mathfrak{h}$. Given that

$$[y, t] = \left[\sum_{k=1}^n a_k e_{kk}, \sum_{i,j=1}^n b_{ij} e_{ij} \right] = \sum_{i,j=1}^n (a_i - a_j) b_{ij} e_{ij},$$

we see that $[y, t] \in \mathfrak{h}$ if and only if $b_{ij} = 0$ for all $i \neq j$. In other words $b \in \mathfrak{h}$. \square

Example 3.9.7. Let us consider $V = k^n$ and $\{e_i\}_{1 \leq i \leq n}$ the canonical base. Let us define the following full flags $\mathcal{F} = (V_i)_{1 \leq i \leq n}$ and $\mathcal{F}' = (V'_i)_{1 \leq i \leq n}$, by

$$V_i := \text{Span}(e_1, \dots, e_i) \quad \text{and} \quad V'_i := \text{Span}\{e_n, \dots, e_{n+1-i}\}$$

We have seen in the example 3.7.14 that the subspace $\mathfrak{b}(\mathcal{F})$ of matrices stabilising the flag \mathcal{F} is the Lie algebra of upper triangular matrices, and an easy calculation shows that $\mathfrak{b}(\mathcal{F}') = \mathfrak{t}$ is the Lie algebra of diagonal matrices. This implies that $\mathfrak{diag}(\mathcal{F}, \mathcal{F}') := \mathfrak{b}(\mathcal{F}) \cap \mathfrak{b}(\mathcal{F}') = \mathfrak{t}$ is a Cartan subalgebra. Furthermore, we have remarked in example 3.7.8 that the space $\mathfrak{n}(\mathcal{F})$ of matrices x such that $x \cdot V_i \subset V_{i-1}$ is the nilpotent Lie subalgebra of strictly upper triangular matrices. As before, an easy calculation shows that $\mathfrak{n}' := \mathfrak{n}(\mathcal{F}')$ is the nilpotent Lie algebra of strictly lower triangular matrices. We have

$$\begin{aligned} \mathfrak{gl}_n &= \mathfrak{n} \oplus \mathfrak{diag}(\mathcal{F}, \mathcal{F}') \oplus \mathfrak{n}' \\ &= \mathfrak{n} \oplus \mathfrak{t} \oplus \mathfrak{n}'. \end{aligned}$$

We also want to remark that $\mathfrak{b} = \mathfrak{b}(\mathcal{F}) = \mathfrak{n} \oplus \mathfrak{t}$.

3.9.1 Conjugacy of Cartan subalgebras

In the preceding subsection we have introduced Cartan subalgebras. In this subsection we aim to prove we essentially have a unique Cartan subalgebra by proving that they are all *conjugated*. In the literature this result is called **Chevalley's theorem** on conjugacy of Cartan subalgebras. To prove this theorem we give first some properties about the Killing form and the weight space decomposition.

Let \mathfrak{g} be a finite dimensional Lie algebra, and let us consider the characteristic polynomial

$$P_x(\lambda) := \det_{\mathfrak{g}}(\text{ad}(x) - \lambda) = (-\lambda)^{\dim(\mathfrak{g})} + p_{\dim(\mathfrak{g})-1}(x)(-\lambda)^{\dim(\mathfrak{g})-1} + \dots + \det(\text{ad}(x))$$

of $\text{ad}(x)$ for some $x \in \mathfrak{g}$. We can even see that the constant term of $P_x(\lambda)$ is zero because $\text{ad}(a)(a) = [a, a] = 0$. For any $1 \leq j \leq \dim(\mathfrak{g})$, it is known that p_j is a homogeneous polynomial on \mathfrak{g} of degree $\dim(\mathfrak{g}) - j$.

Definition 3.9.8. We define the **rank** of \mathfrak{g} as the smallest integer r such that p_r is not identically zero. An element $x \in \mathfrak{g}$ is called **regular** if $p_r(x) \neq 0$. The nonzero polynomial $p_r(x)$ of degree $\dim(\mathfrak{g}) - r$ is called the **discriminant** of \mathfrak{g} .

From now on, we will denote by $\text{rk}(\mathfrak{g})$ the positive integer r introduced in the previous definition.

Proposition 3.9.9. Let \mathfrak{g} be a Lie algebra, the set $\mathfrak{g}_{\text{reg}}$ of regular elements is a connected, dense open subset of \mathfrak{g} .²³

Proof. First of all, if C is the closed subset of \mathfrak{g} defined by the vanishing of the polynomial $p_{\text{rk}(\mathfrak{g})}$. By definition $\mathfrak{g}_{\text{reg}} = \mathfrak{g} \setminus C$, in other words $\mathfrak{g}_{\text{reg}}$ is open. Finally, given that C has non empty interior, we can conclude that $\mathfrak{g}_{\text{reg}}$ is dense. \square

We recall for the reader that for any $x \in \mathfrak{g}$ we have the following properties:

- (i) \mathfrak{g} is the direct sum of the $\mathfrak{g}_{[\gamma]}(\text{ad}(x))$,
- (ii) we have $[\mathfrak{g}_{\gamma_1}(\text{ad}(x)), \mathfrak{g}_{\gamma_2}(\text{ad}(x))] \subset \mathfrak{g}_{[\gamma_1 + \gamma_2]}(\text{ad}(x))$.
- (iii) From (ii), we can conclude that $\mathfrak{g}_{[0]}(\text{ad}(x))$ is a Lie subalgebra.

We have two easy exercises.

Lemma 3.9.10. For every $x \in \mathfrak{g}$, we have $\dim(\mathfrak{g}_{[0]}(\text{ad}(x))) \geq \text{rk}(\mathfrak{g})$. The equality is satisfied if x is regular.

Lemma 3.9.11. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} and suppose that there is a regular element $x \in \mathfrak{g}$ such that $x \in \mathfrak{h}$. Then $\mathfrak{h} = \mathfrak{g}_{[0]}(\text{ad}(x))$.

We end this subsection with the following discussion. Let \mathfrak{h} be a nilpotent Lie subalgebra of a semi-simple Lie algebra \mathfrak{g} (we recall for the reader that this is equivalent of saying that $B(\bullet, \bullet)$ is non-degenerated). We fix $x \in \mathfrak{g}_{\text{reg}}$. By [79, Lemma 19.8.5 (iii)], we have that $\mathfrak{g}_{[0]}(\text{ad}(x))$ is the unique Cartan subalgebra of \mathfrak{g} containing x . In particular, \mathfrak{g} has Cartan subalgebras. Moreover, we have seen in the proof of the theorem 3.8.11 that $\mathfrak{h} \subset \mathfrak{g}_{[0]}(\text{ad}(t))$ for every $t \in \mathfrak{h}$, therefore \mathfrak{h} is a Cartan subalgebra if and only if $\mathfrak{h} = \mathfrak{g}_0^{\mathfrak{h}}$ ([79, Theorem 19.8.6 (ii)]).

²³Here we fix a base of \mathfrak{g} and we identify $\mathfrak{g} \simeq k^{\dim}$. We recall for the reader that we are allowed to fix a basis because the induced Zariski topology doesn't depend of this basis if we identify \mathfrak{g} with $S(\mathfrak{g}^*)$ (the symmetric algebra of the algebraic dual of \mathfrak{g}).

From now on and until the end of this subsection, we will suppose that \mathfrak{h} is a **Cartan subalgebra**. The preceding discussion and theorem 3.8.11 tell us that if \mathfrak{h} is a Cartan Lie subalgebra then \mathfrak{g} admits the decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda \subset \mathfrak{h}^* \setminus \{0\}} \mathfrak{h} \oplus \mathfrak{g}_\lambda^{\mathfrak{h}}, \quad (37)$$

where $\Lambda \subset \mathfrak{h}^* \setminus \{0\}$ is the **set of roots**, consisting of those weights $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{g}_\lambda^{\mathfrak{h}} \neq 0$. Furthermore, for $\lambda, \nu \in \Lambda$, and $x \in \mathfrak{g}_\lambda^{\mathfrak{h}}$, $y \in \mathfrak{g}_\nu^{\mathfrak{h}}$, then by $[\mathfrak{g}_\lambda^{\mathfrak{h}}, \mathfrak{g}_\nu^{\mathfrak{h}}] \subset \mathfrak{g}_{\lambda+\nu}^{\mathfrak{h}}$ we have

$$(\text{ad}(x) \circ \text{ad}(y))^N(\mathfrak{g}_\nu^{\mathfrak{h}}) \subset \mathfrak{g}_{\lambda+N\nu}^{\mathfrak{h}}.$$

As $\lambda \neq 0$ and $\text{char}(k) \neq 0$, the set $\{\lambda + N\nu \mid N \in \mathbb{Z}_{>0}\}$ is an infinity set of distinct functionals. Given that there are only finitely many non-zero root spaces $\mathfrak{g}_\lambda^{\mathfrak{h}}$, there exists N such that $(\text{ad}(x) \circ \text{ad}(y))^N(\mathfrak{g}_\nu^{\mathfrak{h}}) = 0$. In other words, for $\lambda \in \Lambda \cup \{0\}$, the restriction of the Killing form $B(\bullet, \bullet)|_{\mathfrak{g}_\lambda^{\mathfrak{h}} \times \mathfrak{g}_{-\lambda}^{\mathfrak{h}}}$ is non-degenerated (here we uses the hypothesis of \mathfrak{g} being semi-simple). Now, Engel's theorem and a classical inductive argument on $\dim(\mathfrak{g})$, as we have already done in the notes, proves that for every $\lambda \in \Lambda$, there exists a basis of $\mathfrak{g}_\lambda^{\mathfrak{h}}$ such that for any $x, y \in \mathfrak{h}$ the matrix of $\text{ad}(x) \circ \text{ad}(y)|_{\mathfrak{g}_\lambda^{\mathfrak{h}}}$ with respects to this basis is upper triangular with unique eigenvalue $\lambda(x)\lambda(y)$ (cf. [79, Proposition 19.3.9 (iii) and theorem 19.8.6]). The **root decomposition** (37) and the preceding statement imply that for $x, y \in \mathfrak{h}$, we have the relation

$$B(x, y) = \sum_{\lambda \in \Lambda} \dim(\mathfrak{g}_\lambda^{\mathfrak{h}}) \lambda(x) \lambda(y). \quad (38)$$

Let us finally prove that Cartan subalgebras are commutative. Let $\lambda \in \Lambda$, and $x, y \in \mathfrak{h}$. Since $[x, \mathfrak{g}_\lambda^{\mathfrak{h}}] \subset \mathfrak{g}_\lambda^{\mathfrak{h}}$, $[y, \mathfrak{g}_\lambda^{\mathfrak{h}}] \subset \mathfrak{g}_\lambda^{\mathfrak{h}}$ (second part of theorem 3.8.11), and

$$(\text{ad}[x, y])|_{\mathfrak{g}_\lambda^{\mathfrak{h}}} = [\text{ad}(x)|_{\mathfrak{g}_\lambda^{\mathfrak{h}}}, \text{ad}(y)|_{\mathfrak{g}_\lambda^{\mathfrak{h}}}],$$

we see that the trace of $(\text{ad}[x, y])|_{\mathfrak{g}_\lambda^{\mathfrak{h}}}$ is zero. But, by (38), we know that this trace is equal to $\dim(\mathfrak{g}_\lambda^{\mathfrak{h}}) \lambda([x, y])$ and so $\lambda([x, y]) = 0$. We recall for the reader that we want to prove that \mathfrak{h} is commutative. In other words, for every $x, y \in \mathfrak{h}$ the bracket $[x, y] = 0$, and that we have proved that $\lambda([x, y]) = 0$ for every $\lambda \in \Lambda$. We will achieve our objective if we prove that Λ generates \mathfrak{h}^* as a vector space, but this is clear because $B(\bullet, \bullet)|_{\mathfrak{h} \times \mathfrak{h}}$ is non degenerated and \mathfrak{h} is a reflexive vector space (being finite-dimensional). Let us clarify this argument. Let us suppose by contradiction that the **root space** Λ does not generates \mathfrak{h}^* , and let us extend Λ to a basis $\{\alpha_j\}$ of \mathfrak{h}^* . By reflexivity $\mathfrak{h}^{**} \simeq \mathfrak{h}$ we can choose a dual basis $\{t_j\}$ of \mathfrak{h} , and by assumption, we can choose j such that α_j is not one of the simple roots. In other words, we have find a *non-zero* vector $t := t_j \in \mathfrak{h}$ such that $\lambda(t) = 0$ for every $\lambda \in \Lambda$. By (38) this implies that $[t, \mathfrak{h}] = \{0\}$ and by non-degeneracy we must have $t = 0$, which is a contradiction.

Summing up, we have proved the following facts. Let \mathfrak{g} be a semi-simple Lie algebra and \mathfrak{h} a Cartan subalgebra.

- (i) **Root decomposition:** $\mathfrak{g} = \bigoplus_{\lambda \in \Lambda} \mathfrak{g}_\lambda^{\mathfrak{h}}$. Under the hypothesis of \mathfrak{h} being a Cartan subalgebra this decomposition is true without the semi-simplicity hypothesis.
- (ii) The elements of Λ span the vector space \mathfrak{h}^* .
- (iii) The Lie algebra \mathfrak{h} is abelian.

Just one more tool before giving proving conjugacy.

Definition 3.9.12. *Let $\phi \in \text{Enf}(V)$ be a nilpotent operator. We define the exponential form by*

$$\exp(\phi) := I + \phi + \frac{1}{2!}\phi^2 + \frac{1}{3!}\phi^3 + \cdots + \frac{1}{n!}\phi^n + \cdots .$$

Given that ϕ is nilpotent this sum is finite.

The following facts are classical easy exercises for the reader [25, Chapter 1 (1.1.12)].

Lemma 3.9.13. *(i) If ϕ_1 and ϕ_2 are commuting nilpotent operators, we have the relation $\exp(\phi_1 + \phi_2) = \exp(\phi_1)\exp(\phi_2)$. In particular, $\exp(\phi_1)$ is always an invertible operator by $\exp(\phi_1)\exp(-\phi_1) = I$.*

*(ii) Let D be a **nilpotent (inner) derivation** of \mathfrak{g} , i.e. for all $x, y \in \mathfrak{g}$ we have the relation*

$$D([x, y]) = [D(x), y] + [x, D(y)].$$

Then $\exp(D)$ is an automorphism of \mathfrak{g} .

Let \mathfrak{g} be a finite-dimensional Lie algebra. Is straightforward to see that using Jacobi identity the map $\text{ad}(x)$ is an (inner) derivation of \mathfrak{g} . In what follows, we will denote by G the subgroup of the group of automorphisms of \mathfrak{g} generated by automorphisms of the form $\exp(x)$ for $x \in \mathfrak{g}$ and such that $\text{ad}(x)$ is nilpotent (properties of this group are discussed in detail in [59, Chapter 7]).

Theorem 3.9.14 (Chevalley's theorem). *Let \mathfrak{g} be a Lie algebra and G as in the preceding paragraph. Then any two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 are conjugated by G , i.e. there exists $\sigma \in G$ such that $\sigma(\mathfrak{h}_1) = \mathfrak{h}_2$.*

Proof. Let \mathfrak{h} be a Cartan subalgebra. We start by recalling that we have a generalized root space decomposition

$$\mathfrak{g} = \bigoplus_{\lambda \in \Lambda \cup \{0\}} \mathfrak{g}_\lambda^{\mathfrak{h}},$$

with $\mathfrak{g}_0^{\mathfrak{h}} = \mathfrak{h}$. The reasoning that we have given in page 55 to prove that the restriction of the Killing form $B(\bullet, \bullet)$ on $\mathfrak{g}_\lambda^{\mathfrak{h}} \times \mathfrak{g}_{-\lambda}^{\mathfrak{h}}$ is non-degenerated, proves that for any $\lambda \in \Lambda$ and $x \in \mathfrak{g}_\lambda^{\mathfrak{h}}$, the endomorphism $\text{ad}(x)$ is nilpotent.²⁴

²⁴In page 55 we use semi-simplicity to conclude the non degeneracy, the reader will realize that to prove nilpotence this hypothesis is not necessary. Nilpotence of \mathfrak{h} will be important here.

Existence of Zariski open subsets: we want to prove that there is a Zariski open subset of \mathfrak{g} consisting of images of elements of \mathfrak{h} under the action of the group G . Let us suppose $\Lambda \cup \{0\} = \{\lambda_i\}_{1 \leq i \leq k}$ and let us take $\{b_j\}_{j=1}^m$ a basis for $\bigoplus_{\lambda_i} \mathfrak{g}_{\lambda_i}^{\mathfrak{h}}$, consisting of a union of basis for each $\mathfrak{g}_{\lambda_i}^{\mathfrak{h}}$ (we are counting $\mathfrak{g}_0^{\mathfrak{h}} = \mathfrak{h}$). We have the following map

$$f : \begin{array}{ccc} \mathfrak{g} & \rightarrow & \mathfrak{g} \\ x = h + \sum_{j=1}^m \gamma_j b_j & \mapsto & f(x) := \exp(\gamma_1 \text{ad}(b_1)) \cdots \exp(\gamma_m \text{ad}(b_m))(h). \end{array}$$

The map f is well defined because each morphism $\text{ad}(b_j)$ is nilpotent by the result of the first paragraph. It is also possible to prove that f is a polynomial map. By [51, Lemma 8.18] or [18, Appendix I, proposition 3 and 4] we will find the Zariski open subset in question if we find a vector $x \in \mathfrak{g}$ such that the differential $d_x f$ is nonsingular. In other words, the map f is surjective by proposition 3.5.7. Let us try to compute the differential at any arbitrary point $x \in \mathfrak{g}$, evaluated at a point $h + b = t + \sum_j \gamma_j b_j$. We have

$$\begin{aligned} df|_{t=x}(h+b) &= \left. \frac{d}{dt} \right|_{t=0} [f(x+t(h+b))] \\ &= \left. \frac{d}{dt} \right|_{t=0} [\exp(t\gamma_1 \text{ad}(b_1)) \cdots \exp(t\gamma_m \text{ad}(b_m))(x+th)]. \end{aligned}$$

So, to compute this derivative, it suffices to expand the function to those monomials of first order in t . We find

$$\begin{aligned} df|_{t=x}(h+b) &= \left. \frac{d}{dt} \right|_{t=0} \left[\left(\prod_{j=1}^m (I + t\gamma_j \text{ad}(b_j)) \right) (x+th) \right] \\ &= \left. \frac{d}{dt} \right|_{t=0} \left[(x+th) + \sum_{j=1}^m t\gamma_j (\text{ad}(b_j))(x) \right] \\ &= h + \sum_{j=1}^m \gamma_j [b_j, x] \\ &= h + [b, x]. \end{aligned}$$

Thus the linear operator $df|_{t=x}$ restricts to the identity on \mathfrak{h} and $-\text{ad}(x)$ on $\bigoplus_i \mathfrak{g}_{\lambda_i}^{\mathfrak{h}}$. On each space $\mathfrak{g}_{\lambda_i}^{\mathfrak{h}}$, the only eigenvalue of $-\text{ad}(x)$ is $-\lambda_i(x)$. Therefore if we find $h \in \mathfrak{h}$ such that $\lambda_i(h) \neq 0$ for each i , then $df|_{t=h}$ will act invertible on each generalized root space and so it will be a nonsingular operator of \mathfrak{g} .

Given that the functionals $\lambda_i \in \mathfrak{h}^*$ are nonzero, for every i we can find $h_i \in \mathfrak{h}$ such that $\lambda_i(h_i) \neq 0$. We have an identification

$$\begin{array}{ccc} \mathbb{A}_k^{|\Lambda|} & \rightarrow & \text{Span}(\{\lambda_i\}) \\ (\gamma_i) & \mapsto & h := \sum_i \gamma_i h_i. \end{array}$$

For each i , the condition $\lambda_i(h) \neq 0$ defines a nonempty Zariski open subset. The intersection of these subsets gives us the desired h . All in all, we can apply [51, Lemma 8.18], to find a nonempty Zariski open subset $U(\mathfrak{h})$ of the image $f(\mathfrak{g})$ of f . By definition $U(\mathfrak{h})$ consists of points of the form $\sigma(h)$ for some $\sigma \in G$ and $h \in \mathfrak{h}$. We remark for the reader that the arguments just presented hold for any Cartan subalgebra of \mathfrak{g} .

Let us take two Cartan subalgebras \mathfrak{h}_1 and \mathfrak{h}_2 . As we have pointed out, there exist two Zariski open subsets $U(\mathfrak{h}_1)$ and $U(\mathfrak{h}_2)$. Moreover, by definition, the subset $\mathfrak{g}_{\text{reg}}$ of regular elements is also a nonempty Zariski open subset, and the intersection $U(\mathfrak{h}_1) \cap \mathfrak{g}_{\text{reg}}U(\mathfrak{h}_2)$ is nonempty. In other words, there exists a regular element $x \in \mathfrak{g}$, vectors $h_i \in \mathfrak{h}_i$, and automorphisms $\sigma_i \in G$ such that $\sigma_1(h_1) = x = \sigma_2(h_2)$. By definition $h_1 = \sigma^{-1}(x)$ is also regular, and lemma 3.9.11 tells us that $\mathfrak{h} = \mathfrak{g}_{[0]}(\text{ad}(h_1))$. Analogously $\mathfrak{h}_2 = \mathfrak{g}_{[0]}(\text{ad}(h_2))$. If $\sigma := \sigma_2^{-1}\sigma_1 \in G$, we have

$$\sigma(\mathfrak{h}_1) = \sigma(\mathfrak{g}_{[0]}(\text{ad}(h_1))) = \mathfrak{g}_{[0]}(\text{ad}(\sigma(h_1))) = \mathfrak{g}_{[0]}(\text{ad}(h_2)) = \mathfrak{h}_2.$$

□

3.10 Complete irreducibility

Our next objective will be to classify complex semi-simple Lie algebras and to study their representations. In order to introduce these concepts we propose to treat first the case of $\mathfrak{g} = \mathfrak{sl}_2$. The arguments in this subsection follow word by word the ones given in [63]. We start with the following important theorem which can be stated for a general semisimple Lie algebra \mathfrak{g} .

Theorem 3.10.1. Complete irreducibility *Let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of semisimple Lie algebra \mathfrak{g} , and let us suppose that $W \subset V$ is a subrepresentation. Then there exists a subrepresentation W' that is complementary to W .*

Proof. We start by remarking that given that \mathfrak{g} is semisimple, we know that $\mathfrak{g} \simeq \rho(\mathfrak{g})$. So $\rho(\mathfrak{g})$ is semisimple and therefore the restriction of the Killing form $\kappa_V(\bullet, \bullet)$ to $\rho(\mathfrak{g})$ is non degenerated.

From the previous paragraph, we can take μ_1, \dots, μ_r being a basis for $\rho(\mathfrak{g})$, and $\mu_1^\vee, \dots, \mu_r^\vee$ a basis with respect to the Killing form. This allows us to consider $K \in \text{End}(V)$ defined by:

$$K(v) := \sum_{i=1}^r \mu_i(\mu_i^\vee(v)).$$

Its trace is calculated as follows:

$$\text{Tr}(K) = \sum_{i=1}^r \text{Tr}(\mu_i \circ \mu_i^\vee) = \sum_{i=1}^r \kappa(\mu_i, \mu_i^\vee) = r = \dim(\rho(\mathfrak{g})).$$

For the second equality we have used the identification $\rho(\mathfrak{g}) \simeq \rho(\mathfrak{g})^\vee$ via the Killing form. Now, given that W is preserved by $\rho(\mathfrak{g})$, we see that K preserves W . We will consider the following cases.

W is irreducible of co-dimension 1. By applying Shur's lemma [26, Lemma 1.7], we have that K is multiplication by a scalar on W , which must be non-zero because $\text{Tr}(K) > 0$. Given that W is of co-dimension 1, we can conclude that $V = W \oplus \text{Ker}(K)$ as \mathfrak{g} -modules.

W is of co-dimension 1 but not irreducible. Let us deal with this case by using induction on the dimension of V , being the case $\dim(V) = 1$ trivial. Now, for the inductive step, we use the fact that W is not irreducible to take $W_0 \subseteq W$ a non trivial subrepresentation of W . By the inductive hypothesis, we can find $W_0 \subset W_1 \subset W$ such that $V/W_0 = W/W_0 \oplus W_1/W_0$. Given that W_1 is also a representation which contains W_0 as a subrepresentation. So, using again the inductive hypothesis, we can find $U \subset W_1$ such that $W_1 = W_0 \oplus U$, and a dimensional argument allows us to conclude that that $V = W \oplus U$.

W does not have co-dimension 1. Passing to a quotient space, if necessary, we can suppose that W is irreducible. Let us consider the restriction map

$$\mathbf{res} : \mathrm{Hom}_k(V, W) \rightarrow \mathrm{Hom}_k(W, W).$$

An easy calculation shows that \mathbf{res} is even a morphism of \mathfrak{g} -modules, if $\mathrm{Hom}_k(V, W)$ is endowed with the \mathfrak{g} -structure

$$(xf)(v) := xf(v) - f(xv),$$

and $\mathrm{Hom}_k(W, W)$ is clearly a \mathfrak{g} -submodule. Now, let us consider the \mathfrak{g} -submodule $\mathrm{Hom}_{\mathfrak{g}}(W, W)$ of $\mathrm{Hom}_k(W, W)$ of \mathfrak{g} -morphisms. This is

$$\mathrm{Hom}_{\mathfrak{g}}(W, W) := \{f \in \mathrm{Hom}_k(W, W) \mid f(xw) = xf(w) \text{ for all } x \in \mathfrak{g}\}.$$

By the Schur's lemma, $\mathbf{rest}^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W))$ consists of those \mathfrak{g} -morphisms $V \rightarrow W$ whose restriction to W is equal to multiplication by a scalar on W . So, if $h : V \rightarrow W$ is the identity on W but zero elsewhere, then

$$\tilde{f}(v) := \begin{cases} f(v) & \text{if } v \in V \setminus W, \\ 0 & \text{if } v \in W \end{cases}$$

clearly lies in $\mathrm{Ker}(\mathbf{rest})$ and $f = \tilde{f} + ah$, for some scalar $a \in k$. This immediately implies that $\mathrm{Ker}(\mathbf{rest})$ is a \mathfrak{g} -submodule of $\mathbf{rest}^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W))$ with co-dimension one, and one of the previous cases implies that $\mathbf{rest}^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W)) = U \oplus \mathrm{Ker}(\mathbf{rest})$, for some $U \subset \mathbf{rest}^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W))$. Given that W is irreducible, the Schur's lemma also implies that \mathbf{res} is surjective (we have already use this in the prove!), and therefore it maps U surjectively onto $\mathrm{Hom}_{\mathfrak{g}}(W, W)$. In particular, we can find $\nu \in U$, such that $\mathbf{rest}(\nu) = Id$; which implies that $\mathrm{Ker}(\nu) \cap W = 0$. Moreover, **\mathfrak{g} -acts on U by zero because U is one-dimensional.** In other words

$$0 = (x\nu)(v) := x(\nu(v)) - \nu(xv),$$

and we can conclude that $\mathrm{Ker}(\nu)$ is a \mathfrak{g} -submodule of V . Let us finally prove that $V = W \oplus \mathrm{Ker}(\nu)$. To do that, we may extend a basis $\{w_1, \dots, w_s\}$ of W to a basis $\{w_1, \dots, w_s, w'_{s+1}, \dots, w'_r\}$ of V , and to prove that $\nu(w'_i) = 0$ for $s+1 \leq i \leq r$. In fact, by construction, $h(w'_i) = 0$ for all $s+1 \leq i \leq r$ and therefore $\nu(w'_i) = 0$. This ends the proof of the theorem. \square

Corollary 3.10.2. *A semisimple Lie algebra is a direct sum of simple Lie algebras.*

Remark 3.10.3. *Theorem 3.10.1 tells us that to classify the representations of \mathfrak{sl}_2 (and in general of any complex semisimple Lie algebra), we only need to identify the irreducible representations, because the other representations will be direct sums of these.*

3.10.1 Invariance of the Jordan canonical form

Let us recall that in the theorem 2.2.1 we have introduced the **Jordan canonical form** for the Lie algebra $\text{End}(V)$. In this subsection we aim to prove the following important fact which will be used in the classification of the representations of $\mathfrak{sl}_{2,\mathbb{C}}$. Throughout this subsection \mathfrak{g} will denote a (realisable²⁵) semisimple Lie algebra. We recall for the reader that this means that $\mathfrak{g} \subseteq \text{End}(W)$, for some complex finite-dimensional vector space W . Theorem 2.2.1 tells us that for every $x \in \mathfrak{g}$ there exists a semisimple $x_s \in \text{End}(W)$ and a nilpotent $x_n \in \text{End}(W)$ such that $x = x_s + x_n$ in $\text{End}(W)$. Let us see that this decomposition occurs in fact in \mathfrak{g} .

Lemma 3.10.4. *Let $\mathfrak{g} \subseteq \text{End}(W)$ be a Lie algebra. If \mathfrak{g} is semisimple, then for any $x \in \mathfrak{g}$, we have $x_s, x_n \in \mathfrak{g}$ as well.*

Proof. Given that \mathfrak{g} is supposed to be semisimple, corollary 3.8.5 tells that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. In particular, every $x \in \mathfrak{g}$ is a commutator and therefore has zero trace. In other words, if

$$\mathfrak{sl}(W) := \{\varphi \in \text{End}(W) \mid \text{Tr}(\varphi) = 0\},$$

then $\mathfrak{g} \subset \mathfrak{sl}(W)$. Moreover, by definition of x_s and x_n , we know that if $x \in \mathfrak{g}$ has zero trace, then x_s, x_n have zero trace as well. So $x_s, x_n \in \mathfrak{sl}(W)$.

Let us suppose for a while that W is not irreducible and for any subrepresentation $U \subset W$ we consider

$$\mathfrak{s}_U := \{\varphi \in \text{End}(W) \mid \varphi(U) \subseteq U \text{ and } \text{Tr}(\varphi|_U) = 0\}.$$

The discussion given at the beginning of the proof tells us that $\mathfrak{g} \subseteq \mathfrak{s}_U$, and also if $x \in \mathfrak{s}_U$ then $x_s, x_n \in \mathfrak{s}_U$. On the other hand, **It is also possible to prove that $p(x)\mathfrak{g} \subseteq \mathfrak{g}$ for any polynomial $p(t) \in \mathbb{C}[t]$** . In particular, by theorem 2.2.1, we can conclude that $x_s\mathfrak{g} \subseteq \mathfrak{g}$ and $x_n\mathfrak{g} \subseteq \mathfrak{g}$. In other words x_s and x_n are elements of the **normalizer** $\mathfrak{n} = \{\varphi \in \text{End}(V) \mid [\varphi, \mathfrak{g}] \subseteq \mathfrak{g}\}$. Let us consider the Lie algebra

$$\mathfrak{g}' := \left(\bigcap_{U \subseteq V} \mathfrak{s}_U \right) \cap \mathfrak{n}.$$

By definition of the **normalizer**, we know that \mathfrak{g} is an ideal in $\mathfrak{g}' \subseteq \mathfrak{n}$. Therefore, by theorem 3.10.1, there exists a subalgebra $\mathfrak{l} \subseteq \mathfrak{g}'$ such that $\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{l}$. Since $x_s, x_n \in \mathfrak{g}'$ by the first part of the theorem, we will get the result if we prove that $\mathfrak{g}' = \mathfrak{g}$. In other words, let us see that $\mathfrak{l} = 0$. Let $\varphi \in \mathfrak{l}$ and $U \subseteq V$ be an irreducible representation. By our definition of \mathfrak{g}' we know that $\varphi \in \mathfrak{s}_U$, which implies that φ preserves U , and by the Schur's lemma we have that $\varphi|_U$ is multiplication by a scalar γ , but this scalar must be equal to zero because $\text{Tr}(\varphi|_U) = 0$. This completes the proof of the lemma. \square

Theorem 3.10.5. *Let $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of a (realisable) semisimple Lie algebra \mathfrak{g} . If $x = x_s + x_n$ is the Jordan canonical form of $x \in \mathfrak{g}$, then $\rho(x_s) + \rho(x_n)$ is the Jordan canonical form of $\rho(x)$.*

²⁵In fact, by semisimplicity, we know that the adjoint representation ad embeds \mathfrak{g} in $\text{End}(\mathfrak{g})$.

Proof. The idea of the proof consists in proving that $\rho(x_s)$ is semisimple and $\rho(x_n)$ is nilpotent. This facts and the uniqueness in the Jordan decomposition give us $\rho(x)_s = \rho(x_s)$ and $\rho(x)_n = \rho(x_n)$.

Let us start by considering the special case when \mathfrak{g} is simple. In this case ρ induces an isomorphism $\mathfrak{g} \simeq \rho(\mathfrak{g})$ and therefore $\rho(x_s)$ is semisimple and $\rho(x_n)$ is nilpotent. On the other hand, if \mathfrak{g} is semisimple, then by theorem 3.10.1 we have \mathfrak{g} can be decomposed as a direct sum of simple Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$. So

$$\rho(\mathfrak{g}) = \rho(\mathfrak{g}_1) \oplus \rho(\mathfrak{g}_2) \oplus \cdots \oplus \rho(\mathfrak{g}_k)$$

where every $\rho(\mathfrak{g}_i)$ must be a simple algebra in $\rho(\mathfrak{g})$. This clearly implies that $\rho(x_s)$ is semisimple and $\rho(x_n)$ is nilpotent. \square

3.11 Irreducible representations of $\mathfrak{sl}_{2,\mathbb{C}}$

In this section we will classify the representations of $\mathfrak{sl}_{2,\mathbb{C}}$. In light of the theorem 3.10.1 and given that $\mathfrak{sl}_{2,\mathbb{C}}$ is in particular semisimple, it is enough to concentrate our effort in classifying its irreducible representations $\rho : \mathfrak{sl}_{2,\mathbb{C}} \rightarrow \text{End}(V)$. To do that, let us start by recalling that

$$\mathfrak{sl}_{2,\mathbb{C}} = \mathfrak{e} \oplus \mathfrak{h} \oplus \mathfrak{f}$$

where \mathfrak{e} , \mathfrak{h} and \mathfrak{f} are the 1-dimensional vector spaces spanned by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Moreover, we have the following relations

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

Now, by theorem 3.10.5 we know that $\rho(h)$ is diagonalizable and we can order the distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of h by their real part in a non-decreasing way.²⁶ We will denote by W_λ the eigenspace corresponding to λ . In particular $W_\lambda = 0$ if λ is not an eigenvalue of h .

Lemma 3.11.1. *For every $1 \leq i \leq k$, we have $eW_{\lambda_i} \subseteq W_{\lambda_i+2}$. In particular $eW_{\lambda_{k-1}} = eW_{\lambda_k} = 0$.*

Proof. Let $v \in W_{\lambda_i}$. The lemma clearly follows from the relations

$$\begin{aligned} [h, e]v &= 2ev \\ hev - ehv &= 2ev \\ hev - \lambda_i ev &= 2ev \\ hev &= (\lambda_i + 2)ev \end{aligned}$$

\square

²⁶For the sake of the discussion we will drop ρ from the notation. We will warn the reader if a confusion is possible.

The previous lemma indicates us the action of \mathfrak{e} on the eigenspaces of h . The reader might expect that \mathfrak{f} sends W_{λ_i} into W_{λ_i-2} . This being the case, let us reference it as a lemma whose proof almost follows the same relations given in the preceding lemma.

Lemma 3.11.2. *For every i , we have $fW_{\lambda_i} \subset W_{\lambda_i-2}$. In particular, there exists $N \in \mathbb{N}$ such that $F^N v = 0$ for all $v \in V$.*

Proof. The first part of the lemma has been already sketched. From this and given that $V = \bigoplus W_{\lambda_i}$ we conclude the second part of the lemma. \square

Now that we have understood how \mathfrak{e} and \mathfrak{f} act on the eigenspaces of h , we need to study the relations between these eigenspaces and to specify the values of the $\lambda_1, \dots, \lambda_k$. To do that, we start by enunciating the following technical result. The interested reader can find a very explicit proof in [63, Lemma 4.2.1]. We will also remark that in the proof of the lemma the author shows the following relation which we will use later

$$ef^k = f^{k-1}(h - 2(k-1)). \quad (39)$$

Lemma 3.11.3. *Let $v \in \ker(e)$. There exists a polynomial $p_k(t)$ such that $e^k f^k v = p_k(h)v$. Moreover*

$$p_k(h) = \prod_{j=1}^k j(h - (j-1)I_2),$$

where I_2 is the identity matrix of order 2.

Let us use the previous information to specify the eigenspaces W_{λ_i} . To do that, we introduce first the following notation.

Definition 3.11.4. *Let N_{λ_i} be the minimal positive integer such that $f^{N_{\lambda_i}} W_{\lambda_i} = 0$. We set*

$$S_{\lambda_i} := W_{\lambda_i} + fW_{\lambda_i} + \dots + f^{N_{\lambda_i}-1} W_{\lambda_i}.$$

By lemma 3.11.2 this is a subset of

$$W_{\lambda_i} + W_{\lambda_i-2} + \dots + W_{\lambda_i-2(N_{\lambda_i}-1)}.$$

Lemma 3.11.5. *If λ_k is an eigenvalue of maximal real value, then $S_{\lambda_k} = V$.*

Proof. The idea is to prove that S_{λ_k} is a $\mathfrak{sl}_{2,\mathbb{C}}$ -submodule of V which implies that $S_{\lambda_k} = V$ by irreducibility of V .

It is clear that S_{λ_k} is preserved under the action of f , and given that $f^i W_{\lambda_k} \subseteq W_{\lambda_k}$ we can conclude

$$hf^i W_{\lambda_k} = (\lambda_k - 2i)f^i W_{\lambda_k} = f^i W_{\lambda_k}.$$

This proves that h preserves S_{λ_k} . Let us finally show that $eS_{\lambda_k} \subset S_{\lambda_k}$. This follows from the relation (39) given that we have

$$\begin{aligned} eS_{\lambda_k} &= e(W_{\lambda_k} + fW_{\lambda_k} + \dots + f^{N_{\lambda_k}-1} W_{\lambda_k}) \\ &= eW_{\lambda_k} + efW_{\lambda_k} + \dots + ef^{N_{\lambda_k}-1} W_{\lambda_k} \\ &= 0 + hW_{\lambda_k} + f(h-2)W_{\lambda_k} + \dots + f^{N_{\lambda_k}-2}(h-2(N_{\lambda_k}-2))W_{\lambda_k} \\ &\subseteq W_{\lambda_k} + fW_{\lambda_k} + \dots + f^{N_{\lambda_k}-2} W_{\lambda_k} \\ &\subseteq S_{\lambda_k} \end{aligned}$$

This ends the proof of the lemma. \square

Finally by definition of S_{λ_k} and the previous lemma, we can conclude

Corollary 3.11.6. *The h eigenspaces W_{λ_i} are precisely $W_{\lambda_k}, fW_{\lambda_k}, \dots, f^{N_{\lambda_k}-1}W_{\lambda_k}$. In particular, the only eigenvalue of h are $\lambda_k, \lambda_k - 2, \dots, \lambda_k - 2(k-1)$.*

Lemma 3.11.7. *If W_{λ_i} is a non-zero eigenspace of h , then $\dim(W_{\lambda_i}) = 1$.*

Proof. Let us start by proving that $\dim(W_{\lambda_k}) = 1$. Let $0 \neq v \in W_{\lambda_k}$. The subspace $U := \mathbf{Span}(v) + \dots + f^{N_{\lambda_k}-1}\mathbf{Span}(v)$ is a $\mathfrak{sl}_{2,\mathbb{C}}$ -submodule of V and therefore $U = V$. This implies that $W_{\lambda_k} = \mathbf{Span}(v)$.

Finally, by corollary 3.11.6 we know that for any i there exists j such that $W_{\lambda_i} = f^j W_{\lambda_k}$. This proves the statement for the general case. \square

From corollary 3.11.6 if we know λ_k then we know all the eigenvalues of h . The preceding lemma also tells us that all the eigenvalues are different and so there are n distinct eigenvalues given by $\lambda_k, \lambda_k - 2, \dots, \lambda_k - 2(n-1)$. Let us give an explicit description of λ_k .

Lemma 3.11.8. *If $\dim(V) = n$ then $\lambda_k = n - 1$.*

Proof. Given the $h \in \mathfrak{sl}_{2,\mathbb{C}}$ and the Jordan canonical form is preserved, we have that $\mathrm{Tr}(h) = \sum_{i=0}^n \lambda_i = 0$. The preceding description of the eigenvalues of h , gives us

$$\begin{aligned} 0 &= \lambda_k + \lambda_k - \dots + \lambda_k - 2(n-1) \\ &= n\lambda_k - 2n(n-1)/2 \\ &= \lambda_k - (n-1). \end{aligned}$$

\square

Theorem 3.11.9. *There exists only one isomorphic class of n -dimensional representations of $\mathfrak{sl}_{2,\mathbb{C}}$.*

Proof. If $v \in W_{\lambda_k}$, then $v, fv, \dots, f^{n-1}v$ is a basis for V consisting of eigenvectors of h . Under this basis h is a diagonal matrix with eigenvalues $n-1, n-3, \dots, 1-n$ and f is the matrix of ones below the diagonal and zeros elsewhere.

On the other hand, from the relation (39) we know that

$$ef^i v = f^{i-1}(h - 2(i-1))v = f^{i-1}((n-1) - 2(i-1))v = (n-2i)f^{i-1}v.$$

This means that e acts via the matrix with values $n-2, n-4, \dots, 2-n$ on the line directly above the diagonal and zeros elsewhere. We conclude that given the dimension of the representation we can derive the action of e, f, h , up to a change of basis. \square

3.12 Root systems

This chapter is dedicated to the study and classification of the root systems. We will principally follow [35] and [64].

Definition 3.12.1. Let V be a finite dimensional (real) vector space and $\alpha \in V$ a non-zero vector. A **symmetry with vector** α is an automorphism $s \in \text{Aut}(V)$ such that

$$(i) \quad s(\alpha) = -\alpha.$$

(ii) The set $H := \{\nu \in V \mid s(\nu) = \nu\}$ is an hyperplane of V .

Using the notation $\langle \varphi, \nu \rangle := \varphi(\nu)$, for $\varphi \in V^\vee$ and $\nu \in V$, we have the following easy facts:

$$(i) \quad V = H \oplus \mathbb{R}\alpha.$$

(ii) There is a unique element $\alpha^\vee \in V^\vee$ such that $\langle \alpha^\vee, H \rangle = 0$ and $\langle \alpha^\vee, \alpha \rangle = 2$. In this case

$$s(\nu) = \nu - \langle \alpha^\vee, \nu \rangle \alpha.$$

(iii) If $\langle \alpha^\vee, \alpha \rangle = 2$, then the map s defined by $s(\nu) = \nu - \langle \alpha^\vee, \nu \rangle \alpha$ is a symmetry with vector α .

Lemma 3.12.2. Let $\alpha \in V \setminus \{0\}$ and Δ be a finite subset such that $V = \text{Span}_{\mathbb{R}}(\Delta)$. Then there exists at most one symmetry with vector α leaving Δ invariant.

Proof. Let us suppose that there exists two symmetries s and s' . Let us consider $\varphi := s \circ s'$ and observe that $u(\Delta) = \Delta$; this implies that u is a permutation of R and therefore, there exists $n \in \mathbb{Z}_{>0}$ such that $(u|_{\Delta})^n = \text{id}_{\Delta}$. By hypothesis, we have $u^n = \text{id}$, so we will prove that $u = \text{id}$ if we show that all the eigenvalues of u are 1; this follows from the fact that by hypothesis u induces the identity on the quotient $V/\mathbb{R}\alpha$. \square

Definition 3.12.3. Let V be a real vector space. A subset $\Delta \subseteq V \setminus \{0\}$ of a vector space V is called a **root system** in V if Δ satisfies the following conditions:

(i) for each $\alpha \in \Delta$, there exists a symmetry s_α with vector α leaving Δ invariant,

(ii) for each α and β in Δ , we have $\langle \alpha^\vee, \beta \rangle \in \mathbb{Z}$.

If Δ satisfies the preceding conditions, then the dimension of V is called the **rank of the system** and the elements $\alpha \in R$ are called the **roots of the system**. The element α^\vee is called the **dual root** of α .

Definition 3.12.4. A root system Δ is called **reduced** if the intersection of Δ with $\mathbb{R}\alpha$ for $\alpha \in \Delta$ is the set $\{\alpha, -\alpha\}$.

Remark 3.12.5. If Δ is a non reduced root system and $\alpha \in \Delta$ is a vector satisfying $|\Delta \cap \mathbb{R}\alpha| \geq 2$, then

$$\Delta \cap \mathbb{R}\alpha = \{\pm\alpha, \pm 2\alpha\} \quad \text{or} \quad \Delta \cap \mathbb{R}\alpha = \{\pm\alpha, \pm \frac{1}{2}\alpha\}$$

Example 3.12.6. (i) The only reduced root system of rank 1 is $\{\pm\alpha\}$. This is a root system of type A_1 .

•—•

(ii) There is only one non reduced root system of rank 2. In light on the preceding remark, this must be given by $\{\pm 2\alpha, \pm\alpha\}$.

(iii) We have the following root systems of rank 2:

- **Root system of type $A_1 \times A_1$.** This is given by

$$\Delta = \{\pm\alpha, \pm\beta\} \quad \text{with } \alpha = (1, 0) \text{ and } \beta = (0, 1).$$

- **Root systems of type A_2 .** This is given by

$$\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta)\} \quad \text{with } \alpha = (1, 0) \text{ and } \beta = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

- **Root systems of type $B_2 = C_2$.** This is given by

$$\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta)\} \quad \text{with } \alpha = (1, 0) \text{ and } \beta = (-1, 1).$$

- **Root systems of type G_2 .** This is given by

$$\Delta = \{\pm\alpha, \pm\beta, \pm(\alpha + \beta), \pm(2\alpha + \beta), \pm(3\alpha + \beta), \pm(3\alpha + 2\beta)\}$$

with

$$\alpha = (1, 0) \quad \text{and} \quad \beta = \left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right).$$

The Weyl group 3.12.7. Trough this section V will denote a real-vector space and Δ a root system in V . We will denote by $\text{Aut}(\Delta)$ the group of automorphisms in $\text{GL}(V)$ preserving Δ .

Definition 3.12.8. *The Weyl group of Δ is the subgroup of $\text{Aut}(\Delta)$ generated by the symmetries s_α for $\alpha \in \Delta$. This is denoted by $W(\Delta)$.*

In order to compute root systems it will be important to construct a bilinear form invariant under the action of the Weyl group. To do that, we can fix any positive bilinear form B on V and to construct a new form by

$$(u, v) := \frac{1}{|W(\Delta)|} \sum_{w \in W(\Delta)} B(w(u), w(v)).$$

It is clear that (\bullet, \bullet) is a positive definite symmetric bilinear form invariant under the action of the Weyl group. In particular, V has the structure of an Euclidean space. Moreover, by identifying V with its dual under the preceding bilinear form, it is possible to see that

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}. \tag{40}$$

In other words,

$$s_\alpha(v) = v - 2 \frac{(\alpha, v)}{(\alpha, \alpha)} \alpha.$$

Remark 3.12.9. Let Δ^\vee denote the set of all dual roots α^\vee . Then

- The set Δ^\vee is a root system.
- We have the following dual relations $(\alpha^\vee)^\vee = \alpha$ and $(\Delta^\vee)^\vee = \Delta$.

The root system Δ^\vee is called the **dual root system**. Furthermore the isomorphism

$$\varphi: \begin{array}{ccc} \text{GL}(V) & \rightarrow & \text{GL}(V^\vee) \\ u & \mapsto & {}^t u^{-1} \end{array}$$

restricts to an isomorphism between $W(\Delta)$ and $W(\Delta^\vee)$ because ${}^t s_\alpha = s_{\alpha^\vee}$.

Let us now try to understand the position of the vectors α and β given in the preceding examples. In the next proposition θ will denote the angle between α and β .

Proposition 3.12.10. *If α and β are non colinear, then we have the following 4 possibilities:*

- $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle = 0$ and $\theta = \pi/2$;
- $\langle \alpha^\vee, \beta \rangle = \langle \beta^\vee, \alpha \rangle = -1$ and $\theta = 2\pi/3$;
- $\langle \alpha^\vee, \beta \rangle = -2$, $\langle \beta^\vee, \alpha \rangle = -1$ and $\theta = 3\pi/4$;
- $\langle \alpha^\vee, \beta \rangle = -3$, $\langle \beta^\vee, \alpha \rangle = -1$ and $\theta = 5\pi/6$.

Proof. Using the identification (40) and the well-known formula for the cosinus, we have

$$2 \frac{|\beta|}{|\alpha|} \cos(\theta) = \langle \beta^\vee, \alpha^\vee \rangle.$$

Multiplying for the analogous expression for β , we have

$$4 \cos^2(\theta) = \langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle.$$

The proposition follows after remarking that $\langle \alpha^\vee, \beta \rangle \langle \beta^\vee, \alpha \rangle \in \mathbb{Z}$. □

Remark 3.12.11. In the preceding proposition we excluded the positive values of $\langle \alpha^\vee, \beta \rangle$ and $\langle \beta^\vee, \alpha \rangle$. This because in practice α and β are supposed to be **simple roots** and in this case these values are negative.

Basis and systems of simple roots 3.12.12. In practice, it is usual to study an important subset of the root system instead of the whole system. This subset encodes all the information of the roots.

Definition 3.12.13. *A subset S of Δ is called a **base** of Δ if the following conditions are satisfied:*

- (i) S is a basis for V ;
- (ii) every root $\beta \in \Delta$ can be written as a linear combination of elements in S with only positive coefficients, or only negative coefficients.

3.13 The Bernstein-Gelfand-Gelfand category \mathcal{O}

B Tensor categories and Tannaka duality

Coming soon!

4 Algebraic \mathcal{D} -modules

In the last century, the language of differential operators has been of relevant importance in the theory of partial differential equations and in the theory of representations. In this last field, the strength of the geometric methods that the theory of differential operators has given to the mathematician, it has allowed to achieve important results as the Kazhdan-Lusztig conjecture [32, chapter 12] and the classification of simple modules over the first Weyl algebra (example 4.15.3).

One of the most pleasant properties of the sheaf of differential operators is that over a smooth complex algebraic variety, it has noetherian sections over affine open subsets. In fact, if X is a curve²⁷, then the global sections of the sheaf of differential operators form a (left and right) noetherian \mathbb{C} -algebra ([74]). The question is therefore to know the structure of globally defined differential operators, when X is singular. In fact, Malgrange asks in [48] whether this \mathbb{C} -algebra need be finitely generated and noetherian, question that has been answered by I.N. Bernstein, I.M. Gel'fands and S.I. Gel'fands in [10]. They have shown that if X denotes the surface defined by the equation $x^3 + y^3 + z^3 = 0$, then the ring $\mathcal{D}(X)$ of (regular) differential operators on X is not noetherian. This is one of the reasons to consider smooth complex algebraic varieties. As we have seen, the flag varieties are examples of such a varieties.

Another good reason to consider smooth algebraic varieties is that the definition that we will give for the sheaf of **differential operators** makes sense if the structure sheaf \mathcal{O} gives rise to a family of local *regular rings*. In this case, the ring of differential operators is generated by \mathcal{O} and \mathcal{T} . However, this is not always true. the interested reader can find an example in [22, Chapter 3, exercise 3.8].

Remark 4.0.1. *Later, we will use the theorem of Kashiwara's equivalence (theorem 4.9.1), to give an adequate definition for the category of \mathcal{D} -modules when the variety is singular.*

Throughout this section X will denote an irreducible smooth algebraic variety of dimension d_X over the complex field \mathbb{C} with \mathcal{O}_X the structural sheaf. We will denote by \mathcal{T}_X the **tangent sheaf** (whose sections are called **vector fields**) and which is defined over an arbitrary open subset $U \subseteq X$ by

$$\mathcal{T}_X(U) := \{\theta \in \text{End}_{\mathbb{C}} \mathcal{O}_X(U) \mid \theta(fg) = \theta(f)g + f\theta(g) \ (f, g \in \mathcal{O}_X(U))\}.$$

On the other hand, we know that the diagonal morphism

$$\Delta : X \rightarrow X \times_{\text{Spec}(\mathbb{C})} X$$

²⁷Irreducible affine algebraic curve over \mathbb{C} .

is a closed embedding and therefore we can define the **cotangent sheaf** of X by $\Omega_X^1 := \Delta^*(\mathcal{I}/\mathcal{I}^2)$, where \mathcal{I} is the ideal sheaf of $\Delta(X)$. Moreover, in our case Ω_X^1 is a locally free sheaf of rank d_X .

Finally, we have a derivation $d : \mathcal{O}_X \rightarrow \Omega_X^1$, which satisfies $d(fg) = d(f)g + fd(g)$, for $f, g \in \mathcal{O}_X$, and gives rise to an isomorphism $\text{Hom}_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X) \simeq \mathcal{T}_X$ given by composition with d (the last reasoning is just the global version of [49] section 25 and the local description of d given in [29] remark 8.9.2). Locally, if we take a point $p \in X$, then $\mathcal{T}_{X,p}$ is given by (28).

4.1 Basic definitions

Let us start with the following theorem.

Theorem 4.1.1. *For each point $p \in X$, there exists an affine open neighbourhood V of p , regular functions $x_i \in \mathcal{O}_X(V)$, and vector fields $\partial_{x_i} \in \mathcal{T}_X(V)$ ($1 \leq i \leq d_X$) satisfying the conditions*

$$\begin{cases} [\partial_{x_i}, \partial_{x_j}] = 0, \quad \partial_{x_i}(x_j) = \delta_{ij} \quad (1 \leq i, j \leq d_X), \\ \mathcal{T}_V = \bigoplus_{i=1}^{d_X} \mathcal{O}_V \partial_{x_i}. \end{cases}$$

Moreover, we can choose the functions x_1, x_2, \dots, x_{d_X} so that they generate the maximal ideal \mathfrak{m}_p of the local ring $\mathcal{O}_{X,p}$ at p .

Proof. To soften the notation we will assume that $n = d_X$. It is a consequence of [29, chapter II proposition 8.7] that there exist functions x_1, x_2, \dots, x_n generating the maximal ideal \mathfrak{m}_p of $\mathcal{O}_{X,p}$ and that dx_1, dx_2, \dots, dx_n is a basis of the free $\mathcal{O}_{X,p}$ -module $\Omega_{X,p}^1$. Therefore, there exists an affine open neighbourhood V of p such that $\Omega_X^1(V)$ is a free module with basis dx_1, dx_2, \dots, dx_n over $\mathcal{O}_X(V)$. Taking the dual basis we get $\partial_{x_i}(x_j) = \delta_{ij}$. Finally, writing $[\partial_{x_i}, \partial_{x_j}] = \sum_{l=1}^n g_{ij}^l \partial_{x_l}$, we have $g_{ij}^l = [\partial_{x_i}, \partial_{x_j}]x_l = 0$. Hence $[\partial_{x_i}, \partial_{x_j}] = 0$. \square

Definition 4.1.2. *The set $\{x_i, \partial_{x_i} \mid 1 \leq i \leq d_X\}$ defined over an affine open neighbourhood of p and satisfying the conditions of the last theorem is called a **local coordinate system** at p .*

Hereafter, if there is no risk of confusion, we will use the notation $f \in \mathcal{O}_X$ for a local section f of \mathcal{O}_X . We will also identify \mathcal{O}_X with a subsheaf of $\text{End}_{\mathbb{C}}(\mathcal{O}_X)$ by considering $f \in \text{End}_{\mathbb{C}}(\mathcal{O}_X)$ as the endomorphism of \mathcal{O}_X given by $g \mapsto fg$.

Definition 4.1.3. *The \mathbb{C} -subalgebra of $\text{End}_{\mathbb{C}}(\mathcal{O}_X)$ generated by \mathcal{O}_X and \mathcal{T}_X is called the **sheaf of differential operators** and is denoted by \mathcal{D}_X .*

Theorem 4.1.1 gives us a local description of the sheaf \mathcal{D}_X . If $p \in X$, we can take a coordinate affine neighborhood $(U, \{x_i, \partial_{x_i}\}_{1 \leq i \leq d_X})$ at p . Hence we have

$$\mathcal{D}_U = \bigoplus_{\alpha \in \mathbb{N}^d} \mathcal{O}_U \underline{\partial}^\alpha,$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi index and $\underline{\partial}^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_d}^{\alpha_d}$.

Example 4.1.4. ([22, Chapter 1, section 1]) Let $X = \mathbb{A}_{\mathbb{C}}^n$. In this case, we know that $\mathcal{T}_{\mathbb{A}_{\mathbb{C}}^n} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \partial_{x_i}$. The global sections of the ring of differential operators $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ is known as the **n-th Weyl algebra**.²⁸

By definition of the differentiation of a product $\partial_{x_i} \cdot x_i(f) = x_i \cdot \partial_i(f) + f$, and we see that $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ is the \mathbb{C} -subalgebra of $\text{End}_C(\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n})$ generated by the operators $x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}$. By theorem 4.1.1 this operators satisfy the following **Weyl relations**:

$$[\partial_{x_i}, x_j] = \delta_{i,j} \quad \text{and} \quad [\partial_{x_i}, \partial_{x_j}] = [x_i, x_j] = 0. \quad (41)$$

Remark 4.1.5. Let U be an affine open subset of X and \mathcal{R} the \mathbb{C} -algebra generated by elements

$$\{\tilde{f}, \tilde{\theta} \mid f \in \mathcal{O}_X(U), \theta \in \mathcal{T}_X(U)\}$$

satisfying the following fundamental relations:

- (1) $\tilde{f}_1 + \tilde{f}_2 = \widetilde{f_1 + f_2}, \quad \tilde{f}_1 \tilde{f}_2 = \widetilde{f_1 f_2}$.
- (2) $\tilde{\theta}_1 + \tilde{\theta}_2 = \widetilde{\theta_1 + \theta_2}, \quad [\tilde{\theta}_1, \tilde{\theta}_2] = \widetilde{[\theta_1, \theta_2]}$,
- (3) $\tilde{f} \tilde{\theta} = \widetilde{f \theta}, \quad [\tilde{\theta}, \tilde{f}] = \widetilde{[\theta, f]}$.

The map defined by $\tilde{f} \mapsto f$ and $\tilde{\theta} \mapsto \theta$ defines an isomorphism between \mathcal{R} and $\mathcal{D}_X(U)$.

We end this subsection with the following definition.

Definition 4.1.6. We say that a sheaf \mathcal{M} is a left \mathcal{D}_X -module if $\mathcal{M}(U)$ is endowed with a left $\mathcal{D}_X(U)$ -module structure for each open subset U of X and these actions are compatible with the restriction morphisms.

4.2 Algebraic properties of \mathcal{D} -modules

We start this subsection giving to \mathcal{D}_X a structure of *filtered sheaf of noncommutative rings*. First of all, let $(U, \{x_i, \partial_{x_i}\})$ be a coordinate affine open subset of X . We define the **order filtration \mathbf{F}** of \mathcal{D}_U by

$$F_l \mathcal{D}_U = \sum_{|\alpha| \leq l} \mathcal{O}_U \partial^\alpha, \quad (42)$$

and more generally, for an arbitrary open subset V of X we can define the order filtration F of \mathcal{D}_X over V by

$$\begin{aligned} (F_l \mathcal{D}_X)(V) \\ = \{P \in \mathcal{D}_X(V) \mid \rho_{U,V}(P) \in F_l \mathcal{D}_X(U) \text{ for any affine open subset } U \subseteq V\}, \end{aligned}$$

where $\rho_{U,V} : \mathcal{D}_X(V) \rightarrow \mathcal{D}_X(U)$ is the restriction map and, for convenience, we set $F_l \mathcal{D}_X = 0$ if $l < 0$.

²⁸We will usually abuse of the notation and we will identify $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ with its global sections. This is possible in light of the remark 4.2.7.

Proposition 4.2.1. (i) $\{F_l\}_{l \in \mathbb{N}}$ is an increasing filtration of \mathcal{D}_X such that $\mathcal{D}_X = \bigcup_{l \in \mathbb{N}} F_l \mathcal{D}_X$ and each $F_l \mathcal{D}_X$ is a locally free \mathcal{O}_X -module of finite rank.

(ii) $F_0 \mathcal{D}_X = \mathcal{O}_X$ and $(F_l \mathcal{D}_X)(F_m \mathcal{D}_X) = F_{l+m} \mathcal{D}_X$.

(iii) If $P \in F_l \mathcal{D}_X$ and $Q \in F_m \mathcal{D}_X$, then $[P, Q] \in F_{l+m-1} \mathcal{D}_X$.

Proof. (i) and (ii) are obvious, so we will give the proof of (iii). As the Lie product is a bilinear operation is enough to see this property on an expression of the form $\partial^\alpha \in F_l \mathcal{D}_X$ and $\partial^\beta \in F_m \mathcal{D}_X$. In this case we can apply induction on the equality

$$[\partial^\alpha, \partial^\beta] = \partial^{\alpha_1} [\partial^{\alpha - \alpha_1 e_1}, \partial^\beta] + [\partial^{\alpha_1}, \partial^\beta] \partial^{\alpha - \alpha_1 e_1},$$

where $e_1 = (1, 0, \dots, 0)$ is a multi-index of length l . □

Remark 4.2.2. We have the following formula

$$F_l \mathcal{D}_X = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{O}_X) \mid [P, f] \in F_{l-1} \mathcal{D}_X \ (\forall f \in \mathcal{O}_X)\},$$

and the terms of degree 0 and 1 are \mathcal{O}_X and \mathcal{T}_X respectively.

Example 4.2.3. ([22, chapter 3, lemma 1.1]) Let us prove the last assertion in the previous remark for the Weyl algebra $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$. This is, we want to prove

$$F_1 \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} = \mathcal{T}_{\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}} \oplus \mathcal{O}_{\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}}.$$

Let $P \in F_1 \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ and let us consider $Q = P - P(1) \in F_1 \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$. By definition, if $f \in \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$, then $[Q, f] \in F_0 \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$, and therefore $[[Q, f], g] = 0$ for all $f, g \in \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$. Explicitly, one has

$$(Qf)g - (fQ)g - g(Qf) + g(fQ) = 0.$$

Applying this operator to $1 \in \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$, we get $Q(fg) + fQ(g) + gQ(f) - gfQ(1) = 0$ (because $Q(1) = 1$ by definition of Q). In other words $Q \in \mathcal{T}_{\mathbb{A}_{\mathbb{C}}^n}$, but given that $P = Q + P(1)$ then $P \in \mathcal{T}_{\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}} + \mathcal{O}_{\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}}$. To prove that the sum is in fact a direct sum, we use the fact that the following short exact sequence splits

$$0 \rightarrow \mathcal{T}_{\mathbb{A}_{\mathbb{C}}^n} \rightarrow F_1 \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \xrightarrow{P \mapsto P(1)} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \rightarrow 0.$$

The above results tell us that if we take an affine chart $(U, \{x_i, \partial_i\})$ then for the sheaf of commutative graded rings

$$\mathrm{gr}^F \mathcal{D}_X = \bigoplus_{l=0}^{\infty} \mathrm{gr}_l \mathcal{D}_X,$$

where $\mathrm{gr}_l \mathcal{D}_X = F_l \mathcal{D}_X / F_{l-1} \mathcal{D}_X$, we have

$$\mathrm{gr}_l \mathcal{D}_U = F_l \mathcal{D}_U / F_{l-1} \mathcal{D}_U = \bigoplus_{|\alpha|=l} \mathcal{O}_U \xi^\alpha \quad (\xi_i := \partial_{x_i} \bmod (F_0 \mathcal{D}_U)),$$

and therefore

$$\mathrm{gr}^F \mathcal{D}_U = \mathcal{O}_U[\xi_1, \dots, \xi_n].$$

The previous equality has the following useful reinterpretation (cf. [32, Page 17]). Let T^*X be the **cotangent bundle** of X and let $\pi : T^*X \rightarrow X$ be the projection. Regarding ξ_1, \dots, ξ_n as the coordinate system of the *cotangent space* $\bigoplus_{i=1}^n \mathbb{C}dx_i$, we have that $\pi_*\mathcal{O}_{T^*X}|_U = \mathcal{O}_U[\xi_1, \dots, \xi_n]$.

Now, as we have remarked (and proved in the example 4.2.3) we have a canonical map

$$\mathcal{T}_X \rightarrow \mathrm{gr}_1\mathcal{D}_X \rightarrow \mathrm{gr}\mathcal{D}_X.$$

Given that $\mathrm{gr}\mathcal{D}_X$ is in particular a sheaf of commutative algebras, the universal property of the symmetric algebra gives a unique morphism of sheaves of commutative algebras

$$\pi_*\mathcal{O}_{T^*X} \simeq \mathrm{Sym}(\mathcal{T}_X) \rightarrow \mathrm{gr}\mathcal{D}_X \quad (43)$$

(the first isomorphism is given by [29, Chapter II, section 5, exercise 5.17 (d)]), which is in fact an isomorphism by the reasoning given in the previous paragraphs.

The following example will be fundamental in the subsection 4.10.1.

Example 4.2.4 (The Bernstein filtration). As we have remarked, we have $F_0\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$ is a sheaf of infinite dimensional vector spaces. In this example, we will define a filtration $\mathcal{B} = \{\mathcal{B}_k\}_{k \in \mathbb{N}}$ of $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$, consisting of finite dimensional vector spaces.

Let $P = \sum_{(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^n} \gamma_{(\alpha, \beta)} x^\alpha \underline{\partial}^\beta \in \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$. We define the **degree** of P by

$$\mathrm{deg}(P) := \max\{|\alpha| + |\beta| : \gamma_{(\alpha, \beta)} x^\alpha \underline{\partial}^\beta \text{ is a monomial appearing in } P\},$$

and we denote by \mathcal{B}_k the finite dimension \mathbb{C} -vector space generated by all the differential operators of degree at most k . The resulting filtration is called the **Bernstein filtration**. It is an easy exercise to prove that the canonical map

$$\begin{array}{ccc} \mathrm{gr}^{\mathcal{B}}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) & \rightarrow & \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n] \\ x_i & \mapsto & x_i \\ \partial_{x_i} & \mapsto & \xi_i \end{array} \quad (44)$$

is an isomorphism (cf. [22, Chapter 3, theorem 3.1]).

In the rest of the section, we will show some algebraic and homological properties of certain special categories of \mathcal{D}_X -modules. We will be interested in the category of quasi-coherent \mathcal{O}_X -modules (resp. of coherent \mathcal{O}_X -modules) which will be denoted by $\mathbf{Mod}_{qc}(\mathcal{O}_X)$ (resp. by $\mathbf{Mod}_c(\mathcal{O}_X)$) and we keep the notation $\mathbf{Mod}_{qc}(\mathcal{D}_X)$ for the category of \mathcal{O}_X -quasi-coherent \mathcal{D}_X -modules (resp. $\mathbf{Mod}_c(\mathcal{D}_X)$ denotes the category of \mathcal{O}_X -coherent \mathcal{D}_X -modules).

Remark 4.2.5. *It can be shown that \mathcal{M} is a coherent \mathcal{D}_X -module if and only if it is quasi-coherent over \mathcal{O}_X and locally finitely generated over \mathcal{D}_X (cf. [32, Proposition 1.4.9 (ii)]).*

Definition 4.2.6. *A smooth algebraic variety X is called \mathcal{D}_X -affine if the following conditions are satisfied:*

(a) the global section functor $\Gamma(X, \bullet) : \mathbf{Mod}_{qc}(\mathcal{D}_X) \rightarrow \mathbf{Mod}(\Gamma(X, \mathcal{D}_X))$ is exact,

(b) if $\Gamma(X, \mathcal{M}) = 0$ for $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$, then $\mathcal{M} = 0$.

Remark 4.2.7. Any smooth affine algebraic variety is \mathcal{D} -affine.

Affinity is an important property which helps us to *algebrize* the category of quasi-coherent \mathcal{D} -modules. Let us concretely explain this.²⁹ Let us suppose that X is \mathcal{D}_X -affine and let us take $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$. From the above definition, if we apply the global sections functor to the exact sequence

$$0 \rightarrow \mathcal{M}_0 \rightarrow \mathcal{M} \rightarrow \mathcal{M}/\mathcal{M}_0 \rightarrow 0,$$

where \mathcal{M}_0 is the image of the natural morphism $\mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)} \Gamma(X, \mathcal{M}) \rightarrow \mathcal{M}$, we obtain $\Gamma(X, \mathcal{M}/\mathcal{M}_0) = 0$ and therefore $\mathcal{M} = \mathcal{M}_0$.

For $F \in \mathbf{Mod}(\Gamma(X, \mathcal{D}_X))$ a classical five-lemma argument style shows that the canonical morphisms

$$\alpha_{\mathcal{M}} : \mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)} \Gamma(X, \mathcal{M}) \rightarrow \mathcal{M}, \quad \beta_F : F \rightarrow (\Gamma(X, \mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)} F))$$

are isomorphisms and it is straightforward to verified that the **localization functor** $\mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)} (\bullet)$ is left adjoint to $\Gamma(X, \bullet)$.

Proposition 4.2.8. Let's suppose that X is \mathcal{D}_X -affine.

(i) Every $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ is generated over \mathcal{D}_X by its global sections.

(ii) The functor

$$\Gamma(X, \bullet) : \mathbf{Mod}_{qc}(\mathcal{D}_X) \rightarrow \mathbf{Mod}(\Gamma(X, \mathcal{D}_X))$$

gives an equivalence of categories.

Proof. We have already proved (i). To see (ii) we must check that $\Gamma(X, \bullet)$ is surjective (which comes from $\alpha_{\mathcal{M}}$) and fully faithful. So, let $\psi : \Gamma(X, \mathcal{M}) \rightarrow \Gamma(X, \mathcal{N})$, where $\mathcal{N} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$, be a morphism of $\Gamma(X, \mathcal{D}_X)$ -modules and let's define $\phi := \alpha_{\mathcal{N}} \circ (id_{\mathcal{D}_X} \otimes \psi) \circ \alpha_{\mathcal{M}}^{-1}$. The reader can easily verify that $\Gamma(X, \phi) = \psi$. \square

Now, let us consider the case when $F \in \mathbf{Mod}_f(\Gamma(X, \mathcal{D}_X))$ and $\mathcal{M} \in \mathbf{Mod}_c(\mathcal{D}_X)$. It's clear that $\mathcal{D}_X \otimes_{\Gamma(X, \mathcal{D}_X)} F$ belongs to $\mathbf{Mod}_c(\mathcal{D}_X)$. On the other hand, as \mathcal{M} is generated by its global sections and X is quasi-compact, we see that \mathcal{M} is globally generated by finitely many elements of $\Gamma(X, \mathcal{M})$. This means that we have a surjective morphism $\mathcal{D}_X^m \rightarrow \mathcal{M} \rightarrow 0$ for some $m \in \mathbb{N}$ and therefore $\Gamma(X, \mathcal{M})$ belongs to $\mathbf{Mod}_f(\Gamma(X, \mathcal{D}_X))$. We have

Proposition 4.2.9. Let us suppose that X is \mathcal{D}_X -affine. The equivalence given in proposition 4.2.8 induces the equivalence

$$\mathbf{Mod}_c(\mathcal{D}_X) \simeq \mathbf{Mod}_f(\Gamma(X, \mathcal{D}_X)).$$

²⁹A quick proof of proposition 4.2.8 can be found in [8], where the authors remarks that \mathcal{D} is projective generator of the category $\mathbf{Mod}_{qc}(\mathcal{D}_X)$, and therefore [6, Theorem 1.4] gives the result.

Finally, let us suppose again that $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$. Let us take an affine open covering $\{U_i\}_i$ and let $j_k : U_k \rightarrow X$ be the open embedding. By proposition 4.2.8 we can embed $j_k^* \mathcal{M}$ into an injective object $\mathcal{I}_k \in \mathbf{Mod}_{qc}(\mathcal{D}_{U_k})$ and setting $\mathcal{I} = \bigoplus j_{k*} \mathcal{I}_k$, we get a canonical embedding into an injective object of $\mathbf{Mmod}_{qc}(\mathcal{D}_X)$. Moreover, if $\mathcal{N} \in \mathbf{Mod}_{qc}(\mathcal{O}_{U_k})$ the relation

$$\mathrm{Hom}_{\mathcal{O}_{U_k}}(\mathcal{N}, \mathcal{I}_k) \simeq \mathrm{Hom}_{\mathcal{D}_{U_k}}(\mathcal{D}_{U_k} \otimes_{\mathcal{O}_{U_k}} \mathcal{N}, \mathcal{I}_k),$$

tells us that \mathcal{I}_k is flabby. This proves the following proposition.

Proposition 4.2.10. *Any $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ can be embedded into an injective object \mathcal{I} of $\mathbf{Mod}_{qc}(\mathcal{D}_X)$ which is flabby.*

When X is quasi-projective we have a dual version of the last proposition (cf. [32, proposition 1.4.18]). More exactly,

Proposition 4.2.11. *Let's suppose that X is a quasi-projective variety. Then, any $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ is a quotient of a locally free \mathcal{D}_X -module.*

In the preceding section we have shown that over an affine chart $(U, \{x_i, \partial_{x_i}\}_i)$ we get

$$\mathrm{gr}^F \mathcal{D}_X(U) = \mathcal{O}_X(U)[\xi_1, \dots, \xi_{d_X}], \quad \mathrm{gr}^F \mathcal{D}_{X,x} = \mathcal{O}_{X,x}[\xi_1, \dots, \xi_{d_X}].$$

In particular, $\mathrm{gr}^F \mathcal{D}_X(U)$ and $\mathrm{gr}^F \mathcal{D}_{X,x}$ are noetherian rings with global dimension $2d_X$ which implies that both $\mathcal{D}_X(U)$ and $\mathcal{D}_{X,x}$ are left noetherian rings whose global dimensions are smaller than or equal to $2d_X$ (cf. [32, D.2.6]). Applying this fact to the resolution

$$\dots \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_0 \rightarrow \mathcal{M} \rightarrow 0$$

of \mathcal{M} by locally free \mathcal{D}_X -modules given in the proposition 4.2.10 we see that, if we set $\mathcal{Q} = \mathrm{Coker}(\mathcal{P}_{2d_X+1} \rightarrow \mathcal{P}_{2d_X})$ then $\mathcal{Q}|_U$ is a projective object of $\mathbf{Mod}_{qc}(\mathcal{D}_U)$ and we get

Corollary 4.2.12. *Let $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$. There exists a finite resolution*

$$0 \rightarrow \mathcal{P}_{2d_X} \rightarrow \mathcal{P}_{2d_X-1} \rightarrow \dots \rightarrow \mathcal{P}_0 \rightarrow \mathcal{M} \rightarrow 0$$

of \mathcal{M} by locally projective \mathcal{D}_X -modules. If $\mathcal{M} \in \mathbf{Mod}_c(\mathcal{D}_X)$, we can take all \mathcal{P}_i 's to be of finite rank.

Remark 4.2.13. *Actually, one can show that \mathcal{M} admits a finite resolution of length d_X (cf. [32] 2.6.11).*

4.3 Operations with \mathcal{D} -modules

In this section we will define the **transfer bimodules** for a morphism $f : X \rightarrow Y$ of smooth algebraic varieties and the **side-changing operations**. We will construct the objects in full generality, and we will use Weyl algebras to specify the left or right action.

We start with a useful lemma.

Lemma 4.3.1. *Let \mathcal{M} be an \mathcal{O}_X -module. To extend the \mathcal{O}_X -module structure to \mathcal{D}_X is equivalent to give a \mathbb{C} -linear morphism*

$$\nabla : \mathcal{T}_X \rightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}),$$

satisfying the following conditions:

- (1) $\nabla_{f\theta}(s) = f\nabla_{\theta}(s),$
- (2) $\nabla_{\theta}(fs) = \theta(f)s + f\nabla_{\theta}(s),$
- (3) $\nabla_{[\theta_1, \theta_2]}(s) = [\nabla_{\theta_1}, \nabla_{\theta_2}](s).$

Proof. Let's suppose that \mathcal{M} is a \mathcal{D}_X -module. For $\theta \in \mathcal{T}_X$ and $s \in \mathcal{M}$ we define $\nabla_{\theta}(s) := \theta.s$. We must verify the conditions (1) – (3).

The first condition is just the compatibility between the \mathcal{O}_X -module structure and the action of \mathcal{T}_X , and (3) is the definition of the bracket. Finally, by the remark 4.1.5 we get the relation $[\theta, f] = \theta(f)$ and therefore, for $s \in \mathcal{M}$ we see that

$$\theta(f)(s) = [\theta, f](s) = \nabla_{\theta}(fs) - f\nabla_{\theta}(s).$$

Reciprocally, is straightforward to check that the operations $\tilde{f}.s = fs$ and $\tilde{\theta}.s = \nabla_{\theta}(s)$, respect the relations given in remark 4.1.5 and therefore define on \mathcal{M} a structure of \mathcal{D}_X -module compatible with \mathcal{O}_X . \square

Let $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_X)$. In other words, let us suppose that \mathcal{M} is endowed with a **connection**

$$\nabla : \mathcal{T}_X \rightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}).$$

This notions has a dual description. This means that the map $\nabla : \mathcal{T}_X \rightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M})$ is equivalence to a map

$$\begin{aligned} \nabla^{\vee} : \mathcal{M} &\rightarrow \Omega_X^1 \otimes_{\mathbb{C}} \mathcal{M} \\ m &\mapsto \sum_{i=1}^{d_X} (dx_i \otimes \nabla_{\partial_{x_i}}(m)). \end{aligned}$$

In fact, starting from ∇^{\vee} we can reconstruct ∇ by identifying $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X^1, \mathcal{O}_X)$ and then making

$$\begin{aligned} \nabla : \mathcal{T}_X &\rightarrow \mathcal{E}nd_{\mathbb{C}}(\mathcal{M}) \\ \theta &\mapsto [m \mapsto (\theta \otimes 1)\nabla^{\vee}(m)]. \end{aligned}$$

The first condition in lemma 4.3.1 implies that ∇^{\vee} respects the respective \mathcal{O}_X -structures and therefore it defines a map $\nabla^{\vee} : \mathcal{M} \rightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M}$. Let us use (2) in the previous lemma to compute $\nabla^{\vee}(fm)$ with $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$. We have

$$\begin{aligned} \nabla^{\vee}(fm) &= \sum (dx_i \otimes \nabla_{\partial_{x_i}}(m)) \\ &= \sum dx_i \otimes (\partial_{x_i}(f)m + f\nabla_{\partial_{x_i}}(m)) \\ &= f \sum dx_i \otimes \nabla_{\partial_{x_i}}(m) + \sum \partial_{x_i}(f)dx_i \otimes m \\ &= f\nabla^{\vee}(m) + df \otimes m. \end{aligned}$$

Let us finally reinterpret the third condition for the operator ∇^\vee . In fact, this has the following meaningful property. Let us consider

$$\begin{aligned} \nabla_p^\vee : \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{M} &\rightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \mathcal{M} \\ \omega \otimes m &\mapsto d\omega \otimes m + \sum_{i=1}^{d_x} dx_i \wedge \omega \otimes \nabla_{\partial_{x_i}}(m), \end{aligned}$$

and let us remark that

$$\begin{aligned} &\nabla_{p+1}^\vee(\nabla_p^\vee(\omega \otimes m)) \\ &= \nabla_{p+1}^\vee \left(d\omega \otimes m + \sum_{i=1}^{d_x} dx_i \wedge \omega \otimes \nabla_{\partial_{x_i}}(m) \right) \\ &= d^2\omega \otimes m + \sum_{j=1}^{d_x} dx_j \wedge d\omega \otimes \nabla_{\partial_{x_j}}(m) \\ &\quad + \sum_{i=1}^{d_x} \left(d(dx_i \wedge \omega) \otimes \nabla_{\partial_{x_i}}(m) + \sum_{j=1}^{d_x} dx_j \wedge dx_i \wedge \omega \otimes \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}}(m) \right) \\ &= \sum_{j=1}^{d_x} dx_j \wedge d\omega \otimes \nabla_{\partial_{x_j}}(m) + \sum_{i=1}^{d_x} (d^2x_i \wedge \omega - dx_i \wedge d\omega) \otimes \nabla_{\partial_{x_i}}(m) \\ &\quad + \sum_{i=1}^{d_x} \sum_{j=1}^{d_x} dx_j \wedge dx_i \wedge \omega \otimes \nabla_{\partial_{x_j}} \nabla_{\partial_{x_i}}(m) \\ &= \sum_{j=1}^{d_x} dx_j \wedge d\omega \otimes \nabla_{\partial_{x_j}}(m) - \sum_{i=1}^{d_x} dx_i \wedge d\omega \otimes \nabla_{\partial_{x_i}}(m) \\ &= 0. \end{aligned}$$

We remark for the reader that in the fourth equality we have used the third property in lemma 4.3.1. In other words, we have a complex

$$0 \rightarrow \mathcal{M} \xrightarrow{\nabla^\vee} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{M} \cdots \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla_p^\vee} \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \mathcal{M} \cdots \mathcal{O}_X \rightarrow 0 \quad (45)$$

We will go deeper into this complex in the subsection 4.8.

Remark 4.3.2. *We have an analogous version for right \mathcal{D}_X -modules. In this case, it is necessary to define the right \mathcal{D}_X -module structure in terms of ∇ by*

$$s.\theta = -\nabla_\theta(s).$$

Definition 4.3.3. *We say that a \mathcal{D}_X -module \mathcal{M} is an integrable connection if it is locally free of rank finite over \mathcal{O}_X . The category of integrable connections on X will be denoted by $\mathbf{Conn}(X)$.*

Remark 4.3.4. *The category $\mathbf{Conn}(X)$ is an abelian category [32, 1.4.11].*

In contrast with \mathcal{O}_X -modules, it is important to specify if we are dealing with a left or a right \mathcal{D}_X -module. In what follows we will study an \mathcal{O}_X -module that will help us to interchange those structures. To start with, we will denote by $\mathbf{Mod}(\mathcal{D}_X)^{\text{op}}$ the category of right \mathcal{D}_X -modules.

Let's consider the canonical sheaf

$$\Omega_X := \bigwedge^{d_X} \Omega_X^1.$$

In the last section we have identified $\Omega_X^1 \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ and therefore, we can consider Ω_X as the sheaf $\mathcal{A}lt_{\mathcal{O}_X}(\mathcal{T}_X \times \cdots \times \mathcal{T}_X; \mathcal{O}_X)$ of alternating maps. This identification allows us to define a natural action of \mathcal{T}_X on Ω_X , called the **Lie derivative**. If $\theta, \theta_1, \theta_2, \dots, \theta_{d_X} \in \mathcal{T}_X$ and $\omega \in \Omega$, then the Lie derivative is defined by:

$$Lie_{\theta}\omega(\theta_1, \dots, \theta_{d_X}) := \theta(\omega(\theta_1, \dots, \theta_{d_X})) - \sum_{i=1}^{d_X} \omega(\theta_1, \dots, [\theta, \theta_i], \dots, \theta_{d_X}).$$

It is straightforward to verify that Lie satisfies the implicit conditions given in the remark 4.3.2. Hence we can define a structure of a *right* \mathcal{D}_X -module on Ω_X by

$$\omega \cdot \theta := -Lie_{\theta}\omega.$$

Remark 4.3.5. (cf. [32] 1.2.9) Given $\mathcal{M}, \mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X)$ and $\mathcal{M}', \mathcal{N}' \in \mathbf{Mod}(\mathcal{D}_X)^{op}$, we have the following facts:

- (i) $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X)$; $\theta(m \otimes n) := \theta m \otimes n + m \otimes \theta n$,
- (ii) $\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X)^{op}$; $(m' \otimes n)\theta := m'\theta \otimes n - m' \otimes \theta n$,
- (iii) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \in \mathbf{Mod}(\mathcal{D}_X)$; $(\theta\varphi)(s) := \theta(\varphi(s)) - \varphi(\theta(s))$,
- (iv) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}', \mathcal{N}') \in \mathbf{Mod}(\mathcal{D}_X)$; $(\theta\varphi)(s) := -\varphi(s)\theta + \varphi(s\theta)$,
- (v) $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}') \in \mathbf{Mod}(\mathcal{D}_X)$; $(\varphi\theta)(s) := \varphi(s)\theta + \varphi(\theta s)$.

The following lemma is a straightforward calculation using the previous facts.

Lemma 4.3.6. We have the following isomorphisms

$$(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{D}_X} \mathcal{M} \simeq \mathcal{M}' \otimes_{\mathcal{D}_X} (\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}) \simeq (\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{\mathcal{D}_X} \mathcal{N}$$

of \mathbb{C}_X -vector spaces (here \mathbb{C}_X is the constant sheaf).

Let us denote by $\Omega_X^{\otimes -1}$ the dual of the invertible \mathcal{O}_X -module Ω_X . The above remark gives us

Proposition 4.3.7. The functor

$$\Omega_X \otimes_{\mathcal{O}_X} (\bullet) : \mathbf{Mod}(\mathcal{D}_X) \rightarrow \mathbf{Mod}(\mathcal{D}_X)^{op}$$

is an equivalence of categories (well defined by (ii) in the previous remark). Its quasi-inverse is given by

$$\Omega_X^{\otimes -1} \otimes_{\mathcal{O}_X} (\bullet) = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \bullet).$$

The operations described by the above proposition are called **side-changing operations**.

Our next objective is to exhibit some functorial properties of \mathcal{D} -modules. More exactly, for a morphism $f : X \rightarrow Y$ of smooth algebraic varieties, we will introduce two operations called the **inverse image** and the **direct image**.

Let's start with the **inverse image**. Let \mathcal{M} be a left \mathcal{D}_Y -module. By [29] we get a homomorphism of \mathcal{O}_X -modules $f^*\Omega_Y^1 \rightarrow \Omega_X^1$ and therefore, taking the \mathcal{O}_X -dual, we get an \mathcal{O}_X -homomorphism

$$\begin{aligned} \mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) &\rightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*\Omega_Y, f^*\mathcal{O}_Y) = f^*\mathcal{T}_Y. \\ \theta &\mapsto \tilde{\theta} \end{aligned} \quad (46)$$

Based on the preceding homomorphism we can put a left \mathcal{D}_X -module structure on $f^*\mathcal{M}$ by defining

$$\theta(\psi \otimes m) = \theta(\psi) \otimes m + \psi \tilde{\theta}(m),$$

where $\psi \in \mathcal{O}_X$, $\theta \in \mathcal{T}_X$ and $m \in \mathcal{M}$. Taking a local coordinate system $\{y_i, \partial_{y_i}\}$ of Y and writing $\tilde{\theta} = \sum_j \phi_j \otimes \theta_j$ we can see that the action of \mathcal{T}_X can be written as

$$\theta(\psi \otimes m) = \theta(\psi) \otimes m + \psi \sum_{i=1}^{d_Y} \theta(y_i \circ f) \otimes \partial_{y_i} m. \quad (47)$$

Applying the above construction to \mathcal{D}_Y , we have that $f^*\mathcal{D}_Y$ turns out to be a $(\mathcal{D}_X, f^{-1}\mathcal{D}_Y)$ -bimodule which we denote by $\mathcal{D}_{X \rightarrow Y}$. With this definition, it's clear that $f^*\mathcal{M} \simeq \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}\mathcal{M}$ and we get a right exact functor

$$\mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y} f^{-1}(\bullet) : \mathbf{Mod}(\mathcal{D}_Y) \rightarrow \mathbf{Mod}(\mathcal{D}_X). \quad (48)$$

Example 4.3.8. (i) Let $f = (F_1, \dots, F_m) : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^m$ be a polynomial map. We suppose $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} = \mathbb{C}[x_1, \dots, x_n]$ and $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m} = \mathbb{C}[y_1, \dots, y_m]$. If \mathcal{M} is a left $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ -module, then by definition we have

$$f^*\mathcal{M} = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{f^{-1}\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} f^{-1}\mathcal{M}$$

is a module over $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$. We want to extend this action to the whole $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$. To do that, we need to make explicit the action of the operators $\partial_{x_1}, \dots, \partial_{x_n}$, and to prove that it respects the Weyl relations. Let $q \otimes m \in f^*\mathcal{M}$. By (47) we have

$$\partial_{x_i}(q \otimes m) = \partial_{x_i}(q) \otimes m + \sum_{j=1}^m q \partial_{x_i}(F_j) \otimes \partial_{y_j} m.$$

An easy computation using the Leibniz's rule proves that

$$\begin{aligned} \partial_{x_i}(x_i(q \otimes m)) &= \delta_{ij} q \otimes m + x_i \left(\partial_{x_j}(q) \otimes m + \sum_{j=1}^m (q \partial_{x_i}(F_j) \otimes \partial_{y_j} m) \right) \\ &= \delta_{ij} q \otimes m + x_i \partial_{x_j}(q \otimes m). \end{aligned}$$

In other words,

$$[\partial_{x_i}, x_i](q \otimes m) = \delta_{ij}(q \otimes m).$$

The other relations are easy computations.

Finally, we can write the inverse image $f^*\mathcal{M}$ in a slightly different way as follows

$$f^*\mathcal{M} = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{f^{-1}\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} f^{-1}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \otimes_{f^{-1}\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} f^{-1}\mathcal{M}.$$

In this case $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^m} = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{f^{-1}\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} f^{-1}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ is a $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} - f^{-1}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ -bimodule (it is clear that the left and right structures are compatible).

(ii) For a closed embedding $i : X \rightarrow Y$, at any point we can choose an affine chart $(U, \{y_k, \partial_{y_k}\}_{1 \leq k \leq d_Y})$ such that $y_{d_X+1} = \dots = y_{d_Y} = 0$ are well-defining equations of X and $\{x_k = y_k \circ i, \partial_{x_k}\}$ are local coordinates of X . The morphism (46) is given by $\partial_{x_k} \mapsto \partial_{y_k}$. In this particular case, $\mathcal{D}_{X \rightarrow Y}$ is locally described by

$$\mathcal{D}_{X \rightarrow Y} \simeq \mathcal{D}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{d_X+1}, \dots, \partial_{d_Y}].$$

(iii) ([22, Chapter 15]) For further reference, let us compute the transfer bimodule $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{n+m}}$ in the very special case when i is the **standard embedding**

$$\begin{aligned} i : \mathbb{A}_{\mathbb{C}}^n &\rightarrow \mathbb{A}_{\mathbb{C}}^n \times_{\text{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^m \\ x &\mapsto (x, 0), \end{aligned}$$

where 0 is the origin of $\mathbb{A}_{\mathbb{C}}^m$. Basic algebraic geometry tells us that the functions y_1, \dots, y_m in $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{m+n}}$ generates the ideal $\mathcal{I}_{(y)}$ of $\mathbb{A}_{\mathbb{C}}^n$. In other words

$$\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} = i^{-1}(\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{m+n}}) / \mathcal{I}_{(y)}.$$

The previous (canonical) isomorphism implies that if $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{m+n}})$, then

$$\begin{aligned} i^*\mathcal{M} &= \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{i^{-1}\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{m+n}}} i^{-1}\mathcal{M} \\ &\simeq i^{-1}(\mathcal{M} / \mathcal{I}_{(y)}\mathcal{M}) \end{aligned} \tag{49}$$

as $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ -modules. This can be proven as follows. First of all, the isomorphism ϕ in (49) is given by $\phi(q \otimes m) = q\bar{m}$. By (47) we have

$$\begin{aligned} \partial_{x_i}(q \otimes m) &= \partial_{x_i}q \otimes m + \sum_{j=1}^n q \partial_{x_i}(x_j) \otimes \partial_{x_j}m \\ &= \partial_{x_i}q \otimes m + q \otimes \partial_{x_i}m. \end{aligned}$$

The right hand-side is sent by ϕ to

$$\partial_{x_i}q \otimes \bar{m} + q \overline{\partial_{x_i}m} (= \partial_{x_i}(q\bar{m})).$$

This proves that (49) is an isomorphism of $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ -modules. In particular

$$\begin{aligned} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{n+m}} &= i^{-1}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} / \mathcal{I}_{(y)}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}}) \\ &\simeq i^{-1}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}]) \\ &\simeq \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}], \end{aligned} \tag{50}$$

where we have used the canonical identification $i^{-1}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \simeq \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ (this is the isomorphism given in the second item).

(iv) ([22, Chapter 16, section 1]) Let us describe the transfer bimodule when $\pi : X \times_{\text{Spec}(\mathbb{C})} Y \rightarrow Y$ is the second projection. In this case, if $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m})$, then

$$\begin{aligned}\pi^* \mathcal{M} &= \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{m+n}} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} \pi^{-1} \mathcal{M} \\ &\simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m} \otimes_{\pi^{-1} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} \pi^{-1} \mathcal{M} \\ &\simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \pi^{-1} \mathcal{M}.\end{aligned}$$

The last equality follows from $\pi^{-1} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m} = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}$ if they are considered as subrings in $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^{m+n}}$. Now, the description of the Weyl algebra given in the example 4.1.4 tells us that as $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ -modules $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \simeq \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} / \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}(\partial_{x_i})_{1 \leq i \leq n}$ (both considered as subsheaves of $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}}$), and therefore

$$\begin{array}{ccc} \pi^* \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \simeq \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \pi^{-1} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} & \xrightarrow{\simeq} & \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{m+n}} / \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{m+n}}(\partial_{x_i})_{1 \leq i \leq n} \\ P \otimes Q & \mapsto & P.Q \end{array}$$

We remark that in the last isomorphism we have used the canonical map

$$\begin{array}{ccc} \pi^{-1} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} & \rightarrow & \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} \\ Q & \mapsto & 1 \otimes Q. \end{array}$$

Before introducing the **direct image**, we need the following notion. Morally, to define direct images for \mathcal{D} -modules is more natural for right \mathcal{D} -modules than for left \mathcal{D} -modules. The idea is to somehow *transport* the right action to the left. Let us take an affine chart $(U; \{x_i, \partial_i\}_{1 \leq i \leq d_X})$. We define a **transposition** τ of \mathcal{D}_U as follows. For $P(x, \partial) = \sum_{\alpha} a_{\alpha}(x) \partial^{\alpha} \in \mathcal{D}_U$ we set

$$\tau(P(x, \partial)) := \sum_{\alpha} (-1)^{|\alpha|} \partial^{\alpha} a_{\alpha}(x), \quad (51)$$

τ satisfies $\tau(P.Q) = \tau(Q).\tau(P)$ and therefore, for a right \mathcal{D}_U -module \mathcal{M} we can define a structure of left \mathcal{D}_U -module via τ which we will denote by \mathcal{M}^{τ} (let us note that $\mathcal{M}^{\tau} = \mathcal{M}$ as sheaves of abelian groups).

Remark 4.3.9. *The definition of τ a priori depends of the coordinated system taken. The global notion is just proposition 4.3.7.*

Definition 4.3.10. *We have a $(f^{-1} \mathcal{D}_Y, \mathcal{D}_X)$ -bimodule defined by*

$$\mathcal{D}_{Y \leftarrow X} := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1} \mathcal{O}_Y} f^{-1} \Omega_Y^{\otimes -1}$$

Example 4.3.11. (i) ([32, Example 1.3.5]) Using the same notation of example 4.3.8 for a closed embedding $i : X \rightarrow Y$ we have that locally

$$\mathcal{D}_{Y \leftarrow X} \simeq \mathbb{C}[\partial_{d_X+1}, \dots, \partial_{d_Y}] \otimes_{\mathbb{C}} \mathcal{D}_X.$$

(ii) As we have remarked, at the level of the Weyl algebras we have

$$\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m \leftarrow \mathbb{A}_{\mathbb{C}}^n} = (\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^m})^{\tau} \simeq f^{-1} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \otimes_{f^{-1} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^m}} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}.$$

The last isomorphism is given by [22, Chapter 16, lemma 2.2], and this is clearly a $f^{-1} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} - \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ -bimodule. In the following items we will work out some particular cases that we will use later.

(iii) ([22, Chapter 16, section 3]) Let us consider $\pi : X \times_{\text{Spec}(\mathbb{C})} Y \rightarrow Y$ the projection. The fourth item in example 4.3.8 gives us

$$\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m \leftarrow \mathbb{A}_{\mathbb{C}}^{n+m}} = \left(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} / \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}}(\partial_{x_i})_{1 \leq i \leq n} \right)^{\tau} \simeq \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} / (\partial_{x_i})_{1 \leq i \leq n} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} \\ \tau(\overline{P}) \quad \leftarrow \quad \overline{P}$$

The last isomorphism is an easy exercise (the interested reader can also find a proof in [22, Chapter 16, proposition 2.1]).

(iv) ([22, Chapter 17, (1.1)]) Let us consider now the standard embedding $i : \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \rightarrow \mathbb{A}_{\mathbb{C}}^n \times_{\text{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^m$. By definition and the third item in example 4.3.8 (relation (50)), we have

$$\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m} \leftarrow \mathbb{A}_{\mathbb{C}}^n} = \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}] \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}. \quad (52)$$

Let us use the identification $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} = \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ (we recall for the reader that $i^{-1}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) = \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$) to specify the left $i^{-1}\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ action. It is enough to specify the action of y_j on $\partial^\beta \in \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}]$. We have

$$y_j \cdot \partial^\beta = -\beta_j \partial^{\beta - \beta_j e_j} \quad (53)$$

where e_j is the multi-vector with 1 in the j -th coordinate and zero elsewhere.

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties and $\mathcal{N} \in \mathbf{Mod}(\mathcal{D}_X)$. In terms of $\mathcal{D}_{Y \leftarrow X}$ a first tentative of **direct image** for left \mathcal{D}_X -modules is

$$f_*(\mathcal{N}) = f_*(\mathcal{D}_{Y \rightarrow X} \otimes_{\mathcal{D}_X} \mathcal{N}). \quad (54)$$

We remark for the reader that by definition $\mathcal{D}_{Y \rightarrow X} \otimes_{\mathcal{D}_X} \mathcal{N}$ is a left $f^{-1}\mathcal{D}_Y$ -module, and therefore $f_*(\mathcal{D}_{Y \rightarrow X} \otimes_{\mathcal{D}_X} \mathcal{N})$ is a left \mathcal{D}_Y -module via the canonical morphism $\mathcal{D}_Y \rightarrow f_*f^{-1}\mathcal{D}_Y$. In the next subsection we will generalise this notion to the respective derived categories. The point here is that the left exact functor f_* and the right exact functor \otimes are involved.

4.4 Homological properties of \mathcal{D} -modules

In this subsection we will define several functors on derived categories of \mathcal{D} -modules and show some fundamental properties concerning them. At the end of the subsection we will introduce the **Spencer resolution**, the **Koszul resolution** of \mathcal{O}_X and the **de Rham complex**. The second one is an important tool in the proof of *Kashiwara's equivalence*. For a general study on derived categories the reader is invited to look up the appendix (cf. [32]).

Let's suppose that \mathcal{R} is a sheaf of rings. Throughout this section we will denote by $\mathbf{D}(\mathcal{R})$, $\mathbf{D}^+(\mathcal{R})$, $\mathbf{D}^-(\mathcal{R})$ and $\mathbf{D}^b(\mathcal{R})$ the derived categories associated to the abelian category $\mathbf{Mod}(\mathcal{R})$. The key behind the following definitions comes from the fundamental fact that any object $\mathcal{M}^\bullet \in \mathbf{D}^+(\mathcal{R})$ (resp. $\mathbf{D}^-(\mathcal{R})$) is quasi-isomorphic to a complex \mathcal{I}^\bullet (resp. \mathcal{P}^\bullet) of injective (resp. flat) \mathcal{R} -modules belonging to $\mathbf{D}^+(\mathcal{R})$ (resp. $\mathbf{D}^-(\mathcal{R})$). Moreover, if $\mathbf{D}_{qc}^b(\mathcal{D}_X)$ denotes the bounded derived category associated to the abelian category $\mathbf{Mod}_{qc}(\mathcal{D}_X)$, then by proposition 4.2.10 and corollary 4.2.12, any object of $\mathbf{D}(\mathcal{D}_X)$ (resp. $\mathbf{D}_{qc}^b(\mathcal{D}_X)$) is represented by a bounded complex of flat \mathcal{D}_X -modules (resp. locally projective \mathcal{D}_X -modules belonging to $\mathbf{Mod}_{qc}(\mathcal{D}_X)$).

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. As a classical result in algebraic geometry we know that the functor f^* is right exact and the functor f_* is left exact, consequently if $\mathcal{M}^\bullet \in \mathbf{D}^b(\mathcal{D}_Y)$ and $\mathcal{N}^\bullet \in \mathbf{D}^b(f^{-1}\mathcal{D}_Y)$ we can define the functors

$$Rf_* : \mathbf{D}^b(f^{-1}(\mathcal{D}_Y)) \rightarrow \mathbf{D}^b(\mathcal{D}_Y) \quad (\mathcal{N}^\bullet \mapsto f_*(\mathcal{N}^\bullet))$$

by using an injective resolution on \mathcal{N}^\bullet , and

$$Lf^* : \mathbf{D}^b(\mathcal{D}_Y) \rightarrow \mathbf{D}^b(\mathcal{D}_X) \quad (\mathcal{M}^\bullet \mapsto \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}(\mathcal{D}_Y)}^{\mathbf{L}} f^{-1}\mathcal{M}^\bullet)$$

by using a flat resolution of \mathcal{M}^\bullet .

We call Lf^* the **inverse image**. We also use the **shifted inverse image**

$$f^\dagger := Lf^*[d_X - d_Y] : \mathbf{D}^b(\mathcal{D}_Y) \rightarrow \mathbf{D}^b(\mathcal{D}_X),$$

defined by

$$f^\dagger \mathcal{M}^\bullet = Lf^* \mathcal{M}^\bullet[d_X - d_Y]$$

The shifted inverse image will play an important role in Kashiwara's equivalence.

Let $g : Y \rightarrow Z$ be another morphism of smooth algebraic varieties. As a result of the fact that \mathcal{D} is a locally free \mathcal{O} -module we obtain an isomorphism

$$\mathcal{D}_{X \rightarrow Z} \simeq \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1}\mathcal{D}_Y}^{\mathbf{L}} f^{-1}\mathcal{D}_{Y \rightarrow Z}$$

of $(\mathcal{D}_X, (g \circ f)^{-1}\mathcal{D}_Z)$ -bimodules and therefore $L(g \circ f)^* \mathcal{M}^\bullet = Lf^*(Lg^*(\mathcal{M}^\bullet))$. In conclusion,

Proposition 4.4.1. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of smooth algebraic varieties. Then we have*

$$L(g \circ f)^* \simeq Lf^* \circ Lg^*, \quad (g \circ f)^\dagger \simeq f^\dagger \circ g^\dagger.$$

The following functors will play an important role in the next section and they will be considered in depth in the last part of this work.

Definition 4.4.2. *For a closed embedding $i : X \rightarrow Y$ of smooth algebraic varieties we define a left exact functor*

$$i^\natural : \mathcal{M}(\mathcal{D}_Y) \rightarrow \mathcal{M}(\mathcal{D}_X)$$

by

$$i^\natural \mathcal{M} := \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M}).$$

Let $i : X \rightarrow Y$ be a closed embedding. Before proving the following important results ([32, Propositions 1.5.14 and 1.5.16]) we will introduce the **Koszul's resolution** of the $i^{-1}\mathcal{O}_Y$ -module \mathcal{O}_X . Their respective proofs will be given after the relation (55) below.

Proposition 4.4.3. *Let $i : X \rightarrow Y$ be a closed embedding of smooth algebraic varieties. Set $d = \text{codim}_Y(X)$.*

- (i) *For $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_Y)$ we have $H^j(Li^*\mathcal{M}) = 0$ unless $-d \leq j \leq 0$.*

(ii) For $\mathcal{M}^\bullet \in \mathbf{D}^+(\mathcal{D}_Y)$ we have a canonical isomorphism

$$Li^*\mathcal{M} \simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M}^\bullet)[d]$$

in $\mathbf{D}^b(\mathcal{D}_X)$.

Proposition 4.4.4. *Let $i : X \rightarrow Y$ be a closed embedding. Then we have*

$$i^\dagger \mathcal{M}^\bullet \simeq Ri^\sharp \mathcal{M}^\bullet$$

for any $\mathcal{M}^\bullet \in \mathbf{D}^+(\mathcal{D}_Y)$.

4.5 The Koszul complex for a closed embedding

In this subsection we will give a description of the cohomology sheaves of $Li^*\mathcal{M}$ for $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ and $i : X \rightarrow Y$ a closed embedding (cf. [32] for a local description of the \mathcal{D}_X -module structure). Taking the notation given in example 4.3.8 we have a locally free resolution ([81, Corollary 4.5.5])

$$0 \rightarrow K_{d_Y-d_X} \rightarrow \dots \rightarrow K_0 := i^{-1}\mathcal{O}_Y \xrightarrow{i^\sharp} \mathcal{O}_X \rightarrow 0$$

where

$$K_j = \bigwedge^j \left(\bigoplus_{k=d_X+1}^{d_Y} i^{-1}\mathcal{O}_Y dy_k \right)$$

and the morphism $K_j \rightarrow K_{j-1}$ is given by

$$f dy_{k_1} \wedge \dots \wedge dy_{k_j} \mapsto \sum_{p=1}^j (-1)^{p+1} y_{k_p} f dy_{k_1} \wedge \dots \wedge \widehat{dy_{k_p}} \wedge \dots \wedge dy_{k_j}.$$

Taking the tensor product with $i^{-1}\mathcal{D}_Y$ over $i^{-1}\mathcal{O}_Y$ we get a locally free resolution of the right $i^{-1}\mathcal{D}_Y$ -module $\mathcal{D}_{X \rightarrow Y}$ and hence, by (48), we can conclude that $Li^*\mathcal{M}$ is represented by the complex

$$\dots \rightarrow 0 \rightarrow K_{d_Y-d_X} \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{M} \rightarrow K_0 \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{M} \rightarrow 0 \rightarrow \dots \quad (55)$$

The proof of the proposition 4.3.7 will need an appropriate notion of **direct images**. Let us briefly introduce this functor in the next subsection.

4.6 Direct images

We start this subsection by defining the following functors.

$$\mathbf{D}^b(\mathcal{D}_X) \ni \mathcal{M}^\bullet \mapsto \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbf{L}} \mathcal{M}^\bullet \in \mathbf{D}^b(f^{-1}\mathcal{D}_Y),$$

$$\mathbf{D}^b(f^{-1}\mathcal{D}_Y) \ni \mathcal{N}^\bullet \mapsto Rf_*(\mathcal{N}^\bullet) \in \mathbf{D}^b(\mathcal{D}_Y),$$

by using a flat resolution of \mathcal{M}^\bullet and an injective resolution of \mathcal{N}^\bullet . Taking the composition we obtain the functor

$$\int_f : \mathbf{D}^b(\mathcal{D}_X) \rightarrow \mathbf{D}^b(\mathcal{D}_Y)$$

given by

$$\int_f \mathcal{M}^\bullet = Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet),$$

and for an integer k we set

$$\int_f^k \mathcal{M}^\bullet := H^k \left(\int_f \mathcal{M}^\bullet \right).$$

The previous functor is denoted by f_+ in [9]. This is a more classical notation. In this notes we will follow the notation given in [32].

Proof of proposition 4.3.7. The first statement follows from the previous representation of the complex $Li^*\mathcal{M}$. To show (ii) it is sufficient to achieve the isomorphism

$$R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{D}_Y) \simeq \mathcal{D}_{X \rightarrow Y}[-d]. \quad (56)$$

In fact, if we assume this relation, then

$$\begin{aligned} Li^*\mathcal{M} &= \mathcal{D}_{X \rightarrow Y} \otimes_{i^{-1}\mathcal{D}_Y}^{\mathbb{L}} i^{-1}\mathcal{M} \\ &\stackrel{(56)}{\simeq} R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{D}_Y) \otimes_{i^{-1}\mathcal{D}_Y}^{\mathbb{L}} i^{-1}\mathcal{M}[d] \\ &\simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M})[d]. \end{aligned}$$

The last isomorphism is proved as follows (cf. [32, Lemma 2.6.13]). First of all, let us note that the $i^{-1}\mathcal{D}_Y$ -module structure of $i^{-1}\mathcal{M}$ allows us to construct a canonical morphism

$$R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{D}_Y) \otimes_{i^{-1}\mathcal{D}_Y}^{\mathbb{L}} i^{-1}\mathcal{M} \rightarrow R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M}). \quad (57)$$

Now, using the explicit local description of $\mathcal{D}_{Y \leftarrow Y}$ given in the example 4.3.11 items (i) and (ii), and for $i : X \rightarrow Y$ being a closed embedding, we have that $\mathcal{D}_{Y \leftarrow X}$ is coherent as an $i^{-1}\mathcal{D}_Y$ -module. This implies that in (57) we can substitute $\mathcal{D}_{Y \rightarrow X}$ by $i^{-1}\mathcal{D}_Y$, and (57) becomes an isomorphism since both sides are equal to $i^{-1}\mathcal{M}$.

Let us prove (56). In fact, it will be easier to prove an equivalent form. This is, by using the side-changing relations (56), it is enough to establish the following relation

$$R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{\text{op}}}(\mathcal{D}_{X \rightarrow Y}, i^{-1}\mathcal{D}_Y) \simeq \mathcal{D}_{Y \leftarrow X}[-d].$$

We have

$$\begin{aligned} R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{\text{op}}}(\mathcal{D}_{X \rightarrow Y}, i^{-1}\mathcal{D}_Y) &\simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{\text{op}}}(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y, i^{-1}\mathcal{D}_Y) \\ &\simeq R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{D}_Y) \\ &\simeq i^{-1}\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_Y} R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y) \end{aligned}$$

The problem being local, we can assume that Y is endowed with local coordinates $\{y_i, \partial_{y_i}\}_{1 \leq i \leq m}$, such that X is given by the system of equations $y_{m+1} = \dots = y_n = 0$ and $\{x_i, \partial_{x_i}\}_{1 \leq i \leq m}$ is a coordinate system for m .

By using the Koszul's resolution of the $i^{-1}\mathcal{O}_Y$ -module \mathcal{O}_X , we can see that $R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y)$ is represented by the complex

$$[K_0^\vee \rightarrow K_1^\vee \rightarrow \cdots \rightarrow K_d^\vee],$$

where $K_j^\vee := \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(K_j, i^{-1}\mathcal{O}_Y)$. It is clear that K_d^\vee is a locally free $i^{-1}\mathcal{O}_Y$ -module of rank one. Moreover, by definition, we dispose of a canonical perfect paring

$$K_j \otimes_{i^{-1}\mathcal{O}_Y} K_{d-j} \rightarrow K_d$$

for each j . Hence the canonical morphism induced by the paring

$$K_j \rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(K_{d-j}, K_d) \simeq K_{d-j}^\vee \otimes_{i^{-1}\mathcal{O}_Y} K_d$$

is an isomorphism (the last isomorphism comes from the fact that K_{d-j} is locally free of finite rank over $i^{-1}\mathcal{O}_Y$). Finally, taking duals we get

$$(K_{d-j}^\vee \otimes_{i^{-1}\mathcal{O}_Y} K_d)^\vee \simeq K_{d-j} \otimes_{i^{-1}\mathcal{O}_Y} K_d^\vee \xrightarrow{\simeq} K_j^\vee. \quad (58)$$

We can use the previous information to reduce the complex (58) as follows:

$$[K_0^\vee \rightarrow \cdots \rightarrow K_d^\vee] \simeq [K_d \rightarrow \cdots \rightarrow K_0] \otimes_{i^{-1}\mathcal{O}_Y} K_d^\vee \simeq \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} K_d^\vee[-d].$$

Now, as $i^{-1}\mathcal{O}_Y$ -modules and by definition of K_d , we can make the following (local) identifications $K_d = i^{-1}\mathcal{O}_Y d_{y_1} \wedge \cdots \wedge d_{y_d} \xrightarrow{fd_{y_i} \mapsto f} i^{-1}\mathcal{O}_Y$ and also

$$\begin{aligned} i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X &\rightarrow \mathcal{O}_X \\ gdy_1 \wedge \cdots \wedge dy_n \otimes fdx_1 \wedge \cdots \wedge dx_m &\mapsto gf \end{aligned}$$

From (58) and the previous identifications, we can conclude that

$$\begin{aligned} R\mathcal{H}om_{i^{-1}\mathcal{D}_Y^{\text{op}}}(\mathcal{D}_{X \rightarrow Y}, i^{-1}\mathcal{D}_Y) &\simeq i^{-1}\mathcal{D} \otimes_{i^{-1}\mathcal{O}_Y} R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{O}_X, i^{-1}\mathcal{O}_Y) \\ &\simeq i^{-1}\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_Y} \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} K_d^\vee[-d] \\ &\simeq i^{-1}\mathcal{D}_Y \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\Omega_Y^{\otimes -1} \otimes_{i^{-1}\mathcal{O}_Y} \Omega_X[-d] \\ &\simeq \Omega_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}(\mathcal{D}_Y \otimes_{\mathcal{O}_Y} \Omega_Y^{\otimes -1})[-d] \\ &\simeq \Omega \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y) \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\Omega_Y^{\otimes -1}[-d] \\ &= \mathcal{D}_{Y \leftarrow X}[-d]. \end{aligned}$$

The last equality is the definition of $\mathcal{D}_{Y \leftarrow X}$. \square

Proof of proposition 4.4.4. By definition, we know that $i^!\mathcal{M} = Li^*\mathcal{M}[-d]$ (being $d := \text{codim}_Y(X)$), and we have just proved that the last one is isomorphic to $R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M})$. We need to prove

$$R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M}) \simeq Ri^!\mathcal{M}.$$

Reduction to sections supported in X . Let $\mathcal{M}^\bullet \in \mathbf{D}^+(\mathcal{D}_Y)$. If $\Gamma_X(\bullet)$ denotes the functor of sections contained in X , we want to prove that the canonical map $i^{-1}R\Gamma_X(\mathcal{M}^\bullet) \rightarrow i^{-1}\mathcal{M}^\bullet$ induces an isomorphism

$$R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}R\Gamma_X(\mathcal{M}^\bullet)) \rightarrow R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M}^\bullet). \quad (59)$$

To do that, let us denote by $j : U \rightarrow Y$ the complementary open embedding ($U := Y \setminus X$). By [29, Chapter II, exercise 1.20 (ii)] for every injective sheaf $\mathcal{I} \in \mathbf{Mod}(\mathcal{D}_X)$, we have an exact sequence

$$0 \rightarrow \Gamma_X(\mathcal{I}) \rightarrow \mathcal{I} \rightarrow j_*j^{-1}\mathcal{I} \rightarrow 0.$$

in particular, fixing a representation $\mathcal{M}^\bullet \simeq \mathcal{I}^\bullet \in \mathbf{D}^+(\mathcal{D}_X)$ of \mathcal{M} by injective modules, the previous exact sequence generalizes to a short exact sequence of complexes

$$0 \rightarrow \Gamma_X(\mathcal{I}^\bullet) \rightarrow \mathcal{I}^\bullet \rightarrow j_*j^{-1}\mathcal{I}^\bullet \rightarrow 0$$

and by proposition C.1.13, we get a distinguished triangle

$$R\Gamma_X(\mathcal{I}^\bullet) \rightarrow \mathcal{I}^\bullet \rightarrow Rj_*j^{-1}\mathcal{I}^\bullet \xrightarrow{+1}.$$

Now, given that $U \subseteq Y$ is an open subset an easy computation shows that $\mathcal{D}_{Y \leftarrow U} = \mathcal{D}_U$ (as right \mathcal{D}_U -modules), and therefore

$$\int_j j^\dagger \mathcal{I}^\bullet = \int_j Lj^* \mathcal{I}^\bullet = \int_j j^{-1} \mathcal{I}^\bullet = Rj_*j^{-1}\mathcal{I}^\bullet$$

In other words, and by the very definition of derived categories and derived functors, we have a canonical triangle

$$R\Gamma_X(\mathcal{M}^\bullet) \rightarrow \mathcal{M}^\bullet \rightarrow \int_j j^\dagger \mathcal{M}^\bullet \xrightarrow{+1}. \quad (60)$$

In order to help the reader with the lecture of the proof, we remark that the previous triangle represents the following long exact sequence

$$\begin{aligned} 0 &\rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\Gamma_X \mathcal{I}^0) \rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{I}^0) \rightarrow \\ &\rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\int_j j^\dagger \mathcal{I}^0) \rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\Gamma_X \mathcal{I}^1) \rightarrow \\ &\rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{I}^1) \rightarrow \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\int_j j^\dagger \mathcal{I}^1) \cdots \end{aligned}$$

So, we will prove that (59) is in fact an isomorphism if we show that

$$R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\int_j j^\dagger \mathcal{M}^\bullet) = 0.$$

This is a consequence of the projection formula [30, Proposition 5.6] and the fact that

$$R\mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\int_j j^\dagger \mathcal{M}^\bullet) = i^\dagger \int_j j^{-1} \mathcal{M} = i^\dagger Rj_*j^{-1} \mathcal{M} \quad (61)$$

In fact, we have the following relations

$$\begin{aligned} i_*(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y}^{\mathbb{L}} i^{-1}Rj_*j^{-1}\mathcal{M}) &= i_*\mathcal{O}_X \otimes_{\mathcal{O}_Y}^{\mathbb{L}} Rj_*j^{-1}\mathcal{M} \\ &= Rj_*(j^{-1}i_*\mathcal{O}_X \otimes_{\mathcal{O}_U}^{\mathbb{L}} j^{-1}\mathcal{M}) \\ &= 0 \quad (j^{-1}i_*\mathcal{O}_X = 0). \end{aligned} \quad (62)$$

To end the proof of the proposition, we need first to prove that

$$i^\natural \mathcal{M} \simeq \mathcal{H}om_{i^{-1}\mathcal{O}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\Gamma_X(\mathcal{M})). \quad (63)$$

The question being local, we can assume that Y is endowed with local coordinates $\{y_i, \partial_{y_i}\}_{1 \leq i \leq n}$ such that $\mathcal{D}_{Y \leftarrow X} = \mathbb{C}[\partial_{y_{m+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathcal{D}_X$. Let $i^{-1}\mathcal{J} \subseteq i^{-1}\mathcal{O}_Y$ be the defining ideal of X , i.e, $i^{-1}\mathcal{J} = \ker(i^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X)$. So

$$i^{-1}\mathcal{J} \cdot 1 \otimes 1 = 0.$$

Given that $\mathcal{D}_{Y \leftarrow X}$ is generated by $1 \otimes 1$ as an $i^{-1}\mathcal{D}_Y$ -module, the previous relation implies that for all $\phi \in \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{M})$ and all $P \in \mathcal{D}_{Y \leftarrow X}$ we have $i^{-1}\mathcal{J} \cdot \phi(P) = 0$. In other words, $\phi(P) \in i^{-1}\Gamma_X(\mathcal{M})$. Let prove now that this isomorphism pass to the derived categories. This is, we will finally show that

$$Ri^{\sharp}\mathcal{M}^{\bullet} \simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}R\Gamma_X(\mathcal{M}^{\bullet})). \quad (64)$$

In light of the isomorphism (63) we only need to verify that $i^{-1}\Gamma_X(\mathcal{I})$ is an injective $i^{-1}\mathcal{D}_Y$ -module, if \mathcal{I} is an injective \mathcal{D}_Y -module. Let \mathcal{N} be an $i^{-1}\mathcal{D}_Y$ -module. We have the following relations

$$\begin{aligned} \mathrm{Hom}_{i^{-1}\mathcal{D}_Y}(\mathcal{N}, i^{-1}\Gamma_X(\mathcal{I})) &\simeq \mathrm{Hom}_{i^{-1}\mathcal{D}_Y}(i^{-1}i_*\mathcal{N}, i^{-1}\Gamma_X(\mathcal{I})) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_Y}(i_*\mathcal{N}, i_*i^{-1}\Gamma_X(\mathcal{I})) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_Y}(i_*\mathcal{N}, \Gamma_X(\mathcal{I})) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_Y}(i_*\mathcal{N}, \mathcal{I}). \end{aligned}$$

The final isomorphism is given by [32, (C.2.13)] and [32, (C. 2.14)] (further explanations are given at the end of the proposition 4.6.3). This ends the proof of the proposition. \square

Before ending this subsection, let us discuss some important properties of the (derived) directimage functor. The proof of the next result can be found in [32, 1.5.21].

Proposition 4.6.1. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of smooth algebraic varieties. Then we have*

$$\int_{g \circ f} = \int_g \int_f.$$

Continuing with the ideas given in 4.3.11 (i.e, $\mathcal{D}_{Y \leftarrow X}$ is a locally free \mathcal{D}_X -module) and keeping in mind the exactness of the functor i_* , for $i : X \rightarrow Y$ a closed embedding, we have the following local description of the cohomology of the complex $\int_i \mathcal{M}$ for $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_X)$:

$$\int_i^0 \mathcal{M} = \mathbb{C}[\partial_{d_X+1}, \dots, \partial_{d_Y}] \otimes_{\mathbb{C}} i_*\mathcal{M} \quad \text{and} \quad \int_i^{k \neq 0} \mathcal{M} = 0. \quad (65)$$

In particular \int_i^0 is exact and sends $\mathbf{Mod}_{qc}(\mathcal{D}_X)$ to $\mathbf{Mod}_{qc}(\mathcal{D}_Y)$.

Example 4.6.2. Let us work out some computations at the level of Weyl algebras.

- (i) We start by considering the standard embedding $i : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{m+n}$ and let us take $\mathcal{N} \in \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$. By (52) and (54) we have

$$\int_i^0 \mathcal{N} = \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}] \otimes_{\mathbb{C}} i_*\mathcal{N}. \quad (66)$$

Let us specify the $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}}$ -action. First of all, by the Weyl relation, we can write $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} = \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$. By (53) we know that $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ acts on $\mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}]$, and $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ acts on $i_*\mathcal{N}$ by hypothesis.³⁰

- (ii) Another important example is the canonical projection onto the second coordinate $\pi : \mathbb{A}_{\mathbb{C}}^{n+m} \rightarrow \mathbb{A}_{\mathbb{C}}^m$. By affinity (remark 4.2.7) and Serre's criteria for affineness, the cohomology of \int_{π} is concentrated in zero degree. Let us explicitly compute this functor. Let $\mathcal{N} \in \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}})$. By the example 4.3.11 we have

$$\int_{\pi}^0 \mathcal{N} = \pi_* \left(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \leftarrow \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}} \otimes_{\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n+m}}} \mathcal{N} \right) \simeq \pi_* \mathcal{N} / (\partial_{x_i})_{1 \leq i \leq n} \pi_* \mathcal{N}.$$

If we identify $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \subseteq \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$, then we have that $\int_{\pi}^0 \mathcal{N}$ is a $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$ -module (this is [22, Chapter 16, section 3]).³¹

Proposition 4.6.3.³² *Let $i : X \rightarrow Y$ be a closed embedding of smooth algebraic varieties.*

- (i) *There exists a functorial isomorphism*

$$R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{M}^{\bullet}, \mathcal{N}^{\bullet} \right) \simeq i_* R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}^{\bullet}, Ri^{\sharp} \mathcal{N}^{\bullet})$$

for $\mathcal{M}^{\bullet} \in \mathbf{D}^-(\mathcal{D}_X)$ and $\mathcal{N}^{\bullet} \in \mathbf{D}^+(\mathcal{D}_Y)$.

- (ii) *the functor $Ri^{\sharp} : \mathbf{D}^b(\mathcal{D}_Y) \rightarrow \mathbf{D}^b(\mathcal{D}_X)$ is right adjoint to the (direct image) functor $\int_i : \mathbf{D}^b(\mathcal{D}_X) \rightarrow \mathbf{D}^b(\mathcal{D}_Y)$.*

Proof. We only have to show the statement (i) because taking $H^0(R\Gamma(Y, \bullet))$ we obtain (ii). From the canonical isomorphism

$$\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}, \mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{N})) \simeq \mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}, i^{-1}\mathcal{N})$$

we obtain ([44, Proposition 2.6.3])

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}^{\bullet}, R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{N}^{\bullet})) \\ \simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^{\bullet}, i^{-1}\mathcal{N}^{\bullet}), \end{aligned}$$

³⁰In fact, if $U \times V \subset \mathbb{A}_{\mathbb{C}}^{n+m}$ is an affine open subset then $i_*\mathcal{N}(U \times V) = \mathcal{N}(i^{-1}(U \times V)) = \mathcal{N}(U)$ which is a $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}(U)$ -module (this is exactly as in [22, Chapter 17, section 1]).

³¹We remark for the reader that if $U \subset \mathbb{A}_{\mathbb{C}}^m$ is an affine open subset, then $\pi_*\mathcal{N}(U) = \mathcal{N}(\pi^{-1}(U)) = \mathcal{N}(\mathbb{A}_{\mathbb{C}}^n \times U)$. This is the reason to identify $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m} \subseteq \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m}$.

³²We follow word by word the arguments given in [32, Proposition 1.5.25]. Some details have been added at the end of the prove.

and from this we can see by definition of i^\sharp , that

$$\begin{aligned}
R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{M}^\bullet, \mathcal{N}^\bullet \right) &\simeq R\mathcal{H}om_{\mathcal{D}_Y} (i_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet), \mathcal{N}) \\
&\simeq R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (i_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet), R\Gamma_X(\mathcal{N}^\bullet)) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_Y} (i_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet), i_*i^{-1}R\Gamma_X(\mathcal{N}^\bullet)) \\
&\simeq i_*R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (i^{-1}i_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet), i^{-1}R\Gamma_X(\mathcal{N}^\bullet)) \\
&\simeq i_*R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet, i^{-1}R\Gamma_X(\mathcal{N}^\bullet)) \\
&\simeq i_*\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}^\bullet, R\mathcal{H}om_{i^{-1}\mathcal{D}_Y} (\mathcal{D}_{Y \leftarrow X}, i^{-1}\Gamma_X(\mathcal{N}^\bullet))) \\
&\simeq R\mathcal{H}om_{\mathcal{D}_X} (\mathcal{M}^\bullet, Ri^\sharp \mathcal{N}^\bullet).
\end{aligned}$$

Let us explain the previous tower of isomorphisms. The first isomorphism is the very definition of the functor \int_i , the second isomorphism is the well-known adjunction property (cf. [32, Proposition C.2.2]) between the functors $\Gamma_X(\bullet)$ and $i_!i^{-1}(\bullet)$, the third isomorphism is $i_*i^{-1} = id$, the fourth isomorphism is the adjunction formula [32, (C.2.15)], the fifth isomorphism is $i^{-1}i_* = id$, the sixth isomorphism is just the adjunction formula between $\mathcal{H}om$ and \otimes . Finally, the last isomorphism is (64). \square

4.7 The Spencer resolution

Let's denote by

$$\Omega_X^k = \bigwedge^k \Omega_X^1 \quad \text{for } 0 \leq k \leq d_X.$$

We have a canonical map

$$\begin{array}{ccc}
\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^0 \mathcal{T}_X (= \mathcal{D}_X) & \rightarrow & \mathcal{O}_X \\
& & \downarrow \\
& & P \\
& & \uparrow \\
& & P(1)
\end{array}$$

and for $0 < k \leq d_X$ we can define a morphism

$$d_k : \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^k \mathcal{T}_X \rightarrow \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{k-1} \mathcal{T}_X$$

by

$$\begin{aligned}
d_k(P \otimes \theta_1 \wedge \cdots \wedge \theta_k) &= \sum_i (-1)^{i+1} P \theta_i \otimes \theta_1 \wedge \cdots \wedge \widehat{\theta}_i \wedge \cdots \wedge \theta_k \\
&+ \sum_{i < j} (-1)^{i+j} P \otimes [\theta_i, \theta_j] \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta}_i \wedge \cdots \wedge \widehat{\theta}_j \wedge \cdots \wedge \theta_k.
\end{aligned}$$

A standard argument in homological algebra proves that $d_k \circ d_{k-1} = 0$.

In order to show that the last complex, denoted by $\mathrm{Sp}_X^\bullet(\mathcal{D}_X)$ (we use the notation given in [58, Definition 1.4.2]), is acyclic we consider the filtration $\{F_p \mathrm{Sp}_X^\bullet(\mathcal{D}_X)\}$:

³³Here $i_!$ denotes the **proper direct image** functor. We remark for the reader that given that i is a closed embedding $i_! = i_*$ (cf. [38, Proposition 6.9 (i)]), and therefore $i_!i^{-1} = id$.

$$F_p \mathrm{Sp}_X^\bullet(\mathcal{D}_X) = \left[F_{p-d_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \mathcal{T}_X \rightarrow \dots \rightarrow F_p \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^0 \mathcal{T}_X \rightarrow \mathcal{O}_X \right],$$

($F_p \mathcal{O}_X = \mathcal{O}_X$ if $p \geq 0$ and $F_p \mathcal{O}_X = 0$ if $p < 0$) and we will show that the associated graded complex

$$\mathrm{gr} \mathrm{Sp}_X^\bullet(\mathcal{D}_X) = \mathrm{gr}^F \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \mathcal{T}_X \rightarrow \dots \rightarrow \mathrm{gr}^F \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{T}_X \rightarrow \mathrm{gr}^F \mathcal{D}_X \rightarrow \mathcal{O}_X$$

is acyclic. Is straightforward to verify that in the associated graduated complex the morphisms come up from the relation

$$d_k(P \otimes \theta_1 \wedge \dots \wedge \theta_k) = \sum_i (-1)^{i+1} P \theta_i \otimes \theta_1 \wedge \dots \wedge \widehat{\theta}_i \wedge \dots \wedge \theta_k$$

On the other hand, let $\pi : T^*X \rightarrow X$ be the canonical projection and $i : X \rightarrow T^*X$ the closed embedding induced via the zero-section. and let us consider the Koszul resolution of the \mathcal{O}_{T^*X} -module $i_* \mathcal{O}_X$:

$$K^\bullet = \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \mathcal{O}_X} \bigwedge^{d_X} \pi^{-1} \mathcal{T}_X \rightarrow \dots \rightarrow \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \mathcal{O}_X} \bigwedge^0 \pi^{-1} \mathcal{T}_X \xrightarrow{i^\sharp} i_* \mathcal{O}_X \rightarrow 0$$

whose differential is given by

$$\begin{aligned} d : \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \mathcal{O}_X} \bigwedge^p \pi^{-1} \mathcal{T}_X &\rightarrow \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \mathcal{O}_X} \bigwedge^{p-1} \pi^{-1} \mathcal{T}_X \\ \psi \otimes \theta_1 \wedge \dots \wedge \theta_n &\mapsto \sum_{i=1}^k (-1)^{i+1} \psi \sigma_1(\theta_i) \theta_1 \wedge \dots \wedge \widehat{\theta}_i \wedge \dots \wedge \theta_k \end{aligned}$$

Here $\sigma_1(\theta_i) \in \mathrm{gr}(\mathcal{D}_X)$ is the principal symbol of θ_i . Moreover, given that $\mathrm{gr}(\mathcal{D}_X) = \pi_* \mathcal{O}_{T^*X}$, (43), we can conclude $\mathrm{gr} \mathrm{Sp}_X^\bullet(\mathcal{D}_X) \simeq \pi_* K^\bullet$, and given that K^\bullet is acyclic ([81, Corollary 4.5.5]) so is $\pi_* K^\bullet$ because π is an affine morphism.

The complex $\mathrm{Sp}_X^\bullet(\mathcal{D}_X)$ is called the **Spencer resolution of \mathcal{O}_X** . Using the side-changing operation we obtain the complex

$$0 \rightarrow \Omega_X^0 \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \dots \rightarrow \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{D}_X \rightarrow \Omega_X \rightarrow 0. \quad (67)$$

which is nothing more than the complex (45) (cf. [32, Lemma 1.5.27]). We will use this last complex to describe the cohomology of the complexes induced by the functor \int_f , when f is a projection morphism.

4.8 The de Rham complex

Let Y and Z be smooth algebraic varieties and set $X = Y \times Z$. Let $f : X \rightarrow Y$ and $g : X \rightarrow Z$ be the projections. For $\mathcal{M} \in \mathbf{Mod}(\mathcal{O}_Y)$ and $\mathcal{N} \in \mathbf{Mod}(\mathcal{O}_Z)$ we define the bifunctor

$$\mathcal{M} \boxtimes \mathcal{N} := \mathcal{O}_X \otimes_{f^{-1} \mathcal{O}_Y \otimes_{\mathbb{C}} g^{-1} \mathcal{O}_Z} (f^{-1} \mathcal{M} \otimes_{\mathbb{C}} g^{-1} \mathcal{N}) \in \mathbf{Mod}(\mathcal{O}_X)$$

which is exact with respect to both factors and extends to a functor on the derived categories. It is easily seen that this extension, which we denote by $(\bullet) \boxtimes (\bullet)$, sends $\mathbf{D}_{qc}^b(\mathcal{D}_Y) \times \mathbf{D}_{qc}^b(\mathcal{D}_Z)$ (resp. $\mathbf{D}_c^b(\mathcal{D}_Y) \times \mathbf{D}_c^b(\mathcal{D}_Z)$) to $\mathbf{D}_{qc}^b(\mathcal{D}_X)$ (resp. $\mathbf{D}_c^b(\mathcal{D}_X)$). We also note that

$$f^* \mathcal{M} \simeq \mathcal{M} \boxtimes \mathcal{O}_Z, \quad g^* \mathcal{N} \simeq \mathcal{O}_Y \boxtimes \mathcal{N}.$$

Now, let's suppose that $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$. To compute $\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}$ we can use the resolution of the right \mathcal{D}_X -module $\mathcal{D}_{Y \leftarrow X} = \mathcal{D}_Y \boxtimes \Omega_Z$ induced by the resolution of the right \mathcal{D}_Z -module Ω_Z given in the previous section and we define the **de Rham complex** $\mathbf{DR}_{X/Y}(\mathcal{M})$ by

$$(\mathbf{DR}_{X/Y}(\mathcal{M}))^k = \begin{cases} \Omega_{X/Y}^{d_Z+k} \otimes_{\mathcal{O}_X} \mathcal{M} & \text{if } -d_Z \leq k \leq 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential is given by³⁴

$$d(\omega \otimes s) = d\omega \otimes s + \sum_{i=1}^{d_Z} (dz_i \wedge \omega) \otimes \partial_{z_i} s. \quad (68)$$

Here $\{z_i, \partial_{z_i}\}_{1 \leq i \leq d_Z}$ is a local coordinate system of Z . The complex (67) gives us

$$\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M} \simeq \mathbf{DR}_{X/Y}(\mathcal{M}). \quad (69)$$

Proposition 4.8.1. *Let Y and Z be smooth algebraic varieties, and let us consider $f : X = Y \times Z \rightarrow Y$ the projection onto the first factor.*

(i) *For $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_X)$ we have $\int_f \mathcal{M} \simeq Rf_*(\mathbf{DR}_{X/Y}(\mathcal{M}))$.*

(ii) *For $\mathcal{M} \in \mathbf{Mod}(\mathcal{D}_X)$ we have $\int_f^j \mathcal{M} = 0$ unless $-d_Z \leq j \leq d_Z$.*

(iii) *The functor \int_f sends $\mathbf{D}_{qc}^b(\mathcal{D}_X)$ to $\mathbf{D}_{qc}^b(\mathcal{D}_Y)$.*

Proof. The first item follows from the relation (69) and (ii) is a consequence from (i) and the fact that f_* has cohomological dimension d_Z . Finally, to prove (iii) we can suppose that $\mathcal{M} \in \mathbf{D}_{qc}(\mathcal{D}_X)$. In this case, the complex $\mathbf{DR}_{X/Y}(\mathcal{M})$ is a complex of quasi-coherent \mathcal{O}_X -modules and given that X is in particular a noetherian scheme, then we can conclude that $Rf_*(\mathbf{DR}_{X/Y}(\mathcal{M}))$ is a complex of quasi-coherent \mathcal{D}_Y -modules (cf. [29, Chapter II, proposition 5.8]) and (i) gives (iii). \square

4.9 Kashiwara's equivalence

Throughout this subsection $i : X \rightarrow Y$ will denote a closed embedding between smooth algebraic varieties. According to (53) and (65) the functor

$$\int_i^0 : \mathbf{Mod}_{qc}(\mathcal{D}_X) \rightarrow \mathbf{Mod}_{qc}(\mathcal{D}_Y)$$

is an exact functor and the image of a \mathcal{D}_X -module is an object of the full subcategory $\mathbf{Mod}_{qc}^X(\mathcal{D}_Y)$ of quasi-coherent \mathcal{D}_Y -modules (resp. we keep the notation for coherent \mathcal{D}_Y -modules) supported by X . For the proof of the next theorem we will follow the same lines of reasoning given in [32, Theorem 1.6.1] specifying the details.

³⁴We remark for the reader that this is exactly the complex that we have constructed in (45). The connection is given by the \mathcal{D}_X -module structure of \mathcal{M} .

Theorem 4.9.1. (Kashiwara's equivalence). *Let $i : X \rightarrow Y$ be a closed embedding. The functors*

$$\int_i^0 : \mathbf{Mod}_{qc}(\mathcal{D}_X) \rightarrow \mathbf{Mod}_{qc}^X(\mathcal{D}_Y) \quad \text{and} \quad i^\dagger = H^0 i^\dagger : \mathbf{Mod}_{qc}^X(\mathcal{D}_Y) \rightarrow \mathbf{Mod}_{qc}(\mathcal{D}_X)$$

are inverse and define an equivalence of categories (the second equality comes from proposition 4.4.4).

Proof. Let's take $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{Mod}_{qc}^X(\mathcal{D}_Y)$. We need to show that the canonical homomorphisms induced by adjointness (proposition 4.6.3)

$$\int_i^0 H^0 i^\dagger \mathcal{N} \rightarrow \mathcal{N} \quad \text{and} \quad \mathcal{M} \rightarrow H^0 i^\dagger \int_i^0 \mathcal{M}$$

are isomorphisms. Let us clarify these isomorphisms. First of all, by proposition 4.4.4 we have

$$H^0 i^\dagger \mathcal{N} = \mathcal{H}i_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{N})$$

so

$$\int_i^0 H^0 i^\dagger \mathcal{N} = i_* (\mathcal{D}_{Y \leftarrow Y} \otimes_{\mathcal{D}_X} \mathcal{H}om_{i^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X}, i^{-1}\mathcal{N})),$$

and the first isomorphism is defined by the rule

$$\begin{aligned} \int_i^0 H^0 i^\dagger \mathcal{N} &\rightarrow \mathcal{N} \\ P \otimes \varphi &\mapsto \varphi(P). \end{aligned}$$

On the other hand

$$\begin{aligned} H^0 i^\dagger \int_i^0 \mathcal{M} &= \mathcal{H}om_{i^{-1}\mathcal{D}}(\mathcal{D}_{Y \leftarrow X}, i^{-1}i_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M})) \\ &= \mathcal{H}om_{i^{-1}\mathcal{D}}(\mathcal{D}_{Y \leftarrow X}, \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X} \mathcal{M}), \end{aligned}$$

and the second isomorphism is given by

$$\begin{aligned} \mathcal{M} &\rightarrow H^0 i^\dagger \int_i^0 \mathcal{M} \\ m &\mapsto [P \mapsto m \otimes P]. \end{aligned}$$

Now, since the previous problem is a local question, we can suppose that Y is affine and that it is endowed with local coordinates $\{y_k, \partial_{y_k}\}_{1 \leq k \leq d_Y}$ as in example 4.3.8. Moreover, as we have remarked along the examples, we can even suppose that X is defined by a system of equations $y_{m+1} = \dots = y_{d_Y} = 0$. Let X_l be the closed subvariety of Y defined by $y_{m+1} = \dots = y_l = 0$, for $m < l \leq d_X$. The chain of closed embeddings

$$X = X_{d_Y} \xrightarrow{i_{d_Y}} X_{d_Y-1} \xrightarrow{i_{d_Y-1}} \dots \xrightarrow{i_2} X_1 \xrightarrow{i_1} Y,$$

and the relations $\int_i = \int_{i_1} \dots \int_{i_{d_Y}}$ and $i^\dagger = i_{d_Y}^\dagger \dots i_1^\dagger$ (propositions 4.4.1 and 4.6.1) tell us that we can suppose that X is the hypersurface $y_{d_Y} = 0$. We set $y = y_{d_Y}$, $\partial = \partial_{y_{d_Y}}$ and $\theta = y\partial$. Let us first study the cohomology of the

complex $i^\dagger \mathcal{N}$. As we have remarked in subsection 4.5, and given that we are in codimension 1, the Koszul complex of the $i^{-1} \mathcal{O}_Y$ -module \mathcal{O}_X is given by

$$\cdots \rightarrow 0 \rightarrow i^{-1} \mathcal{O}_Y dy \xrightarrow{f dy \rightarrow y f} i^{-1} \mathcal{O}_Y \xrightarrow{i^\sharp} \mathcal{O}_X \rightarrow 0 \rightarrow \cdots$$

and therefore $Li^* \mathcal{N}$ is represented by the complex

$$\cdots \rightarrow 0 \rightarrow i^{-1} \mathcal{N} \xrightarrow{y} i^{-1} \mathcal{N} \rightarrow 0 \rightarrow \cdots \quad (70)$$

In consequence, and considering the shifting of degree -1 , we get

$$H^0 i^\dagger \mathcal{N} = H^{-1} Li^* \mathcal{N} = \ker(y) \quad \text{and} \quad H^1 i^\dagger \mathcal{N} = H^0 Li^* \mathcal{N} = \text{coker}(y).$$

Let us consider the eigenspaces $i^{-1} \mathcal{N}^j := \{s \in i^{-1} \mathcal{N} \mid \theta s = j s\}$ ($j \in \mathbb{Z}$). The relation $[\partial, y] = 1$ clearly implies that $y i^{-1} \mathcal{N}^j \subset i^{-1} \mathcal{N}^{j+1}$ and $\partial i^{-1} \mathcal{N}^j \subset i^{-1} \mathcal{N}^{j-1}$, so θ induces an isomorphism $j \times : i^{-1} \mathcal{N}^j \rightarrow i^{-1} \mathcal{N}^j$ for every $j \neq 0$. Moreover, since $\partial y = \theta + 1$ then $\partial y : i^{-1} \mathcal{N}^j \rightarrow i^{-1} \mathcal{N}^j$ is an isomorphism for every $j \neq -1$. In particular, if $j < -1$, then

$$i^{-1} \mathcal{N}^j \xrightarrow{y} i^{-1} \mathcal{N}^{j+1} \xrightarrow{\partial} i^{-1} \mathcal{N}^j$$

are both isomorphisms for $j < -1$.

Let us see by induction that

$$\ker(y^k : i^{-1} \mathcal{N} \rightarrow i^{-1} \mathcal{N}) \subset \bigoplus_{j=1}^k i^{-1} \mathcal{N}^{-j}. \quad (71)$$

For $k = 1$, the relation $ys = 0$ gives us $\theta s = (\partial y - 1)s = -s$.

Let us suppose $k > 1$. Then for $s \in \ker(y^k)$ we have $y^k s = y^{k-1}(ys) = 0$ and $ys \in \bigoplus_{j=1}^{k-1} i^{-1} \mathcal{N}^{-j}$ by hypothesis of induction. Hence $\partial ys \in \bigoplus_{j=2}^k i^{-1} \mathcal{N}^{-j}$ and

$$\theta s + s = y \partial s + s = \partial ys \in \bigoplus_{j=2}^k i^{-1} \mathcal{N}^{-j}. \quad (72)$$

On the other hand, we have $y^{k-1}(\theta s + ks) = \partial y^k s = 0$. Therefore by the hypothesis of induction

$$\theta s + ks \in \bigoplus_{j=1}^{k-1} i^{-1} \mathcal{N}^{-j}. \quad (73)$$

From (72) and (73) we get $ks + \theta s - \partial ys = (k-1)s \in \bigoplus_{j=1}^k i^{-1} \mathcal{N}^{-j}$ (recall that $[\partial, y] = 1$), which implies that $s \in \bigoplus_{j=1}^k i^{-1} \mathcal{N}^{-j}$ because $k > 1$. Moreover, the fact that \mathcal{N} is a quasi-coherent \mathcal{O}_Y -module supported in X tells us that (71) is an equality

$$i^{-1} \mathcal{N} = \bigoplus_{j=1}^{\infty} i^{-1} \mathcal{N}^{-j}.$$

In particular $H^1 i^\dagger \mathcal{N} = 0$, and also

$$i^{-1} \mathcal{N} = \mathbb{C}[\partial] \otimes_{\mathbb{C}} i^{-1} \mathcal{N}^{-1} \quad \text{and} \quad H^0 i^\dagger \mathcal{N} = i^{-1} \mathcal{N}^{-1}$$

From this we get the following relations. First of all

$$\begin{aligned} \int_i^0 H^0 i^\dagger \mathcal{N} &= \int_i^0 i^{-1} \mathcal{N}^{-1} \\ &= i_* (\mathbb{C}[\partial] \otimes_{\mathbb{C}} i^{-1} (\mathcal{N}^{-1})) \\ &= i_* i^{-1} \mathcal{N} \\ &= \mathcal{N} \end{aligned}$$

On the other hand, it is easy to see that $(\mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* \mathcal{M})^{-1} = i_* \mathcal{M}$, so

$$H^0 i^\dagger \int_i^0 \mathcal{M} = i^{-1} ((\mathbb{C}[\partial] \otimes_{\mathbb{C}} i_* \mathcal{M})^{-1}) = i^{-1} i_* \mathcal{M} = \mathcal{M}.$$

This ends the proof of the theorem. \square

A natural question is that if the preceding results remains true when considering the respective derived categories. We will give a positive answer to this question in the next corollary. We will denote by $\mathbf{D}_{qc}^{b,X}(\mathcal{D}_Y)$ (resp. $\mathbf{D}_c^{b,X}(\mathcal{D}_Y)$) the subcategory of $\mathbf{D}_{qc}^b(\mathcal{D}_Y)$ (resp. $\mathbf{D}_c^b(\mathcal{D}_Y)$) consisting of complexes \mathcal{N}^\bullet whose cohomology sheaves are supported by X .

Corollary 4.9.2. *The functor*

$$\int_i : \mathbf{D}_{qc}^b(\mathcal{D}_X) \rightarrow \mathbf{D}_{qc}^{b,X}(\mathcal{D}_Y)$$

is an equivalence of triangulated categories. Its quasi-inverse is given by

$$Ri^\dagger = i^\dagger : \mathbf{D}_{qc}^{b,X}(\mathcal{D}_Y) \rightarrow \mathbf{D}_{qc}^b(\mathcal{D}_X)$$

Proof. Let $\mathcal{M}^\bullet \in \mathbf{D}_{qc}^b(\mathcal{D}_X)$ and $\mathcal{N}^\bullet \in \mathbf{D}_{qc}^{b,X}(\mathcal{D}_Y)$. We need to prove that the canonical morphisms

$$\mathcal{M}^\bullet \rightarrow i^\dagger \int_i \mathcal{M}^\bullet \quad \text{and} \quad \int_i i^\dagger \mathcal{N}^\bullet \rightarrow \mathcal{N}^\bullet$$

are isomorphisms. We will give the prove for the first one being completely similar the argument for the second morphism. We reason by induction on the **cohomological length**

$$\mathbf{l}(\mathcal{M}^\bullet) := \max\{i \mid H^i(\mathcal{M}^\bullet) \neq 0\} - \min\{j \mid H^j(\mathcal{M}^\bullet) \neq 0\}$$

of \mathcal{M}^\bullet . If $\mathbf{l}(\mathcal{M}^\bullet) = 0$, then $\mathcal{M}^\bullet = \mathcal{M}[k]$ for some $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ and $k \in \mathbb{Z}$. In other words, we may assume $\mathcal{M}^\bullet = \mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ and the assertion follows from the preceding theorem. Let us assume that $\mathbf{l}(\mathcal{M}^\bullet) > 0$. By definition of the **truncated complexes** (Definition C.1.12), there exists $k \in \mathbb{Z}$ such that

$$\mathbf{l}(\tau^{\leq k} \mathcal{M}^\bullet) < \mathbf{l}(\mathcal{M}^\bullet) \quad \text{and} \quad \mathbf{l}(\tau^{> k}(\mathcal{M}^\bullet)) < \mathbf{l}(\mathcal{M}^\bullet).$$

By applying $i^\dagger \int_i$ to the distinguished triangle

$$\tau^{\leq k} \mathcal{M}^\bullet \rightarrow \mathcal{M}^\bullet \rightarrow \tau^{> k} \mathcal{M}^\bullet \xrightarrow{+1}$$

induced from the exact sequence (106) and the proposition C.1.13 in the appendix, we obtain the distinguished triangle

$$i^\dagger \int_i \tau^{\leq k} \mathcal{M}^\bullet \rightarrow i^\dagger \int_i \mathcal{M}^\bullet \rightarrow i^\dagger \int_i \tau^{> k} \mathcal{M}^\bullet \xrightarrow{+1}.$$

These triangles are connected via the following commutative diagram

$$\begin{array}{ccccc} \tau^{\leq k} \mathcal{M}^\bullet & \longrightarrow & \mathcal{M}^\bullet & \longrightarrow & \tau^{> k} \mathcal{M}^\bullet \xrightarrow{+1} \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_3 \\ i^\dagger \int_i \tau^{\leq k} \mathcal{M}^\bullet & \longrightarrow & i^\dagger \int_i \mathcal{M}^\bullet & \longrightarrow & i^\dagger \int_i \tau^{> k} \mathcal{M}^\bullet \xrightarrow{+1} \end{array}$$

By hypothesis of induction α_1 and α_2 are isomorphisms. Hence α_3 is also an isomorphism (cf. [43, Proposition 10.1.15]). \square

4.10 Characteristic varieties and holonomic \mathcal{D} -modules

In this section we will develop one of the central definitions in this lectures, namely, the notion of a *holonomic* \mathcal{D} -module. For this, we will globalize to the theory of \mathcal{D}_X -modules some facts of the theory of graded rings (cf. [22] and [32]).

We start with the following property [32, Proposition 2.1.1].

Proposition 4.10.1. [Definition]. *Let (\mathcal{M}, F) be a filtered \mathcal{D}_X -module. The following condition are equivalent to each other:*

- (i) $gr^F \mathcal{M}$ is coherent over $gr^F \mathcal{D}_X$.
- (ii) $F_i \mathcal{M}$ is coherent over \mathcal{O}_X for each i , and there exists $i_0 \gg 0$ satisfying $F_j \mathcal{D}_X \cdot F_i \mathcal{M} = F_{i+j} \mathcal{M}$, for $j \geq 0$ and $i \geq i_0$.
- (iii) There exist locally a surjective \mathcal{D}_X -homomorphism $\Phi : \mathcal{D}_X^{\oplus m} \rightarrow \mathcal{M}$ and integers n_1, \dots, n_m such that $\Phi(F_{i-n_1} \mathcal{D}_X \oplus \dots \oplus F_{i-n_m} \mathcal{D}_X) = F_i \mathcal{M}$.

If F satisfies one of the above equivalent conditions we say that F is a **good filtration**.

Remark 4.10.2. *A \mathcal{D}_X -module admits a good filtration if and only if it is a coherent \mathcal{D}_X -module.*

Let \mathcal{M} be a coherent \mathcal{D}_X -module and choose a good filtration on it. Let $\pi : T^*X \rightarrow X$ be the cotangent bundle of X . Regarding $\xi_1 := \sigma_1(\partial_{x_1}), \dots, \xi_{d_X} := \sigma_1(\partial_{x_{d_X}})$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^{d_X} \mathbb{C} dx_i$, we know that $\mathcal{O}_U[\xi_1, \dots, \xi_{d_X}]$ is identified with the sheaf of algebras $\pi_* \mathcal{O}_{T^*X}|_U$ and therefore we obtain a canonical identification $gr^F \mathcal{D}_X \simeq \pi_* \mathcal{O}_{T^*X}$, (43). We define the **characteristic variety** of \mathcal{M} , denoted by $\mathbf{Ch}(\mathcal{M})$, as the support of the coherent \mathcal{O}_{T^*X} -module

$$\widetilde{gr^F \mathcal{M}} := \mathcal{O}_{T^*X} \otimes_{\pi^{-1} \pi_* \mathcal{O}_{T^*X}} \pi^{-1}(gr^F \mathcal{M}).$$

Remark 4.10.3. *The characteristic variety does not depend on the choice of a good filtration F ([32, Theorem 2.2.1]).*

By a classic result in algebraic geometry (cf. [32, Lemma 3.3]) we have the following

Theorem 4.10.4. *For a short exact sequence $0 \rightarrow \mathcal{M} \rightarrow \mathcal{L} \rightarrow \mathcal{N} \rightarrow 0$ of coherent \mathcal{D}_X -modules, we have*

$$\mathbf{Ch}(\mathcal{L}) = \mathbf{Ch}(\mathcal{M}) \cup \mathbf{Ch}(\mathcal{N}).$$

Let us introduce the notion of **multiplicity** for $\mathcal{M} \in \mathbf{Mb}_c(\mathcal{D}_X)$. Let C be an irreducible component of the support of \mathcal{M} which will be denoted by $\mathbf{Supp}(\mathcal{M})$. Let U be an affine open subset of X such that $\overline{C \cap U} = C$ and let's denote the defining ideal of $C \cap U$ by $\mathfrak{p}_C \subset \mathcal{O}_U(U)$. Taking the localization at \mathfrak{p}_C we get an artinian $\mathcal{O}_U(U)_{\mathfrak{p}_C}$ -module $\mathcal{M}(U)_{\mathfrak{p}_C}$ whose length we denote by $m_C(\mathcal{M})$. We call it the **multiplicity** of \mathcal{M} along C . The next fact is just a global version of the *additive* property of the length.

Proposition 4.10.5. *Let $0 \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow \mathcal{L} \rightarrow 0$ be an exact sequence of coherent \mathcal{D}_X -modules. Then for an irreducible subvariety C of T^*X such that C is an irreducible component of $\mathbf{Ch}(\mathcal{N})$ we have*

$$m_C(\widehat{gr^F(\mathcal{N})}) = m_C(\widehat{gr^F(\mathcal{M})}) + m_C(\widehat{gr^F(\mathcal{L})})$$

Moreover, if we define the **characteristic cycle** of \mathcal{M} by

$$\mathbf{CC}(\mathcal{N}) = \sum_{C \in \mathbf{Supp}(\mathbf{Ch}(\mathcal{N}))} m_C(\widehat{gr^F(\mathcal{N})})C,$$

and $\mathbf{CC}_d(\mathcal{N})$ the sum taking only the irreducible components of degree d , then in particular for $d = \dim \mathbf{Ch}(\mathcal{N})$ we have

$$\mathbf{CC}_d(\mathcal{N}) = \mathbf{CC}_d(\mathcal{M}) + \mathbf{CC}_d(\mathcal{L}).$$

4.10.1 Holonomcity for Weyl algebras

In order to give a good intuition to the reader about the notion of holonomicity and to simplify the proof of the **Bernstein inequality** (4.11.1), we propose to study first the particular case when $X = \mathbb{A}_{\mathbb{C}}^n$. By remark 4.2.7 we can work on the algebraic side by taking global sections.

From now on, we will denote by $\mathbf{A}_n := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ the **n-th Weyl algebra**. We recall for the reader that this algebra has the following description

$$\mathbf{A}_n = \mathbb{C}[x_1, \dots, x_n, \partial_{x_1}, \dots, \partial_{x_n}] / ([\partial_{x_i}, x_j] - \delta_{ij}).$$

In the previous sections we have said that a **filtered module** \mathcal{M} is built up from pieces $F_d \mathcal{M}$ having a structure of \mathcal{O}_X -module as well. This property is important when dealing with computations of characteristic varieties. Nevertheless, we can not usually control the dimensions of the pieces $F_d \mathcal{M}$ (considered as \mathbb{C} -vectors spaces), and there is not hope in introducing a good notion of (algebraic) **dimension** and therefore neither the notion of **holonomicity**. To solve this drawback we will consider the **Bernstein filtration** $\{\mathcal{B}_k\}_{k \in \mathbb{N}}$ defined

in the example 4.2.4. We recall for the reader that this is a filtration of the Weyl algebra \mathbf{A}_n , and that \mathcal{B}_k consists of those differential operators of degree at most k . We also recall that

$$\deg(P) := \max\{|\alpha| + |\beta| : (\alpha, \beta) \in \mathbb{N}^{2n}, \exists \gamma \in \mathbb{C} \text{ s.t. } \gamma x^\alpha \partial^\beta \text{ appears in } P\}.$$

From now on, we will say that M is a **filtered** left \mathbf{A}_n -module, if M is endowed with a filtration $F = \{F_k\}_{k \in \mathbb{N}}$ of *finite-dimensional* \mathbb{C} -vector spaces, and which is compatible with the Bernstein filtration in the sense that $\mathcal{B}_i \cdot F_j \subseteq F_{i+j}$. Moreover, we will say that the filtration is **good** if the resulting graded module $\text{gr}^F(M)$ is a finitely generated $\text{gr}^{\mathcal{B}}(\mathbf{A}_n) = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$ -module (the last identification is (44)).

An important class of \mathbf{A}_n -modules for which it is possible to construct a good filtration are those *finitely generated* \mathbf{A}_n -modules. In fact, if M is such an \mathbf{A}_n -module, then we may find a finite-dimensional vector space F_0 with a basis consisting of a set of generators for M . Defining

$$F_k := \mathcal{B}_k \cdot F_0,$$

the resulting filtration is in fact a good filtration of M as the reader can easily prove.

Remark 4.10.6. *The preceding filtration is characterised by the existence of a positive integer number N_0 such that $\mathcal{B}_i \cdot F_l = F_{i+l}$, for all $l \geq N_0$. Moreover, a filtration that satisfies this condition is a good filtration (cf. [22, Chapter 8, theorem 3.1]).*

Example 4.10.7. (i) Using the Weyl relations we easily see that

$$\mathcal{B}_k(\mathbf{A}_{n+m}) = \sum_{i+j=k} \mathcal{B}_i(\mathbf{A}_n) \mathcal{B}_j(\mathbf{A}_m)$$

(ii) If (M, F) and (N, G) are good filtered \mathbf{A}_n and \mathbf{A}_m modules, respectively. Then

$$T_k := \sum_{i+j=k} F_i \otimes_{\mathbb{C}} G_j$$

defines a good filtration of $M \otimes_{\mathbb{C}} N$ as an \mathbf{A}_{n+m} -module. Let us just specify the left \mathbf{A}_{n+m} -action, because the rest easily follows from (i) and the previous remark. In fact, the action is induced via the canonical identification $\mathbf{A}_{n+m} = \mathbf{A}_n \otimes_{\mathbb{C}} \mathbf{A}_m$ by $P \otimes Q \cdot n \otimes m := Pm \otimes Qn$.

The following result is the starting point of the results and notions of this subsections. The reader can find a proof in [29, Chapter 1, theorem 7.5], [65, Chapter 2, section 3, theorem 2] or in [22, Chapter p, theorem 1.1].

Theorem 4.10.8 (Hilbert-Serre). *Let $M = \bigoplus_{k \in \mathbb{N}} M_k$ be finitely generated graded module over a polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. There exists a polynomial $\chi_M(t) \in \mathbb{Q}[t]$ and a positive integer N such that*

$$\sum_{i=1}^l \dim_{\mathbb{C}}(M_i) = \chi_M(l)$$

for all $l \geq N$.

The polynomial $\chi_M(t)$ is known as the **Hilbert polynomial**.

Let us suppose that M is a finitely generated left \mathbf{A}_n -module and that F is a good filtration. Let us denote by $\chi_M^F(t) \in \mathbb{Q}[t]$ the Hilbert polynomial of the graded module $\text{gr}^F(M)$ over the polynomial ring $\mathbb{C}[\underline{x}, \underline{x}']$ in $2n$ variables. The previous theorem tells that for $l \gg 0$

$$\chi_M^F(l) = \sum_{i=1}^l \dim_{\mathbb{C}}(F_i/F_{i-1}) = \dim_{\mathbb{C}}(F_l) \quad (74)$$

Definition 4.10.9. Let M be a finitely generated \mathbf{A}_n -module and F a good filtration. The **dimension** $\mathbf{d}(M)$ of M is the degree of the Hilbert polynomial $\chi_M^F(t)$. If $a_{\mathbf{d}(M)}$ is the leading coefficient of $\chi_M^F(t)$, we define the **multiplicity** of M by

$$\mathbf{m}(M) := \mathbf{d}(M)! a_{\mathbf{d}(M)}.$$

Both numbers are non-negative integers.

Remark 4.10.10. Even if a priori the preceding definitions seem to depend on the filtration F , it is possible to prove that the dimension and multiplicity of a finitely generated \mathbf{A}_n -module M are **invariants** of M . In other words, they do not depend on the filtration (cf. [22, Chapter 9, section 2]).

Example 4.10.11. Let us work out some basic examples.

- (i) **Dimension of the Weyl algebra.** We already know that the Bernstein filtration \mathcal{B} is a good filtration for the Weyl algebra \mathbf{A}_n . In order to compute its dimension, we can use (74). By definition we have ($l \gg 0$)

$$\begin{aligned} \chi_{\mathbf{A}_n}^{\mathcal{B}}(l) &= \dim_{\mathbb{C}}(\mathcal{B}_l) \\ &= \text{Non-negative solutions of } |\alpha| + |\beta| \leq l \\ &= \binom{l+2n}{2n} \end{aligned}$$

This implies that $\chi_{\mathbf{A}_n}^{\mathcal{B}}(t) = \binom{t+2n}{2n}$ has degree $2n = \mathbf{d}(\mathbf{A}_n)$.

- (ii) **Dimension of the ring of polynomials.** If \mathcal{S}_k denotes the vector space of polynomials of degree at most k . Then $\mathcal{S} = \{\mathcal{S}_k\}_{k \in \mathbb{N}}$ is a good filtration of $\mathbb{C}[x_1, \dots, x_n]$. In particular for $l \gg 0$

$$\chi_{\mathbb{C}[\underline{x}]}^{\mathcal{S}}(l) = \dim_{\mathbb{C}}(\mathcal{S}_l) = \binom{n+l}{n}.$$

As before this implies that $\chi_{\mathbb{C}[\underline{x}]}^{\mathcal{S}}(t) = \binom{t+n}{n}$ is a polynomial of degree $n = \mathbf{d}(\mathbb{C}[\underline{x}])$.

- (iii) **Fourier transform.** Let us start by considering the following automorphism of the Weyl algebra

$$\begin{aligned} \mathcal{F} : \mathbf{A}_n &\rightarrow \mathbf{A}_n \\ x_i &\mapsto \partial_{x_i} \\ \partial_{x_j} &\mapsto -x_j \end{aligned} \quad (75)$$

The automorphism \mathcal{F} is called the **Fourier transform**. It clearly preserves the Bernstein filtration, and given a finitely generated \mathbf{A}_n -module M , we can twist it and to obtain a finitely generated left \mathbf{A}_n -module $M_{\mathcal{F}}$ as follows. As vector spaces $M_{\mathcal{F}} = M$ and the left \mathbf{A}_n -structure is given by $P \bullet m = \mathcal{F}(P)m$. Let us prove that $\mathbf{d}(M_{\mathcal{F}}) = \mathbf{d}(M)$. To do that we start by taking a vector space F_0 whose basis is a set of generators for M . It is clear that $F_k := \mathcal{B}_k \cdot F_0$ is a good filtration for M . But given that $M_{\mathcal{F}}$ is also generated by F_0 and \mathcal{F} preserves the Bernstein filtration we have that $G_k := \mathcal{B}_k \bullet F_0 = F_k$ is a good filtration for $M_{\mathcal{F}}$. In particular, the associated Hilbert polynomials are the same, and therefore M and $M_{\mathcal{F}}$ have the same dimension and multiplicity.

Lemma 4.10.12. *Let M be a finitely generated left \mathbf{A}_n -module. Then $\mathbf{d}(M) \leq 2n$.*

Proof. It is not hard to prove that $\mathbf{d}(M) = \max\{\mathbf{d}(N), \mathbf{d}(M/N)\}$ for N a submodule of M (cf. [22, Chapter 9, theorem 3.2]). From this information and a standard inductive argument over the short exact sequence

$$0 \rightarrow \mathbf{A}_n \rightarrow \mathbf{A}_n^{\oplus m} \rightarrow \mathbf{A}_n^{\oplus(m-1)} \rightarrow 0$$

we can conclude that $\mathbf{d}(\mathbf{A}_n^{\oplus m}) = 2n$. Now, by hypothesis there exists a surjective morphism $\varphi : \mathbf{A}_n^{\oplus m} \rightarrow M$. From the previous reasoning we have that

$$2n = \mathbf{d}(\mathbf{A}_n^{\oplus m}) = \max\{\mathbf{d}(M), \mathbf{d}(\ker(\varphi))\}.$$

□

We actually have a lower bound for the dimension of finitely generated left \mathbf{A}_n -module. This is first version of the so-called Bernstein inequality.

Theorem 4.10.13 (Bernstein inequality). *Let M be a finitely generated left \mathbf{A}_n -module. Then $\mathbf{d}(M) \geq n$.*

Proof. Let F_0 be a vector space having a basis consisting of generators of M , and let us consider the filtration defined by $F_k := \mathcal{B}_k \cdot F_0$. This is a good filtration for M and such that $F_0 \neq 0$. We claim that the canonical morphism

$$\begin{array}{ccc} \mathcal{B}_k & \rightarrow & \text{Hom}_{\mathbb{C}}(F_k, F_{2k}) \\ P & \mapsto & [m \mapsto Pm] \end{array}$$

is injective. To prove this, let us take $0 \neq P \in \mathcal{B}_k$ and let us see that $P \cdot F_k \neq 0$. We proceed by induction. For $k = 0$, we have $\mathcal{B}_0 = \mathbb{C}$ and the statement is equivalent to $F_0 \neq 0$, which is true by construction. Let us now take $k \geq 1$ and let us suppose that the result is true for F_{k-1} . We proceed by contradiction and we assume that $P \cdot F_k = 0$, for some non-zero $P \in \mathcal{B}_k$. The reader might note that this implies that $P \notin \mathcal{B}_0$. Hence an expression for P must contain a monomial $\gamma x^\alpha \partial^\beta$ with $\gamma \in \mathbb{C}$ and $|\alpha| + |\beta| > 0$. If $\alpha_k \neq 0$, then $\alpha_k \gamma x^{\alpha - e_i} \partial^\beta$ is a monomial appearing in $[P, \partial_{x_k}]$ and $[P, \partial_{x_k}]$ is a non-zero element of \mathcal{B}_{k-1} .

On the other hand, since $P \cdot F_k = 0$, we have

$$[P, \partial_{x_k}]F_{k-1} \subseteq P \partial_{x_k} F_{k-1} \subseteq P \cdot F_k = 0,$$

which is clearly a contradiction. Let us back to the proof of the theorem. From the preceding injection we have for $k \gg 0$

$$\begin{aligned}\chi_{\mathbf{A}_n}^{\mathcal{B}}(k) &= \dim_{\mathbb{C}}(\mathcal{B}_k) \leq \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}}(F_k, F_{2k})) \\ &\leq \dim_{\mathbb{C}}(F_k^{\vee} \otimes_{\mathbb{C}} F_{2k}) \\ &= \dim_{\mathbb{C}}(F_k) \cdot \dim_{\mathbb{C}}(F_{2k}) \\ &= \chi_M^F(k) \cdot \chi_M^F(2k),\end{aligned}$$

which clearly implies that $2n = \deg(\chi_{\mathbf{A}_n}^{\mathcal{B}}(k)) \leq \deg(\chi_M^F(k) \cdot \chi_M^F(2k)) = 2\mathbf{d}(M)$. \square

Example 4.10.14. (i) We actually can attain the bounds. As we have showed in the example 4.10.11, $\mathbf{d}(\mathbf{A}_n) = 2n$ and $\mathbf{d}(\mathbb{C}[x_1, \dots, x_n]) = n$.

- (ii) If I is a non zero left ideal of \mathbf{A}_n , then $\mathbf{d}(A/I) \leq 2n - 1$. To prove this, it is enough to consider the case when $I = \mathbf{A}_n \cdot P$ is a principal ideal. The reason is that for a general non-zero ideal I and $P \in I$ we have that \mathbf{A}_n/I is a quotient of $A/A \cdot P$ and $\mathbf{d}(\mathbf{A}_n/I) \leq \mathbf{d}(\mathbf{A}_n/\mathbf{A}_n \cdot P)$ by the first paragraph in the proof of the lemma 4.10.12.

Let us consider the short exact sequence

$$0 \rightarrow \mathbf{A}_n \xrightarrow{(\cdot) \cdot P} \mathbf{A}_n \rightarrow \mathbf{A}_n/\mathbf{A}_n \cdot P \rightarrow 0.$$

If $\mathbf{d}(\mathbf{A}_n/\mathbf{A}_n \cdot P) = \mathbf{d}(\mathbf{A}_n) = 2n$, then from [22, Chapter 9, theorem 3.2]

$$m(\mathbf{A}_n) = m(\mathbf{A}_n) + m(\mathbf{A}_n/\mathbf{A}_n \cdot P)$$

which is clearly impossible.

- (ii) **Dimension of a tensor product.** Let M be a finitely generated \mathbf{A}_n -module and N a finitely generated \mathbf{A}_m -module. We want to show that $\mathbf{d}(M \otimes_{\mathbb{C}} N) = \mathbf{d}(M) + \mathbf{d}(N)$. We start by given a description of the graded module $\mathrm{gr}^T(M \otimes_{\mathbb{C}} N)$ associated to the filtration $\{T_k\}$ defined in the example 4.10.7 (ii). To do that, we suppose that F and G are good filtrations for M and N , respectively, and we remark that the kernel of the canonical surjective map

$$\begin{array}{ccc} \kappa_{ij} : F_i \otimes_{\mathbb{C}} G_j & \mapsto & \mathrm{gr}_i^F(M) \otimes_{\mathbb{C}} \mathrm{gr}_j^G(N) \\ m \otimes n & \rightarrow & \bar{m} \otimes \bar{n} \end{array}$$

it is clearly contained in T_{i+j-1} . With this information, we can form the following commutative triangle

$$\begin{array}{ccc} \bigoplus_{i+j=l} (F_i \otimes_{\mathbb{C}} G_j / \ker(\kappa_{ij})) & \longrightarrow & \bigoplus_{i+j=l} (\mathrm{gr}_i^F(M) \otimes_{\mathbb{C}} \mathrm{gr}_j^G(N)) \\ & \searrow & \uparrow \kappa_l \\ & & \mathrm{gr}_l^T(M \otimes_{\mathbb{C}} N). \end{array}$$

Since the morphism on the top is bijective, then so is the morphism κ_l on the right-hand side of the triangle. Let us use this information to compute $\mathbf{d}(M \otimes_{\mathbb{C}} N)$.

Using the isomorphism κ_l we see that

$$\dim_{\mathbb{C}}(T_k) = \dim_{\mathbb{C}} \left(\bigoplus_{i+j=l} (F_i \otimes_{\mathbb{C}} G_j / \ker(\kappa_{ij})) \right) - \dim_{\mathbb{C}}(T_{k-1}).$$

Repeating this process and using the item (ii) of the example 4.10.7, we find that

$$\begin{aligned} \dim_{\mathbb{C}}(T_l) &= \sum_{k=0}^l \sum_{i+j=k} \dim_{\mathbb{C}}(\text{gr}_i^F(M)) \dim_{\mathbb{C}}(\text{gr}_j^G(N)) \\ &\leq \sum_{i=0}^l \dim_{\mathbb{C}}(\text{gr}_i^F(M)) \sum_{j=0}^l \dim_{\mathbb{C}}(\text{gr}_j^G(N)) \\ &= \dim_{\mathbb{C}}(F_l) \dim_{\mathbb{C}}(G_l) \\ &\leq \dim_{\mathbb{C}}(T_{2l}). \end{aligned}$$

In consequence, for $l \gg 0$ we have

$$\chi_{M \otimes N}^T(l) \leq \chi_M^F(l) \chi_N^G(l) \leq \chi_{M \otimes N}^T(2l),$$

and therefore as a polynomials we have

$$\deg(\chi_{M \otimes N}^T(t)) = \deg(\chi_M^F(t)) + \deg(\chi_N^G(t)). \quad (76)$$

We end this subsection with the following important remark from [29, Chapter I, section 7, theorem 7.5]. This results allows us to connect the notions introduced in this subsection with the ones given in previous subsection about the characteristic varieties.

Remark 4.10.15. *Let M be a finitely generated left \mathbf{A}_n -module. Then*

$$\dim(\text{Ch}(M)) = \mathbf{d}(M).$$

Here $\text{Ch}(M) = V(\sqrt{\text{Ann}_{\mathbb{C}[\underline{x}, \underline{\xi}]}(\text{gr}^F(M))}) = \mathbf{Supp}_{\mathbb{C}[\underline{x}, \underline{\xi}]}(\text{gr}^F(M))$. The second equality is a well-known fact in commutative algebra. Its proof is a standard reasoning dealing with the localisation.

4.11 Bernstein inequality

The heart of this subsection is the following result. The idea will be to reduce the computations to ones given in the previous subsection by using the remark 4.10.15.

Proposition 4.11.1 (Global Bernstein inequality). *Let \mathcal{M} be a coherent \mathcal{D}_X -module. Then*

$$\dim \mathbf{Ch}(\mathcal{M}) \geq d_X.$$

To prove the above proposition we will need the following relation ([32, Lemma 2.3.5]). Before the statement let us fix the following notation. Let Z be a smooth closed subvariety of Y and let $i : Z \rightarrow X$ the respective closed embedding. This

morphism gives rise to two morphisms between the cotangent bundles of X and Z :

$$\begin{array}{ccc}
 & Z \times_X T^*X & \\
 \rho_i \swarrow & & \searrow p_2 \\
 T^*Z & & T^*X
 \end{array} \tag{77}$$

Here p_2 is the projection, which is a closed embedding, and $\rho_i : Z \times_X T^*X \rightarrow T^*Z$ is defined as follows. Let \mathcal{J} be the ideal sheaf of X . First of all, the pull-back morphism $i^*\Omega_{X/\mathbb{C}}^1 \rightarrow \Omega_{Z/\mathbb{C}}^1$ (cf. [29, Chapter II, proposition 8.11]) induces a short exact sequence

$$0 \rightarrow \mathcal{T}_Z \rightarrow i^*\mathcal{T}_X \rightarrow \mathcal{N}_{Z/X} \rightarrow 0$$

where $\mathcal{N}_{Z/X} := \mathcal{H}om_{\mathcal{O}_Z}(\mathcal{J}/\mathcal{J}^2, \mathcal{O}_Z)$ is the **normal sheaf** ([29, Chapter II, proposition 8.12]). By smoothness, this is a locally free \mathcal{O}_Z -module of rank $d_X - d_Z (= \text{codim}_X(Z))$. The first morphism defines ρ_i which is in particular a smooth morphism of relative dimension $\text{codim}_X(Z)$.

Locally, if X is endowed with local coordinates $\{x_i, \partial_{x_i}\}_{1 \leq i \leq d}$, Z is the hypersurface $x = x_1 = 0$ and i is given by the morphism of \mathbb{C} -algebras $i : A \rightarrow B$ ($A = \mathcal{O}_X(X)$ and $B = A/I$), then previous diagram is induced by

$$\begin{array}{ccc}
 & B[\xi, \xi_2, \dots, \xi_d] & \\
 \rho_i \nearrow & & \nwarrow p_2 \\
 A[\xi, \xi_2, \dots, \xi_d] & & B[\xi_2, \dots, \xi_d]
 \end{array}$$

$\xi = \sigma_1(\partial_x)$ and $\xi_i = \sigma_1(\partial_{x_i})$, for $2 \leq i \leq d$, are the respective principal symbols.

Lemma 4.11.2. *Let Z be a smooth closed subvariety of X and let \mathcal{M} be a coherent \mathcal{D}_Z -module. Set $\mathcal{N} = \int_i^0 \mathcal{M}$, where $i : Z \hookrightarrow X$ denotes the embedding. Let $\rho_i : Z \times_X T^*X \rightarrow T^*Z$ and $p_2 : Z \times_X T^*X \hookrightarrow T^*X$ be the natural morphisms induced by i . Then we have*

$$\mathbf{Ch}(\mathcal{N}) = p_2 \rho_i^{-1}(\mathbf{Ch}(\mathcal{M})).$$

Proof. The problem being local on Z we can assume that X is endowed with local coordinates $\{x_i, \partial_{x_i}\}_{1 \leq i \leq d}$, and an inductive argument allows to suppose that Z is a hypersurface of X defined by the equation $x = x_1 = 0$. We also set $\partial_{x_1} = \partial$ and $\xi = \sigma_1(\partial)$ the principal symbol of ∂ . Moreover, by affineness, we can consider the global situation. We set $M := \Gamma(Z, \mathcal{M})$.

Now, let us recall that after the example 4.3.8 we know that

$$N := \Gamma\left(X, \int_i^0 \mathcal{M}\right) = \mathbb{C}[\partial] \otimes_{\mathbb{C}} M.$$

With this information, and after fixing a good filtration F of M (with $F_{-1}M = 0$), we can define a filtration G of N by

$$G_j N := \sum_{l=1}^j \sum_{k \leq l} \mathbb{C} \partial^k \otimes_{\mathbb{C}} F_{l-1} M.$$

At his point we can observe that by definition of G , every term $G_k N$ is a coherent $B := \mathcal{O}_Z(Z)$ -module and therefore the second item in the proposition 4.10.1 tells us that G is a good filtration. We also point out that

$$\mathrm{gr}_j^G N = \bigoplus_{l=0}^j \mathbb{C} \xi^l \otimes_{\mathbb{C}} (F_{j-1} M / F_{j-l-1} M)$$

and that $\mathrm{gr}^F M$ is a finitely generated $B[\xi_2, \dots, \xi_d]$ -module. If $A := \mathcal{O}_X(X)$. Then the previous equality clearly implies that

$$\begin{aligned} \mathrm{gr}^G N &= \mathbb{C}[\xi] \otimes_{\mathbb{C}} \mathrm{gr}^F M \\ &= \mathbb{C}[\xi] \otimes_{\mathbb{C}} B[\xi_2, \dots, \xi_d] \otimes_{\mathbb{C}[\xi_2, \dots, \xi_d]} \mathrm{gr}^F M \\ &= B[\xi, \xi_2, \dots, \xi_d] \otimes_{B[\xi_2, \dots, \xi_d]} \mathrm{gr}^F M \end{aligned}$$

as $A[\xi_1, \dots, \xi_d]$ -modules since the ideal of definition $I \subseteq A$ of B acts trivially on $\mathrm{gr}^G(N)$. Tensoring the $B[\xi_2, \dots, \xi_d]$ -module $\mathrm{gr}^F M$ by $B[\xi, \xi_2, \dots, \xi_d]$ corresponds to pull back $\mathrm{gr}^F(\mathcal{M})$ via ρ_i , and the previous sentence correspondence to push forward $\rho_i^*(\mathrm{gr}^F(\mathcal{M}))$ via p_2 (i.e, to consider the resulting module over $A[\xi_1, \dots, \xi_d]$). This proves that

$$\mathbf{Ch}(N) = \mathbf{Supp}(\mathrm{gr}^G(N)) = p_2 \rho_i^{-1}(\mathbf{Supp}(\mathrm{gr}^F(M))) = p_2 \rho_i^{-1}(\mathbf{Ch}(M)).$$

□

Corollary 4.11.3. *Let $i : Z \rightarrow Y$ be a closed embedding, and \mathcal{M} a coherent \mathcal{D}_Z -module. Then the coherent \mathcal{D}_X -module $\mathcal{N} := \int_i^0 \mathcal{M}$ satisfies*

$$\dim(\mathbf{Ch}(\mathcal{N})) = \dim(\mathbf{Ch}(\mathcal{M})) + \dim(X) - \dim(Z).$$

Proof. The proof follows after having remarked that in the proof of the previous lemma, we have shown that ρ_i is a smooth morphism of relative dimension $\mathrm{codim}_X(Z)$ and p_2 is a closed embedding. □

The following reasoning is given in [62] (discussion after the proof of the proposition 13.10).

Proof of proposition 4.11.1. Let \mathcal{M} be a quasi-coherent \mathcal{D}_X -module. Since the question is local, we can assume that X is affine. By smoothness, we can find a closed embedding $i : X \rightarrow \mathbb{A}_{\mathbb{C}}^d$ into an affine space. By corollary 4.11.3 we know that

$$\begin{aligned} \dim(\mathbf{Ch}(\mathcal{M})) - \dim(X) &= \dim\left(\int_i^0 \mathcal{M}\right) - \dim(\mathbb{A}_{\mathbb{C}}^d) \\ &= \dim\left(\int_i^0 \mathcal{M}\right) - d \\ &\geq 0. \end{aligned}$$

The final equality follows from the theorem 4.10.13. □

Definition 4.11.4. *A coherent \mathcal{D}_X -module \mathcal{M} is called a **holonomic** if it satisfies $\dim \mathbf{Ch}(\mathcal{M}) \leq d_X$.*

Example 4.11.5. Let's suppose that \mathcal{M} is an integrable connection. If $F_i\mathcal{M} = 0$ for $i < 0$ and $F_i\mathcal{M} = \mathcal{M}$ for $i \geq 0$, then F defines a good filtration on \mathcal{M} and $\text{gr}^F\mathcal{M} = \mathcal{M}$. Let us denote by $s : T^*X \rightarrow X$ is the zero section. Locally, if X is supposed to be endowed with a system of coordinates $\{x_i, \partial_{x_i}\}_{1 \leq i \leq d}$, then s is described via the comorphism

$$\begin{array}{ccc} s^\sharp : \mathcal{O}_X[\xi_1, \dots, \xi_d] & \rightarrow & \mathcal{O}_X \\ & \xi_i & \mapsto 0 \\ & f & \mapsto f \end{array} \quad (78)$$

Now, by construction $\mathcal{T}_X \subseteq \text{Ann}_{\pi_*\mathcal{O}_{T^*X}}(\text{gr}^F\mathcal{M})$ and by definition

$$\mathbf{Ch}(\mathcal{M}) = \mathbf{Supp}(\mathcal{M}) = V\left(\sqrt{\text{Ann}_{\mathbb{C}[\xi]}(\mathcal{M})}\right)$$

From the previous two relations, we see that if $\mathfrak{p} \in \mathbf{Ch}(\mathcal{M})$ then $\sum_{i=1}^d \mathcal{O}_X[\xi]\xi_i \subseteq \mathfrak{p}$. In other words, we have from (78) that \mathfrak{p} lies in T_X^*X . This proves that $\mathbf{Ch}(\mathcal{M}) = T_X^*X$ and therefore $\dim(\mathbf{Ch}(\mathcal{M})) = \dim(X)$.

Example 4.11.6. (i) From the example 4.10.14 (ii) and the Bernstein inequality 4.10.13, we know that if I is a left ideal of \mathbf{A}_1 then $1 \leq \mathbf{d}(\mathbf{A}_1/I) \leq 1$. So \mathbf{A}_1/I is holonomic.

(ii) If M is a holonomic \mathbf{A}_n -module and N is a holonomic \mathbf{A}_m -module, then $M \otimes_{\mathbb{C}} N$ is a holonomic $\mathbf{A}_{n+m} = \mathbf{A}_n \otimes_{\mathbb{C}} \mathbf{A}_m$ -module by the third item in the example 4.10.14.

From now on, we will denote by $\mathbf{Mod}_{hol}(\mathcal{D}_X)$ the full subcategory of $\mathbf{Mod}(\mathcal{D}_X)$ consisting of **holonomic** \mathcal{D}_X -modules.

4.12 Duality functors

We have introduced the notion of holonomicity via the dimension of the **characteristic variety**. In this section we will give a second description of this category by using the so-called **duality functors**. The definition will be inspired in the following example (cf. [32, Section 2.6]).

Example 4.12.1. Let us take $0 \neq P \in \mathbf{A}_1$ an operator of positive degree. We already know that left \mathbf{A}_1 -module $\mathbf{A}_1/\mathbf{A}_1 \cdot P$ is holonomic. Now, by applying the functor $\text{Hom}_{\mathbf{A}_1}(\mathbf{A}_1, \bullet)$ the short exact sequence³⁵

$$0 \rightarrow \mathbf{A}_1 \xrightarrow{(\bullet) \cdot P} \mathbf{A}_1 \rightarrow \mathbf{A}_1/\mathbf{A}_1 \cdot P \rightarrow 0$$

we get an exact sequence

$$0 \rightarrow \text{Hom}_{\mathbf{A}_1}(\mathbf{A}_1/\mathbf{A}_1 \cdot P, \mathbf{A}_1) \rightarrow \mathbf{A}_1 \xrightarrow{P \cdot (\bullet)} \mathbf{A}_1$$

Where we have used the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{A}_1}(\mathbf{A}_1, \mathbf{A}_1) & \xrightarrow{\varphi \mapsto P\varphi} & \text{Hom}_{\mathbf{A}_1}(\mathbf{A}_1, \mathbf{A}_1) \\ \uparrow & & \downarrow \\ \mathbf{A}_1 & \dashrightarrow & \mathbf{A}_1 \end{array}$$

³⁵Injectivity becomes from the fact that if $QP = 0$, then $\deg(QP) = \deg(Q) + \deg(P) = 0$. Since $\deg(P) > 0$ we have that $Q \in \mathbb{C}$, and therefore $Q \in \ker((\bullet) \cdot P)$ if $Q = 0$.

From this we get the following. First of all

$$\mathrm{Ext}_{\mathbf{A}_1}^0(\mathbf{A}_1/\mathbf{A}_1 \cdot P, \mathbf{A}_1) = \mathrm{Hom}_{\mathbf{A}_1}(\mathbf{A}_1/\mathbf{A}_1 \cdot P, \mathbf{A}_1) = \ker((\bullet) \cdot P : \mathbf{A}_1 \rightarrow \mathbf{A}_1) = 0$$

and the only non-vanish cohomology group is the first one

$$\mathrm{Ext}_{\mathbf{A}_1}^1(\mathbf{A}_1/\mathbf{A}_1 \cdot P, \mathbf{A}_1) \simeq \mathbf{A}_1/P \cdot \mathbf{A}_1$$

which is a right \mathbf{A}_1 -module. To obtain a left \mathbf{A}_1 -module we apply the transposition (51), which is equivalent to tensoring with $(\bullet) \otimes_{\mathbb{C}[x]} \Omega_{\mathbf{A}_1^c}^\vee$. By [22, Chapter 16, proposition 2.1] we get

$$\mathrm{Ext}_{\mathbf{A}_1}^1(\mathbf{A}_1/\mathbf{A}_1 \cdot P, \mathbf{A}_1) \otimes_{\mathbb{C}[x]} \Omega_{\mathbf{A}_1^c}^\vee = \mathbf{A}_1/\mathbf{A}_1 \cdot \tau(P)$$

Definition 4.12.2. We define the **duality functor** $\mathbb{D}_X : \mathbf{D}^-(\mathcal{D}_X) \rightarrow \mathbf{D}^+(\mathcal{D}_X)^{op}$ by

$$\begin{aligned} \mathbb{D}_X(\mathcal{M}^\bullet) &:= R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X] \\ &= R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X]). \end{aligned}$$

Let us remark that a priori the complex $\mathbb{D}_X(\mathcal{M})$ can be infinite, which makes the computations harder. Let us remark that this is not the case (cf. [32, Proposition 2.6.5])

Proposition 4.12.3. (i) The functor \mathbb{D}_X sends $\mathbf{D}_c^b(\mathcal{D}_X)$ to $\mathbf{D}_c^b(\mathcal{D}_X)$.

(ii) $(\mathbb{D}_X)^2 = \mathrm{Id}$ on $\mathbf{D}_c^b(\mathcal{D}_X)$.

Proof. To prove (i) we may assume that $\mathcal{M}^\bullet = \mathcal{M} \in \mathbf{Mod}_c(\mathcal{D}_X)$ is concentrated in zero degree. By definition, it is enough to prove that $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{D}) \in \mathbf{Mod}_c(\mathcal{D}_X)$. Let $U \subseteq X$ be an affine open subset. By the corollary 4.2.12 we can find a resolution $\mathcal{P}^\bullet \rightarrow \mathcal{M}|_U$ of $\mathcal{M}|_U$ by free \mathcal{D}_U -modules of finite rank. Given that U is affine $\mathcal{P}^\bullet(U) \rightarrow \mathcal{M}(U)$ is a resolution of $\mathcal{M}(U)$ by free $\mathcal{D}_X(U)$ -modules of finite rank. From this information we have the following relations

$$\begin{aligned} \mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{D}_X)(U) &= (H^i \mathrm{Hom}_{\mathcal{D}_U}(\mathcal{P}^\bullet, \mathcal{D}_U))(U) \\ &= H^i(\mathrm{Hom}_{\mathcal{D}_U}(\mathcal{P}^\bullet, \mathcal{D}_U)(U)) \\ &= H^i(\mathrm{Hom}_{\mathcal{D}_X(U)}(\mathcal{P}^\bullet, \mathcal{D}_U)) \\ &= H^i(\mathrm{Hom}_{\mathcal{D}_X(U)}(\mathcal{P}^\bullet(U), \mathcal{D}_X(U))) \\ &= \mathrm{Ext}_{\mathcal{D}_X(U)}^i(\mathcal{M}(U), \mathcal{D}_X(U)) \end{aligned} \tag{79}$$

This proves (i). This reasoning also proves the boundedness of the complex $\mathbb{D}_X(\mathcal{M})$.

For the second part we follow word by word the proof given in [32, Proposition 2.6.5 (ii)]. To start with, we need to construct a canonical morphism $\mathcal{M}^\bullet \rightarrow \mathbb{D}_X^2(\mathcal{M}^\bullet)$, for $\mathcal{M}^\bullet \in \mathbf{D}^b(\mathcal{D}_X)$. By definition, we have

$$\mathbb{D}_X^2(\mathcal{M}^\bullet) \simeq R\mathrm{Hom}_{\mathcal{D}_X^{op}}(R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X), \mathcal{D}_X),$$

where $R\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X)$ and \mathcal{D}_X are regarded as objects in $\mathbf{D}^b(\mathcal{D}_X)^{op}$ (complexes of right \mathcal{D}_X -modules), and the left multiplication on the right-hand side is

induced from the left multiplication of \mathcal{D}_X on \mathcal{D}_X . Set $\mathcal{H}^\bullet := R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X)$. Applying $H^0(R\Gamma(X, \bullet))$ to

$$R\mathcal{H}om_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X^{op}}(\mathcal{M}^\bullet \otimes_{\mathbb{C}} \mathcal{H}^\bullet, \mathcal{D}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, R\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{H}^\bullet, \mathcal{D}_X)),$$

we obtain

$$\mathrm{Hom}_{\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_X^{op}}(\mathcal{M}^\bullet \otimes_{\mathbb{C}} \mathcal{H}^\bullet, \mathcal{D}_X) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{M}^\bullet, R\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{H}^\bullet, \mathcal{D}_X)).$$

In consequence, the canonical morphism

$$\mathcal{M}^\bullet \otimes_{\mathbb{C}} \mathcal{H}^\bullet = \mathcal{M}^\bullet \otimes_{\bullet} R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X) \rightarrow \mathcal{M}^\bullet$$

in $\mathbf{D}^b(\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}^{op})$ induces a canonical morphism

$$\mathcal{M}^\bullet \rightarrow R\mathcal{H}om_{\mathcal{D}_X^{op}}(\mathcal{H}^\bullet, \mathcal{D}_X) = \mathbb{D}_X^2(\mathcal{M}^\bullet)$$

in $\mathbf{D}^b(\mathcal{D}_X)$. To show that this is an isomorphism we can assume that X is affine because we are dealing with a local question. Moreover, we can even replace \mathcal{M}^\bullet by \mathcal{D}_X and the assertion follows. \square

We have the following global versions of [32, Theorem D.4.3]. The reader can find an explicit proof in [41, 66]. Let \mathcal{M} be a coherent \mathcal{D}_X -module.

- (i) $\mathbf{codim}_{T^*X}(\mathbf{Ch}(\mathcal{E}xt_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1})) \geq i$.
- (ii) $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathcal{D}_X) = 0$ ($i < \mathbf{codim}_{T^*X}(\mathbf{Ch}(\mathcal{M}))$).

We can finally reinterpret holonomicity in terms of the duality functor as follows.

Proposition 4.12.4. *Let \mathcal{M} be a coherent \mathcal{D}_X -module.*

- (i) $H^i(\mathbb{D}_X(\mathcal{M})) = 0$ unless $-(d_X - \mathbf{codim}(\mathbf{Ch}_{T^*X}(\mathcal{M}))) \leq i \leq 0$.
- (i) $\mathbf{codim}_{T^*X}(\mathbf{Ch}(\mathcal{M})) \geq d_X - i$.
- (iii) \mathcal{M} is holonomic if and only if $H^i(\mathbb{D}_X(\mathcal{M})) = 0$ for all $i \neq 0$.
- (iv) If \mathcal{M} is holonomic, then $\mathbb{D}_X(\mathcal{M}) \simeq \mathcal{M}$ is also holonomic.

Proof. We follow word by word the proof given in [32, Corollary 2.6]. The first two statements are equivalent versions of (i) and (ii) in the preceding remark. The fourth item and the "only if" part of (iii) follows from (i), (ii) and the Bernstein inequality. To show the "if" part, we assume that $H^i(\mathbb{D}_X(\mathcal{M})) = 0$, for all $i \neq 0$, i.e., $\mathbb{D}_X(\mathcal{M}) \simeq H^0(\mathbb{D}_X(\mathcal{M}))$ in $\mathbf{D}_c^b(\mathcal{D}_X)$. Let us denote by $\mathcal{M}^\vee := H^0(\mathbb{D}_X(\mathcal{M}))$. By the second statement of the previous proposition, we know that $\mathbb{D}_X(\mathcal{M}^\vee) = \mathbb{D}_X^2(\mathcal{M}) = \mathcal{M}$ and therefore $H^0(\mathbb{D}_X(\mathcal{M}^\vee)) \simeq \mathcal{M}$. Moreover, by (ii), we also have that

$$\mathbf{codim}_{T^*X} \mathbf{Ch}(H^0(\mathbb{D}_X(\mathcal{M})) \simeq \mathcal{M}) \geq d_X,$$

so $\mathbf{D}_X(\mathcal{M}^\vee) \simeq \mathcal{M}$ is a holonomic \mathcal{D}_X -module. \square

Example 4.12.5. (i) Considering the resolution $0 \rightarrow \mathcal{D}_X \rightarrow \mathcal{D}_X \rightarrow 0$ of \mathcal{D}_X we have the following

$$\begin{aligned}\mathbb{D}_X(\mathcal{D}_X) &= [0 \rightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{D}_X) \rightarrow 0] \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X] \\ &= \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1}[d_X].\end{aligned}$$

In particular

$$H^k(\mathbb{D}(\mathcal{D}_X)) = \begin{cases} \mathcal{D}_X \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} & \text{if } k = -d_X, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Let \mathcal{M} be an integrable connection. We consider it as a complex in $\mathbf{D}^-(\mathcal{D}_X)$ concentrated in zero degree. Let us prove that

$$\mathbb{D}_X(\mathcal{M}) \simeq \mathcal{H}om_{\mathcal{O}}(\mathcal{M}, \mathcal{O}_X).$$

To do that, we start by remarking that given that \mathcal{M} is locally free then we can tensor by $(\bullet) \otimes_{\mathcal{O}_X} \mathcal{M}$ the **Spencer resolution** (subsection 4.7) to obtain a resolution of \mathcal{M}

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{\bullet} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \mathcal{M} \rightarrow 0.$$

Let us use this resolution to compute the cohomology group $\mathcal{E}xt_{\mathcal{D}_X}^{d_X}(\mathcal{M}, \mathcal{D}_X)$. We have

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X-1} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{D}_X) & \rightarrow & \mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^{d_X-1} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{D}_X) & \rightarrow & \mathcal{H}om_{\mathcal{O}_X}(\bigwedge^{d_X} \mathcal{T}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \mathcal{D}_X) \\ \downarrow & & \downarrow \\ \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X-1} \otimes_{\mathcal{O}_X} \mathcal{D}_X) & \xrightarrow{\kappa} & \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{D}_X) \end{array}$$

Moreover, using the fact that \mathcal{M} is a locally free \mathcal{O}_X -module, we also have

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{D}_X) &= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega_X^{d_X} \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ &= \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X}) \otimes_{\mathcal{O}_X} \mathcal{D}_X \\ &\xrightarrow{\varphi \otimes P \mapsto \varphi \cdot P} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X}) \rightarrow 0.\end{aligned}$$

The morphism κ and the preceding surjection, we see that

$$\mathcal{E}xt_{\mathcal{D}_X}^{d_X}(\mathcal{M}, \mathcal{D}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X}) \quad (80)$$

and therefore

$$H^0(\mathbb{D}_X(\mathcal{M})) \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \Omega_X^{d_X}) \otimes_{\mathcal{O}_X} \Omega_X^{\otimes -1} \simeq \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{O}_X)$$

The third item in the previous proposition proves the initial claim.

(iii) From the preceding item we have the following relation

$$\mathbb{D}_X(\mathcal{O}_X) = H^0(\mathbb{D}_X(\mathcal{O}_X)) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X.$$

This gives a reason to the shift $[d_x]$ made in the definition of \mathbb{D}_X . This is an adjustment so that \mathbb{D}_X sends \mathcal{O}_X to itself.

Lemma 4.12.6. For $\mathcal{M}^\bullet \in D_c^b(\mathcal{D}_X)$ and $\mathcal{N}^\bullet \in D^b(\mathcal{D}_X)$, we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet.$$

Proof. First of all let's note that we have a canonical morphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet \rightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet). \quad (81)$$

By 4.2.12 and the additive property of $\mathcal{H}om$, for a finite direct sum, we can assume that $\mathcal{M}^\bullet = \mathcal{D}_X$ and the assertion follows since both sides of (81) are isomorphic to \mathcal{N}^\bullet . \square

Proposition 4.12.7. For $\mathcal{M}^\bullet \in D_c^b(\mathcal{D}_X)$, and $\mathcal{N}^\bullet \in D^b(\mathcal{D}_X)$ we have an isomorphism

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}_X(\mathcal{M}^\bullet)) \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{N}^\bullet. \quad (82)$$

in $D^b(\mathbb{C}_X)$. In particular we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{N}^\bullet) \simeq \Omega_X \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet[-d_X]. \quad (83)$$

Proof. Let us see (83). To do that, we remark for the reader that if we prove that

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \simeq \Omega_X[-d_X]$$

then by the previous lemma, we will have

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{N}^\bullet) \simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet \simeq \Omega_X \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet[-d_X]$$

In other words, we can assume that $\mathcal{N}^\bullet = \mathcal{D}_X$ and to use the Spencer resolution to compute $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X)$. We have

$$\begin{aligned} & R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X) \\ &= \left[\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{D}_X) \rightarrow \cdots \rightarrow \mathcal{H}om_{\mathcal{D}_X} \left(\mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d_X} \mathcal{T}_X, \mathcal{D}_X \right) \right] \\ &\simeq \left[\mathcal{D}_X \rightarrow \cdots \rightarrow \mathcal{H}om_{\mathcal{O}_X} \left(\bigwedge^{d_X} \mathcal{T}_X, \mathcal{D}_X \right) \right] \\ &\simeq \left[\mathcal{D}_X \rightarrow \cdots \rightarrow \bigwedge^{d_X} \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{D}_X \right] \\ &\simeq \Omega_X[-d_X]. \end{aligned}$$

The last isomorphism comes from (67). This gives us (83). Now, by definition and the last lemma we have

$$R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{N}^\bullet) \simeq (\Omega_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathbb{D}_X(\mathcal{M}^\bullet)) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{N}^\bullet[-d_X] \quad (84a)$$

$$\simeq \Omega_X \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathbb{D}_X(\mathcal{M}^\bullet) \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{N}^\bullet[-d_X] \quad (84b)$$

$$\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathbb{D}_X(\mathcal{M}^\bullet)) \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathcal{N}^\bullet \quad (84c)$$

The last isomorphism is (83). \square

We end this subsection with the following computation of the **global dimension** of the stalk $\mathcal{D}_{X,x}$. We recall for the reader that in corollary 4.2.12 we have proved that every module $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$ has a finite resolution of length $2d_X$ by locally projective \mathcal{D}_X -modules. We can improve this results as follows (cf. [32, Theorem 2.6.11]).

Theorem 4.12.8. *Let $U \subseteq X$ be an affine open subset and $x \in X$.*

- (i) *The rings $\mathcal{D}_X(U)$ and $\mathcal{D}_{X,x}$ have left and right global dimension d_X .*
- (ii) *If $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$, then admits a resolution*

$$0 \rightarrow \mathcal{P}_{d_X} \rightarrow \cdots \rightarrow \mathcal{P}_1 \rightarrow \mathcal{P}_0 \rightarrow \mathcal{M} \rightarrow 0$$

by locally projective \mathcal{D}_X -modules. If $\mathcal{M} \in \mathbf{Mod}_c(\mathcal{D}_X)$, we can take the \mathcal{P}_i 's to be locally free of finite rank.

Proof. Given that $\mathcal{D}_X(U)$ is a left noetherian ring with finite length global dimension, its global left dimension equals the largest integer m such that there exists a finitely generated $\mathcal{D}_X(U)$ -module M satisfying $\mathrm{Ext}_{\mathcal{D}_X(U)}^m(M, \mathcal{D}_X(U)) \neq 0$. By the Bernstein inequality (theorem) and the first item in the remarks given before the proposition 4.12.4, we have $\mathrm{Ext}_{\mathcal{D}_X(X)}^i(M, \mathcal{D}_X(U)) = 0$ for any finitely generated $\mathcal{D}_X(U)$ -module M and $i > d_X$. Moreover, by (80) and (79) we have

$$\begin{aligned} \mathcal{E}xt_{\mathcal{D}_X}^{d_X}(\mathcal{O}_X, \mathcal{D}_X)(U) &= \mathrm{Ext}_{\mathcal{D}_X(U)}^{d_X}(\mathcal{O}_X(U), \mathcal{D}_X(U)) \\ &= \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(U), \Omega_X(U)) \\ &= \Omega_X(U) \neq 0. \end{aligned}$$

This implies that the (left) global dimension of \mathcal{D}_X is exactly d_X . The statement for the right global dimension follows from the fact that $\mathbf{Mod}(\mathcal{D}_X) = \mathbf{Mod}(\mathcal{D}_X)^{op}$ via the side-changing operations. The statement for $\mathcal{D}_{X,x}$ also follows from the preceding reasoning.

Finally, the second item follows from (i) and corollary 4.2.12. □

4.13 Functorial relations under a proper morphism

The goal of this subsection is to show the following result.

Theorem 4.13.1. *Let $f : X \rightarrow Y$ be a proper morphism. Then we have a canonical isomorphism*

$$\int_f \mathbb{D}_X \rightarrow \mathbb{D}_Y \int_f : D_c^b(\mathcal{D}_X) \rightarrow D_c^b(\mathcal{D}_Y)$$

of functors.

To prove the above theorem we recall for the reader the following notions introduced in the proof of the proposition 4.4.4.

Let Z be a closed subset of X and $U := X \setminus Z$ the complementary open subset. If $i : Z \rightarrow X$ and $j : U \rightarrow X$ denote the embeddings, then for any injective sheaf \mathcal{I} on X we get an exact sequence

$$0 \rightarrow \Gamma_Z(\mathcal{I}) \rightarrow \mathcal{I} \rightarrow j_*j^{-1}\mathcal{I} \rightarrow 0$$

and therefore, for any $\mathcal{M}^\bullet \in D_{qc}^b(\mathcal{D}_X)$ we get a distinguished triangle

$$R\Gamma_Z(\mathcal{M}^\bullet) \rightarrow \mathcal{M}^\bullet \rightarrow Rj_*j^{-1}\mathcal{M}^\bullet \xrightarrow{+1}. \quad (85)$$

We also remark for the reader that the projection formula for \mathcal{O} -modules (cf. [30, Proposition 5.6]) implies

$$i_*(\mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_X}^{\mathbf{L}} i^{-1}Rj_*\mathcal{M}^\bullet|_U) = i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X}^{\mathbf{L}} Rj_*\mathcal{M}^\bullet|_U$$

and therefore

$$i^\dagger \int_j \mathcal{M}^\bullet|_U = 0. \quad (86)$$

(The first equality has been proved in (61) and (62)). Finally, let us prove that

$$R\Gamma_Z(\mathcal{M}^\bullet) \simeq \int_i i^\dagger \mathcal{M}^\bullet. \quad (87)$$

By the very definition $R\Gamma_Z(\mathcal{M}^\bullet) \in \mathbf{D}_{qc}^{b,Z}(\mathcal{D}_X)$, so $R\Gamma_Z(\mathcal{M}^\bullet) \simeq \int_i i^\dagger R\Gamma_Z(\mathcal{M}^\bullet)$ by corollary 4.9.2. This yields to prove that $i^\dagger R\Gamma_Z(\mathcal{M}^\bullet) \simeq i^\dagger \mathcal{M}^\bullet$. By applying i^\dagger to the distinguished triangle (85) it is enough to prove that $i^\dagger Rj_*j^{-1}\mathcal{M}^\bullet = 0$. This follows from 86 because

$$i^\dagger Rj_*j^{-1}\mathcal{M}^\bullet = i^\dagger \int_j \mathcal{M}^\bullet|_U = 0.$$

Now, let $f : X \rightarrow Y$ be a proper morphism of smooth algebraic varieties. We want to construct a morphism

$$Tr_f : \int_f : \mathcal{O}_X[d_X] \rightarrow \mathcal{O}_Y[d_Y] \quad (88)$$

called the **trace map** of f .

First of all, as X is quasi projective, we have an embedding $j : X \rightarrow \mathbb{P}^n$ and therefore we can consider f as the composition between a closed embedding and a projection

$$X \xrightarrow{x \mapsto (f(x), j(x))} Y \times \mathbb{P}^n \longrightarrow Y. \quad (89)$$

The case $f = i$ is just to apply the canonical morphism³⁶ $\int_i i^\dagger \rightarrow id$ (coming from propositions 4.4.4 and 4.6.3) to $\mathcal{O}_{Y \times \mathbb{P}^n}$ and take a shift of degree d_Y . More exactly, we have a morphism

$$\int_i i^\dagger \mathcal{O}_Y \rightarrow \mathcal{O}_Y \quad \text{in } \mathbf{D}^b(\mathcal{D}_Y).$$

By definition $i^\dagger \mathcal{O}_Y = i^* \mathcal{O}_Y[d_X - d_Y] = \mathcal{O}_X[d_X - d_Y]$, so the required morphism Tr_i is obtained after taking the shift $[d_Y]$ in the morphism $\int_i \mathcal{O}_X[d_X - d_Y] \rightarrow \mathcal{O}_Y$.

For the case of the projection $X = \mathbb{P}^n \times Y \rightarrow Y$, by $\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^n} \boxtimes \mathcal{O}_Y$, we can suppose that Y consists of a single point $p : \mathbb{P}^n \rightarrow \mathbf{pt}$. By proposition 4.8.1

³⁶We have given an explicit description at the beginning of the proof of Kashiwara's equivalence.

$$\int_p \mathcal{O}_{\mathbb{P}^n} = Rp_*(DR_{\mathbb{P}^n/\mathbf{pt}}(\mathcal{O}_{\mathbb{P}^n})) = R\Gamma(\mathbb{P}^n, [\mathcal{O}_{\mathbb{P}^n} \rightarrow \dots \rightarrow \Omega_{\mathbb{P}^n}^n])$$

and we have isomorphisms

$$H^0\left(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\right) \simeq \tau^{\geq 0}\left(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\right) \simeq H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}).$$

By [29, Chapter III, corollary 7.13] and the previous isomorphism we get the desired morphism

$$\int_p \mathcal{O}_{\mathbb{P}^n} \rightarrow \tau^{\geq 0}\left(\int_p \mathcal{O}_{\mathbb{P}^n}[n]\right) \simeq \mathbb{C} \simeq \mathcal{O}_{pt}.$$

Finally, from the preceding reasoning, we define the **trace** of f as the composition of

$$\int_f \mathcal{O}_X[d_X] = \int_p \int_i \mathcal{O}_X[d_X] \rightarrow \int_p \mathcal{O}_{\mathbb{P}^n \times Y}[d_Y + d_X] \rightarrow \mathcal{O}_Y[d_Y].$$

Before proving our theorem we introduce the following results. The first one is a sort of **projection formula** whose proof can be found in [32, Corollary 1.7.5].

Lemma 4.13.2. *Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. Then for $\mathcal{M}^\bullet \in \mathbf{D}_{qc}^b(\mathcal{D}_X)$ and $\mathcal{N} \in \mathbf{D}_{qc}^b(\mathcal{D}_Y)$ we have*

$$\int_f (\mathcal{M}^\bullet \otimes_{\mathcal{O}_X}^{\mathbb{L}} Lf^* \mathcal{N}^\bullet) \simeq \left(\int_f \mathcal{M}^\bullet\right) \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathcal{N}^\bullet.$$

Finally, given that any morphism $f : X \rightarrow Y$ of smooth algebraic varieties is a composition of a closed embedding and a projection

$$X \xrightarrow{x \mapsto (f(x), x)} Y \times X \xrightarrow{(y, x) \mapsto y} Y$$

we can conclude by (65) and the propositions 4.6.1 and 4.8.1 the following important proposition

Proposition 4.13.3. *Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. Then \int_f sends $\mathbf{D}_{qc}^b(\mathcal{D}_X)$ to $\mathbf{D}_{qc}^b(\mathcal{D}_Y)$.*

Remark 4.13.4. *The previous lemma can fail if we consider the category $\mathbf{D}_c^b(\mathcal{D}_X)$. For instance, if $i : \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^2$ denotes the standard embedding, then it is clear that $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^2} \in \mathbf{Mod}_c(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^2})$, but by (50) we know that*

$$i^* \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^2} = \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^1} \otimes_{\mathbb{C}} \mathbb{C}[\partial_y]$$

is a free $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^1}$ -module whose basis consists of the set of powers of ∂_y . Since this basis can not be finite, we can conclude that $i^ \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^2}$ is not a coherent $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^1}$ -module.*

Proof of theorem 4.13.1. By definition (**Why?**), we know that

$$\begin{aligned} \int_f \mathbb{D}_X(\mathcal{M}^\bullet) &= Rf_*(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_X) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{D}_{X \rightarrow Y}) \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \Omega_Y^{\otimes -1}[d_X] \\ &= Rf_*(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_{X \rightarrow Y})) \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \Omega_Y^{\otimes -1}[d_X], \end{aligned}$$

(the second equality follows from the lemma 4.12.6) and

$$\mathbb{D}_Y \int_f \mathcal{M}^\bullet = R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_f \mathcal{M}^\bullet, \mathcal{D}_Y \right) \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \Omega_Y^{\otimes -1}[d_Y].$$

Let us construct a canonical morphism

$$\Phi(\mathcal{M}^\bullet) : Rf_*(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_{X \rightarrow Y}[d_X])) \rightarrow R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_f \mathcal{M}^\bullet, \mathcal{D}_Y[d_Y] \right)$$

in $D_{qc}^b(\mathcal{D}_Y^{op})$. By proposition 4.13.2 we have

$$\int_f \mathcal{D}_{X \rightarrow Y}[d_X] = \int_f (\mathcal{O}_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} Lf^* \mathcal{D}_Y[d_Y]) \simeq \left(\int_f \mathcal{O}_X[d_X] \right) \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathcal{D}_Y.$$

The trace map of f induces a canonical morphism $\int_f \mathcal{D}_{X \rightarrow Y}[d_X] \rightarrow \mathcal{D}_Y[d_Y]$ and $\Phi(\mathcal{M}^\bullet)$ is defined as the composite of

$$\begin{aligned} & Rf_*(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, \mathcal{D}_{X \rightarrow Y}[d_X])) \\ & \rightarrow Rf_* R\mathcal{H}om_{f^{-1}\mathcal{D}_Y}(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet, \mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{D}_{X \rightarrow Y}[d_X]) \\ & \rightarrow R\mathcal{H}om_{\mathcal{D}_Y}(Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{M}^\bullet), Rf_*(\mathcal{D}_{Y \leftarrow X} \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{D}_{X \rightarrow Y}[d_X])) \\ & = R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_f \mathcal{M}^\bullet, \int_f \mathcal{D}_{X \rightarrow Y}[d_X] \right) \\ & \rightarrow R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_f \mathcal{M}^\bullet, \mathcal{D}_Y[d_Y] \right). \end{aligned} \tag{90}$$

The first morphism is given by taking the tensor product with $\mathcal{D}_{Y \leftarrow X}$, the second one is just the definition of f_* and the last one is induced by the trace morphism. Let's prove that $\Phi(\mathcal{M}^\bullet)$ is an isomorphism.

By decomposing f as a closed embedding and a projection we can reduce the proof to these cases and we may assume from the beginning that $\mathcal{M}^\bullet = \mathcal{D}_X$ (If f is a closed embedding then this assertion is proposition 4.2.11 and if f is the projection the assertion comes from the fact that a product between a projective space and a smooth affine variety is \mathcal{D} -affine [32, Theorem 1.6.5]).

If $f = i : X \rightarrow Y$ is a closed embedding then $\Phi(\mathcal{M}^\bullet)$ is given by the composition

$$\begin{aligned} Ri_*(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, i^* \mathcal{D}_Y))[d_X] & \simeq R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_i i^* \mathcal{D}_Y \right) [d_X] \\ & \simeq R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_i i^\dagger \mathcal{D}_Y \right) [d_Y] \\ & \rightarrow R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \mathcal{D}_Y \right) [d_Y], \end{aligned}$$

The second isomorphism is just definition of i^\dagger and the last one is induced by the canonical morphism $\int_i i^\dagger \rightarrow id$. The first isomorphism follows from Kashiwara's

equivalence as follows. We have a canonical isomorphism $i^\sharp \int_i i^* \mathcal{D}_Y = \mathcal{D}_Y$ and by proposition 4.6.3 we have

$$\begin{aligned} i_*(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, i^* \mathcal{D}_Y)) &\simeq i_*(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, i^\sharp \int_i i^* \mathcal{D}_Y)) \\ &\simeq R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_i i^* \mathcal{D}_Y \right). \end{aligned}$$

We want to show that the map

$$R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_i i^\dagger \mathcal{D}_Y \right) [d_Y] \rightarrow R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \mathcal{D}_Y \right) [d_Y]$$

is an isomorphism. Let us fix $U := Y \setminus X$ and $j : U \rightarrow Y$ the open embedding. The distinguished triangle (defined by (85) and (87))

$$\int_i i^\dagger \mathcal{D}_Y \rightarrow \mathcal{D}_Y \rightarrow \int_j j^\dagger \mathcal{D}_Y \xrightarrow{+1}$$

tells us that we only need to prove that $R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_j j^* \mathcal{D}_Y \right) = 0$. This is a consequence from the propositions 4.6.1, 4.6.3 and the relations (86) as follows

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_i \mathcal{D}_X, \int_j j^* \mathcal{D}_Y \right) &\simeq i_* R\mathcal{H}om_{\mathcal{D}_X} \left(\mathcal{D}_X, i^\dagger \int_j j^* \mathcal{D}_Y \right) \\ &\simeq i_* i^\dagger \int_j j^* \mathcal{D}_Y = 0 \end{aligned}$$

Finally, let us consider the case $f = p : \mathbb{P}^n \times Y \rightarrow Y$ is the projection. By $\mathcal{D}_X = \mathcal{D}_{\mathbb{P}^n} \boxtimes \mathcal{D}_Y$ we can suppose that $Y = \{\mathbf{pt}\}$ is a single point. In this case, $\mathcal{D}_{\mathbb{P}^n \rightarrow \mathbf{pt}} = \mathcal{O}_{\mathbb{P}^n}$ and $\mathcal{D}_{\mathbf{pt} \leftarrow \mathbb{P}^n} = \Omega_{\mathbb{P}^n}$. From this information we have the following relations

$$\begin{aligned} Rp_*(R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{D}_X, \mathcal{D}_{X \rightarrow Y}[d_X])) &= Rp_*(R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^n}}(\mathcal{D}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}[n])) \\ &= R\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}[n]) \\ &\simeq \mathbb{C}[n] \end{aligned}$$

and

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_p \mathcal{D}_X, \mathcal{D}_Y[d_Y] \right) &= R\mathcal{H}om_{\mathcal{D}_{\mathbf{pt}}} (Rp_*(\Omega_{\mathbb{P}^n} \otimes_{\mathcal{D}_{\mathbb{P}^n}} \mathcal{D}_{\mathbb{P}^n}), \mathcal{D}_{\mathbf{pt}}) \\ &= R\mathcal{H}om_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), \mathbb{C}) \\ &= \mathcal{H}om_{\mathbb{C}}(\mathbb{C}[-n], \mathbb{C}) \\ &= \mathbb{C}[n]. \end{aligned}$$

Hence we only need to show that $\Phi(\mathcal{D}_{\mathbb{P}^n})$ is non trivial. But by (90) and the fact that

$$Rp_*(R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^n}}(\mathcal{D}_{\mathbb{P}^n}, \mathcal{D}_{\mathbb{P}^n \rightarrow \mathbf{pt}})) = R\mathcal{H}om_{\mathcal{D}_{\mathbb{P}^n}}(\mathcal{D}_{\mathbb{P}^n}, \mathcal{D}_{\mathbb{P}^n \rightarrow \mathbf{pt}})$$

the morphism $\Phi(\mathbb{P}^n)[-n]$ is given by

$$\begin{aligned}
RHom_{\mathcal{D}_{\mathbb{P}^n}}(\mathcal{D}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) &\rightarrow RHom_{\mathbb{C}}(\Omega_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n} \otimes_{\mathcal{D}_{\mathbb{P}^n}}^{\mathbf{L}} \mathcal{O}_{\mathbb{P}^n}) \\
&\rightarrow RHom_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes_{\mathcal{D}_{\mathbb{P}^n}}^{\mathbf{L}} \mathcal{O}_{\mathbb{P}^n})) \\
&\rightarrow RHom_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), \tau^{\geq n} R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes_{\mathcal{D}_{\mathbb{P}^n}}^{\mathbf{L}} \mathcal{O}_{\mathbb{P}^n})) \\
&\simeq RHom_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathcal{D}_{\mathbb{P}^n}}), \mathbb{C}[-n]) \\
&\simeq RHom_{\mathbb{C}}(R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}), R\Gamma(\mathbb{P}^n, \Omega_{\mathbb{P}^n}))
\end{aligned}$$

(the third morphism is induced via the **trace map** constructed before). This map is induced by the canonical morphism $\mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^n}}(\Omega_{\mathbb{P}^n}, \Omega_{\mathbb{P}^n})$ which is non-trivial. \square

We end this subsection by introducing the following results. The reader can find the the proof of the first proposition in [32, Proposition 3.1.6].

Proposition 4.13.5. *Let \mathcal{M} be a holonomic \mathcal{D}_X -module. Then there exists an open dense subset $U \subset X$ such that $\mathcal{M}|_U$ is coherent over \mathcal{O}_U . In other words, $\mathcal{M}|_U$ is an integrable connection on U .*

Proposition 4.13.6. *Let $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_X)$. For an open dense subset $U \subset X$ suppose that we are giving a holonomic submodule \mathcal{N} of $\mathcal{M}|_U$. Then there exists a holonomic submodule $\tilde{\mathcal{N}}$ of \mathcal{M} such that $\tilde{\mathcal{N}}|_U = \mathcal{N}$.*

Proof of the proposition 4.13.6. By a standard reasoning in algebraic geometry we may assume that \mathcal{M} is coherent and $\mathcal{M}|_U = \mathcal{N}$. Set $\mathcal{L} = H^0(X, \mathbb{D}_X(\mathcal{M}))$. By the second item of the proposition 4.12.4 we have $\text{codim}(\mathbf{Ch}(\mathcal{L})) \geq d_X$ and hence \mathcal{L} is holonomic. Then $\tilde{\mathcal{N}} = \mathbb{D}_X(\mathcal{L})$ is holonomic by the fourth item of proposition 4.12.4.

By definition, $\mathcal{L} = H^0(X, \mathbb{D}_X(\mathcal{M})) \simeq \tau^{\geq 0} \mathbb{D}_X(\mathcal{M})$ and by (106) we have a distinguished triangle

$$\tau^{\leq -1} \mathbb{D}_X(\mathcal{M}) \rightarrow \mathbb{D}_X(\mathcal{M}) \rightarrow \mathcal{L} \xrightarrow{+1}.$$

By applying \mathbb{D}_X we obtain

$$\tilde{\mathcal{N}} \rightarrow \mathcal{M} \rightarrow \mathbb{D}_X(\tau^{\leq -1} \mathbb{D}_X(\mathcal{M})) \xrightarrow{+1},$$

and since the duality functors commutes with restrictions to open sets and by holonomicity of $\mathcal{M}|_U$ we have

$$\tilde{\mathcal{N}}|_U = \mathbb{D}_U(\mathcal{L}|_U) = \mathbb{D}^2(\mathcal{M}|_U) = \mathcal{M}|_U = \mathcal{N}.$$

With this, one verifies that the canonical morphism $\tilde{\mathcal{N}} \rightarrow \mathcal{M}$ is injective and we get the result. \square

4.14 Preservation of holonomicity

Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties. In this section we will show that the functors \int_f and f^\dagger preserve the holonomicity. More exactly, if $\mathbf{D}_{hol}(\mathcal{D}_X)$ denotes the full subcategory of $\mathbf{D}(\mathcal{D}_X)$ consisting of complexes of objects $\mathcal{M}^\bullet \in \mathbf{D}(\mathcal{D}_X)$ such that $H^i(\mathcal{M}^\bullet)$ is a holonomic \mathcal{D}_X -module. We have

Theorem 4.14.1. *Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties.*

(i) \int_f sends $D_h^b(\mathcal{D}_X)$ to $D_h^b(\mathcal{D}_Y)$.

(ii) f^\dagger sends $D_h^b(\mathcal{D}_Y)$ to $D_h^b(\mathcal{D}_X)$.

To prove the above theorem we recall for the reader the following notations.

Let's suppose $X = \mathbb{A}_{\mathbb{C}}^n$. In this case, we know that $\mathcal{T}_{\mathbb{A}_{\mathbb{C}}^n} = \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} \partial_{x_i}$ and therefore

$$\mathbf{A}_n := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) = \bigoplus_{\alpha, \beta} \mathbb{C} x^\alpha \partial^\beta$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. We recall that the algebra \mathbf{A}_n is called the **Weyl algebra**. Since $\mathbb{A}_{\mathbb{C}}^n$ is affine, the **global sections functor** induces the following equivalences of categories

$$\mathbf{Mod}_{qc}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) \rightarrow \mathbf{Mod}(\mathbf{A}_n), \quad \mathbf{Mod}_c(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) \rightarrow \mathbf{Mod}_f(\mathbf{A}_n).$$

By afineness (remark 4.2.7) and given that the **Fourier transform** (example 4.10.11 (iii)) induces an auto-equivalence of the category $\mathbf{Mod}(\mathbf{A}_n)$, we have the following auto-equivalence of the category $\mathbf{Mod}_{qc}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$

$$\begin{array}{ccccccc} \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) & \rightarrow & \mathbf{Mod}(\mathbf{A}_n) & \xrightarrow{\mathcal{F}} & \mathbf{Mod}(\mathbf{A}_n) & \rightarrow & \mathbf{Mod}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}) \\ \mathcal{M} & \mapsto & M := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \mathcal{M}) & \mapsto & M_{\mathcal{F}} & \mapsto & \mathcal{M}_{\mathcal{F}} := \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n} \otimes_{\mathbf{A}_n} M_{\mathcal{F}}. \end{array}$$

We have the following technical lemmas.

Lemma 4.14.2. *Let us consider*

$$p : \mathbb{A}_{\mathbb{C}}^1 \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^{n-1} \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1} \quad \text{and} \quad i : \mathbb{A}_{\mathbb{C}}^{n-1} \rightarrow \mathbb{A}_{\mathbb{C}}^n$$

the projection and the closed embedding. For $\mathcal{M} \in \mathbf{Mod}_{qc}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ we have

$$H^k \left(\int_p \mathcal{M} \right) \simeq H^k(Li^* \widehat{\mathcal{M}})$$

for any k .

Proof. Let us denote $M := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{M})$. By proposition 4.8.1 (i) we have

$$\int_p \mathcal{M} \simeq Rp_* (\mathbf{DR}_{\mathbb{A}_{\mathbb{C}}^n / \mathbb{A}_{\mathbb{C}}^{n-1}}(\mathcal{M})).$$

To compute the complex $Rp_* (\mathbf{DR}_{\mathbb{A}_{\mathbb{C}}^n / \mathbb{A}_{\mathbb{C}}^{n-1}}(\mathcal{M}))$ we consider the commutative diagram

$$\begin{array}{ccc} \mathbb{A}_{\mathbb{C}}^1 \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^{n-1} & \xrightarrow{p} & \mathbb{A}_{\mathbb{C}}^{n-1} \\ \downarrow q & & \downarrow \\ \mathbb{A}_{\mathbb{C}}^1 & \longrightarrow & \mathrm{Spec}(\mathbb{C}). \end{array}$$

By [29, Chapter II, proposition 8.10] we have

$$\Omega_{\mathbb{A}_{\mathbb{C}}^n/\mathbb{A}_{\mathbb{C}}^{n-1}}^1 = q^* \left(\Omega_{\mathbb{A}_{\mathbb{C}}^1/\mathrm{Spec}(\mathbb{C})}^1 \right) = \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n} dx_1.$$

We can use this information to identify

$$\begin{array}{ccc} \Omega_{\mathbb{A}_{\mathbb{C}}^n/\mathbb{A}_{\mathbb{C}}^{n-1}}^1 \otimes_{\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}} \mathcal{M} & \xrightarrow{\cong} & \mathcal{M} \\ f dx_1 \otimes m & \mapsto & fm \end{array}$$

(as $\mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}$ -modules) and the **de Rham complex** has the shape $[\mathcal{M} \xrightarrow{\partial_{x_1}} \mathcal{M}]$ by (68). We have proved

$$\int_p \mathcal{M} \simeq R p_* (\mathbf{DR}_{\mathbb{A}_{\mathbb{C}}^n/\mathbb{A}_{\mathbb{C}}^{n-1}}(\mathcal{M})) = [p_* \mathcal{M} \xrightarrow{\partial_{x_1}} p_* \mathcal{M}].$$

From this we get

$$\Gamma \left(\mathbb{A}_{\mathbb{C}}^{n-1}, H^k \left(\int_p \mathcal{M} \right) \right) = \begin{cases} \ker(M \xrightarrow{\partial_{x_1}} M) & (k = -1) \\ \mathrm{coker}(M \xrightarrow{\partial_{x_1}} M) & (k = 0) \\ 0 & (k \neq 0, -1), \end{cases}$$

and therefore

$$\begin{aligned} \Gamma \left(\mathbb{A}_{\mathbb{C}}^{n-1}, H^k \left(\widehat{\int_p \mathcal{M}} \right) \right) &= \begin{cases} \ker(\widehat{M} \xrightarrow{x_1} \widehat{M}) & (k = -1) \\ \mathrm{coker}(\widehat{M} \xrightarrow{x_1} \widehat{M}) & (k = 0) \\ 0 & (k \neq 0, -1), \end{cases} \\ &= \Gamma(\mathbb{A}_{\mathbb{C}}^{n-1}, H^k(Li^* \widehat{\mathcal{M}})). \end{aligned}$$

For the second equality the reader can use the Koszul's complex and to apply the same reasoning that we have given in the theorem of Kashiwara's equivalence (complex (70)). \square

The following lemma is an immediate consequence of remark 4.10.15 and the example 4.10.11 (iii).

Lemma 4.14.3. *A coherent $\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}$ -module \mathcal{M} is holonomic if and only if $\mathcal{M}_{\mathcal{F}}$ is holonomic.*

We have the following technical lemma. We follow word by word the proof given in [22, Chapter 10, lemma 3.1].

Lemma 4.14.4. *Let $(M, F = \{F_k\})$ be a good filtered finitely generated left \mathbf{A}_n -module. Let us suppose that there exist constants c_1 and c_2 such that for $l \gg 0$*

$$\dim_k(F_l) \leq c_1 j^n / n! + c_2 (j+1)^{n-1}.$$

Then M is a holonomic \mathbf{A}_n -module.

Before starting the proof, we remark for the reader that this lemma has a more general version without the hypothesis of finitude. This property makes part of the proof of this general statement, and it is proved by remarking that every finitely generated submodule of M has multiplicity at most c_1 .

Proof. Let us consider the Hilbert polynomial $\chi_M^F(t) \in \mathbb{Q}[t]$ of M associated to the filtration F . Using the polynomial bound of the hypothesis, we see that for $l \gg 0$ we have

$$\chi_M^F(l) \leq c_1 j^n / n! + c_2 (j+1)^{n-1}.$$

In particular, the degree of $\chi_M^F(t)$ cannot exceed n . Hence $\mathbf{d}(M) \leq n$ and from the Bernstein inequality we can conclude that $\mathbf{d}(N) = n$. \square

Lemma 4.14.5. *Let us consider the open embedding $j : (\mathbb{A}_{\mathbb{C}}^n \setminus \{0\}) \times \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^n$. If $\mathcal{M} \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$, then so is $H^0(\int_j j^\dagger \mathcal{M})$.*

Proof. Let us denote by $U := (\mathbb{A}_{\mathbb{C}}^n \setminus \{0\}) \times \mathbb{A}_{\mathbb{C}}^n$. Given that j is an open embedding it is affine and the functors j^* and j_* are exact. We have the following relations. First of all

$$\begin{aligned} j^\dagger \mathcal{M} &= Lj^* \mathcal{M} = \mathcal{D}_{U \times \rightarrow \mathbb{A}_{\mathbb{C}}^n} \otimes_{j^{-1} \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n}} j^{-1} \mathcal{M} = \mathcal{O}_U \otimes_{j^{-1} \mathcal{O}_{\mathbb{A}_{\mathbb{C}}^n}} j^{-1} \mathcal{M} \\ &= \mathcal{M}|_U. \end{aligned}$$

On the other hand,

$$\int_j j^\dagger \mathcal{M} = j_* (\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n \leftarrow U} \otimes_{\mathcal{D}_U} \mathcal{M}|_U) = j_*(\mathcal{M}|_U).$$

Let $M := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{M}) \in \mathbf{Mod}_f(\mathbf{A}_n)$. From the previous relations we have that

$$\Gamma\left(\mathbb{A}_{\mathbb{C}}^n, \int_j j^\dagger \mathcal{M}\right) = \mathbb{C}[x, x_1^{-1}] \otimes_{\mathbb{C}[x]} M = M_{x_1}$$

considered as a $\mathbb{C}[x]$ -module via the canonical morphism $\mathbb{C}[x] \rightarrow \mathbb{C}[x, x_1^{-1}]$. By using the remark (4.10.15) it is enough to prove that M_{x_1} is holonomic. Let F be a good filtration of M and for every $k \in \mathbb{N}$ we define

$$G_k := \{f/x_1^k \mid \deg(f) \leq 2k\}.$$

Let us first verify that $G = \{G_k\}$ is a filtration of M_{x_1} .

Let $f/x_1^k \in M_{x_1}$ such that $\deg(f) = s$. Given that $f/x_1^k = fx_1^s/x_1^{s+k}$ and $\deg(fx_1^s) = 2s \leq 2(s+k)$ we see that $f/x_1^k \in G_{s+k}$. From this we can conclude that M_{x_1} is the union of all the G_k .

Now, let us take $f/x_1^k \in G_k$. Since $\deg(x_i f x_1) = 2 + \deg(f) \leq 2 + 2k = 2(k+1)$, we have that $x_i(f/x_1^k) = x_i f x_1 / x_1^{k+1} \in G_{k+1}$. Differentiating f/x_1^k with respect to x_i , we get

$$(x_1 \partial_{x_i}(f) - k f \delta_{i1}) / x_1^{k+1}$$

with $\deg(x_1 \partial_{x_i}(f) - k f \delta_{i1}) \leq 2(k+1)$, so $\partial_{x_i}(f/x_1^k) \in G_{k+1}$. From this reasoning we see that $\mathcal{B}_1 \cdot G_k \subseteq G_{k+1}$. By induction and given that $\mathcal{B}_1^i = \mathcal{B}_i$, we also have $\mathcal{B}_i \cdot G_k \subseteq G_{k+i}$. To end the proof that G is a filtration of M_{x_1} we need to see that G_k has finite dimension, but this follows from the fact that

$$\dim_{\mathbb{C}}(G_k) \leq \dim_{\mathbb{C}}(\mathcal{B}_{2k}) = \binom{2k+n}{n}.$$

Moreover, given that the two terms of highest degree in k of this binomial number are

$$2^n k^n / n! \quad \text{and} \quad 2^{n-1} (n+1) n k^{n-1} / 2(n!)$$

it follows that

$$\dim_{\mathbb{C}}(G_k) \leq 2^n k^n / n! + 2^{n-1} (n+1) n (k+1)^{n-1} / n!.$$

From lemma 4.14.4 we can conclude that M_{x_1} is holonomic. \square

Lemma 4.14.6. *Let \mathcal{M} and \mathcal{N} be coherent \mathcal{D}_X -modules. Then*

$$\text{Ch}(\mathcal{M} \boxtimes \mathcal{N}) = \text{Ch}(\mathcal{M}) \times \text{Ch}(\mathcal{N}).$$

Proof. If (\mathcal{M}, F_1) and (\mathcal{N}, F_2) are good filtrations of \mathcal{M} and \mathcal{N} . Then

$$F_k(\mathcal{M} \boxtimes \mathcal{N}) := \sum_{i+j=k} F_i(\mathcal{M}) \boxtimes F_j(\mathcal{N}) \subset \mathcal{M} \boxtimes \mathcal{N}$$

is good filtration for $\mathcal{M} \boxtimes \mathcal{N}$. With this is clear that

$$\text{gr}^F(\mathcal{M} \boxtimes \mathcal{N}) = \text{gr}^{F_1}(\mathcal{M}) \boxtimes \text{gr}^{F_2}(\mathcal{N}),$$

which implies the result. \square

Lemma 4.14.7. *Let $i : X \rightarrow Y$ be a closed embedding. Then for $\mathcal{M}^\bullet \in \mathbf{D}_c^b(\mathcal{D}_X)$ we have*

$$\mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_X) \Leftrightarrow \int_i \mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_Y).$$

Proof. The proof is essentially to reduce to lemma 4.11.2 and note that the morphism ρ (appearing in the cited lemma) can be taken smooth with fibers of dimension 1 (why?). \square

Proof of theorem 4.14.1. First of all we reduce the proof of (i) to the case $f = p : \mathbb{C}^n \rightarrow \mathbb{C}^{n-1}$ is the projection. We can assume in (i) that $\mathcal{M}^\bullet \in \mathbf{Mod}_h(\mathcal{D}_X)$ and considering f as the composition of a closed embedding and a projection (see for example (89)), we can use the above lemma to reduce to the case when f is the projection $f : X \times Y \rightarrow Y$. Moreover, given that the problem is local on Y , we may assume that Y is affine. Let us take $X = \bigcup_{i=0}^r X_i$ an affine open covering of X and let us denote by $j_{i_0 \dots i_k} : \bigcap_{p=0}^k X_{i_p} \rightarrow X$ ($0 \leq i_0 < \dots < i_k \leq r$) the open embedding. We know by [29, Chapter III, theorem 4.5] that \mathcal{M} is quasi-isomorphic to the Čech complex $C^\bullet(\mathcal{M})$ with

$$C^k(\mathcal{M}) = \bigoplus_{i_0 < \dots < i_k} j_{i_0 \dots i_k}^* \mathcal{M} |_{\bigcap_{p=0}^k X_{i_p}} \simeq \bigoplus_{i_0 < \dots < i_k} \int_{j_{i_0 \dots i_k}} j_{i_0 \dots i_k}^* \mathcal{M}$$

(to achieve the previous isomorphism the reader can follow the same lines of reasoning given at the beginning of the proof of the lemma 4.14.5) and therefore it will be sufficient to show $\int_{f \circ j_{i_0 \dots i_k}} j_{i_0 \dots i_k}^* \mathcal{M} \in \mathbf{D}_h^b(\mathcal{D}_Y)$ which also implies that we can assume from the beginning that X is affine. Let us fix closed embeddings $\alpha : X \rightarrow \mathbb{A}_{\mathbb{C}}^n$, $\beta : Y \rightarrow \mathbb{A}_{\mathbb{C}}^m$, and consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X \times Y & \xrightarrow{\alpha \times \beta} & \mathbb{A}_{\mathbb{C}}^{n+m} \\ \downarrow f & & & & \downarrow p \\ Y & \xrightarrow{\beta} & & & \mathbb{A}_{\mathbb{C}}^m \end{array}$$

where g is the graph embedding. By proposition 4.6.1 we have the relation

$$\int_{\beta} \int_f \mathcal{M} = \int_p \int_{(\alpha \times \beta) \circ g} \mathcal{M},$$

and since $\int_{(\alpha \times \beta) \circ g} \mathcal{M} \in \mathbf{Mod}_h(\mathcal{D}_{\mathbb{C}^{m+n}})$ by the previous lemma, and the fact that $\mathbb{A}_{\mathbb{C}}^{n+m} \rightarrow \mathbb{A}_{\mathbb{C}}^n$ is a composite of morphisms $\mathbb{A}_{\mathbb{C}}^k \rightarrow \mathbb{A}_{\mathbb{C}}^{k-1}$, we only need to consider the case when f is the projection $\mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1}$. Refreshing the notation we have $f = p : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^{n-1}$ is the projection, $\mathcal{M} \in \mathbf{Mod}_h(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ and we want to show $\int_p \mathcal{M} \in \mathbf{D}_h^b(\mathbb{A}_{\mathbb{C}}^n)$. The lemmas 4.14.2 and 4.14.3 tell us that it is enough to show that $i^\dagger \mathcal{M} \in \mathbf{D}_h^b(\mathbb{A}^{n-1})$.

Under the hypothesis and notations in the lemma 4.14.5, and according to (85) and (87) we have a distinguished triangle

$$\int_i i^\dagger \mathcal{M} \rightarrow \mathcal{M} \rightarrow \int_j j^\dagger \mathcal{M} \xrightarrow{+1} \quad (91)$$

which induces the exact sequence

$$0 \rightarrow H^0 \left(\int_i i^\dagger \mathcal{M} \right) \rightarrow \mathcal{M} \rightarrow H^0 \left(\int_j j^\dagger \mathcal{M} \right) \rightarrow H^1 \left(\int_i i^\dagger \mathcal{M} \right) \rightarrow 0 \quad (92)$$

Since $H^0(\int_j j^\dagger \mathcal{M}) \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ by the lemma 4.14.5, we can conclude that $\int_i i^\dagger \mathcal{M} \in \mathbf{D}_h^b(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ and therefore $i^\dagger \mathcal{M} \in \mathbf{D}_h^b(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^{n-1}})$ by the previous lemma.

Finally, let us prove that (i) implies (ii). Decomposing f as a closed embedding and a projection we can assume first that f is the projection $X \times Y \rightarrow Y$. In this case $f^* \mathcal{M} \simeq \mathcal{O}_X \boxtimes \mathcal{M}$ and this is holonomic by lemma 4.14.6

$$\mathbf{Ch}(\mathcal{O}_X \boxtimes \mathcal{M}) = \mathbf{Ch}(\mathcal{O}_X) \times \mathbf{Ch}(\mathcal{M}) = T_X^* X \times \mathbf{Ch}(\mathcal{M}).$$

For a closed embedding we can use the distinguished triangle (91)

$$\int_i i^\dagger \mathcal{M} \rightarrow \mathcal{M} \rightarrow \int_j j^\dagger \mathcal{M} \xrightarrow{+1}.$$

Given that $j^\dagger \mathcal{M} = \mathcal{M}|_U \in \mathbf{Mod}_h(\mathcal{D}_U)$, then (i) tells us that $\int_j j^\dagger \mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_Y)$. The same argument given in (92) implies that $\int_i i^\dagger \mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_Y)$. The previous lemma finally gives us $i^\dagger(\mathcal{M}^\bullet) \in \mathbf{D}_h^b(\mathcal{D}_X)$. \square

Example 4.14.8. Given that the proof of the theorem 4.14 is quite complicated, in this example we will try to explain this result in the special case when

$$f = (f_1, \dots, f_m) : \mathbb{A}_{\mathbb{C}}^n \rightarrow \mathbb{A}_{\mathbb{C}}^m$$

is a polynomial map, i.e, $f_1, \dots, f_m \in \mathbb{C}[x_1, \dots, x_n]$ are polynomials. In other words, we want to prove that if $\mathcal{M} \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$ (resp. $\mathcal{N} \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m})$), then $\int_f^0 \mathcal{M} \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^m})$ (resp. $f^* \mathcal{N} \in \mathbf{Mod}_{hol}(\mathcal{D}_{\mathbb{A}_{\mathbb{C}}^n})$).

Remark 4.10.15 tells us that we can work in the global situation $M := \Gamma(\mathbb{A}_{\mathbb{C}}^n, \mathcal{M})$ a holonomic \mathbf{A}_n -module and $N := \Gamma(\mathbb{A}_{\mathbb{C}}^m, \mathcal{N})$ a holonomic \mathbf{A}_m -module. Let us summing up what we already know.

- (i) f can be decomposed as the composition of a (standard) closed embedding, an isomorphism and a projection

$$\mathbb{A}_{\mathbb{C}}^n \xrightarrow{i(x):=(x,0)} \mathbb{A}_{\mathbb{C}}^n \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^m \xrightarrow{\varphi(x,y):=(x,y+f(y))} \mathbb{A}_{\mathbb{C}}^n \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{A}_{\mathbb{C}}^m \xrightarrow{\pi(x,y):=y} \mathbb{A}_{\mathbb{C}}^m$$

- (ii) The fourth statement in the example 4.3.8 tells us that

$$\pi^*(N) = \mathbb{C}[x_1, \dots, x_n] \otimes_{\mathbb{C}} N.$$

Hence $\pi^*(N)$ is holonomic by (76).

- (iii) By (66) we can deduce that

$$i_*(M) = \Gamma(\mathbb{A}_{\mathbb{C}}^m, \int_i^0 \mathcal{M}) = \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}] \otimes_{\mathbb{C}} M.$$

Since $\mathbb{C}[y_1, \dots, y_m]_{\mathcal{F}} = \mathbb{C}[\partial_{y_1}, \dots, \partial_{y_m}]$ we can conclude as before that $i_*(M)$ is holonomic.

We will concentrate the effort in proving that **inverse images** of (standard) embedding and **inverse images** of (standard) projections preserves holonomicity. In light of the proposition 4.6.1 we can work out the case of a (standard) embedding assuming that $m = n + 1$.

Claim 1: if $K \in \mathbf{Mod}_{hol}(\mathbf{A}_{n+1})$ which does not contains any non-zero element with support on $\mathbb{A}_{\mathbb{C}}^n$, then $i^*(K) \in \mathbf{Mod}_{hol}(\mathbf{A}_n)$.

Proof. We start by recalling for the reader that

$$i^*(K) = K/yK$$

by the third statement in the example 49. Let F be a good filtration for K , and for every $k \in \mathbb{N}$ let us consider

$$G_k := (F_k + yK)/yK (\simeq F_k/F_k \cap yK).$$

The key point to understand here is that we are building a good filtration for K/yK as an \mathbf{A}_n -module, from a good filtration of an \mathbf{A}_{n+1} -module. It is straightforward to see that G is actually a filtration of K/yK and we will use the lemma 4.14.4 to see that this is actually a good filtration. We start with the following relations

$$\begin{aligned} \dim_{\mathbb{C}}(G_k) &= \dim_{\mathbb{C}}(F_k) - \dim_{\mathbb{C}}(F_k \cap yK) \\ &\leq \dim_{\mathbb{C}}(F_k) - \dim_{\mathbb{C}}(yF_{k-1}) \\ &= \dim_{\mathbb{C}}(F_k) - \dim_{\mathbb{C}}(F_{k-1}). \end{aligned}$$

The first equality is clear, the inequality follows from the fact that $yF_{k-1} \subseteq F_k \cap yK$ and the second equality uses the fact that $F_{k-1} \xrightarrow{y} yF_{k-1}$ is injective by hypothesis. So, translating the previous relations in terms of the Hilbert polynomial $\chi_K^F(t) \in \mathbb{Q}[t]$, we find that

$$\dim_{\mathbb{C}}(G_k) \leq \chi_K^F(k) - \chi_K^F(k-1).$$

Given that leading coefficient of $\chi_K^F(t) - \chi_K^F(t-1)$ is $\mathbf{m}(K)t^{n-1}/n!$, we can conclude that there exists $c \in \mathbb{Q}$ such that

$$\dim_{\mathbb{C}}(G_k) \leq \mathbf{m}(K)k^{n-1}/(n-1)! + c(k-1)^{n-1}.$$

This ends the proof of the first claim. \square

(iv) If $K \in \mathbf{Mod}_{hol}(\mathbf{A}_{n+1})$, then $i^*K \in \mathbf{Mod}_{hol}(\mathbf{A}_n)$.

Proof. Let

$$\Gamma_{\mathbb{A}_{\mathbb{C}}^n}(K) := \{m \in K \mid \exists N \in \mathbb{N} \text{ such that } y^N m = 0\}.$$

and put $K' := K/\Gamma_{\mathbb{A}_{\mathbb{C}}^n}(K)$. We remark for the reader that, via the Weyl relations, it is easy to see that K is an \mathbf{A}_{n+1} -module and using the preceding claim we also see that $i^*(K') \in \mathbf{Mod}_{hol}(\mathbf{A}_n)$. So, all what we have to do is to prove that the canonical morphism

$$K/yK \rightarrow K'/yK'$$

is an isomorphism. This is an easy computation (cf. [22, Chapter 18, lemma 1.1]). \square

Let us suppose again that $n, m \in \mathbb{N}$ are arbitrary.

(v) If $L \in \mathbf{Mod}_{hol}(\mathbf{A}_{m+n})$, then $\pi_*L \in \mathbf{Mod}_{hol}(\mathbf{A}_m)$.

Proof. The second statement of the example 4.6.2 tells us that

$$\pi_*L = L / \sum_{i=1}^n \partial_{x_i} L = \left(L / \sum_{i=1}^n x_i L \right)_{\mathcal{F}}$$

In particular

$$\mathbf{d}(\pi_*L) = \mathbf{d} \left(\left(L / \sum_{i=1}^n x_i L \right)_{\mathcal{F}} \right) = \mathbf{d} \left(L / \sum_{i=1}^n x_i L \right).$$

However,

$$L / \sum_{i=1}^n x_i L \simeq j^*L$$

where $j : \mathbb{A}_{\mathbb{C}}^m \rightarrow \mathbb{A}_{\mathbb{C}}^{n+m}$ is the standard embedding $j(y) = (0, y)$. This ends the proof of the claim. \square

All in all, we can easily use the decomposition given in (i) and the statements from (ii) to (v) to prove that the inverse image functor and the direct image functor preserve holonomicity.

We end this section with the following **adjunction formula**. Let $f : X \rightarrow Y$ be a morphism of smooth algebraic varieties, we define new functors

$$\begin{aligned} \int_{f!} &:= \mathbb{D}_Y \int_f \mathbb{D}_X : \mathbf{D}_h^b(\mathcal{D}_X) \rightarrow \mathbf{D}_h^b(\mathcal{D}_Y). \\ f^\star &:= \mathbb{D}_X f^\dagger \mathbb{D}_Y : \mathbf{D}_h^b(\mathcal{D}_Y) \rightarrow \mathbf{D}_h^b(\mathcal{D}_X). \end{aligned}$$

Theorem 4.14.9. *For $\mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_X)$ and $\mathcal{N}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_Y)$ we have a natural isomorphism*

$$\begin{aligned} R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_{f!} \mathcal{M}^\bullet, \mathcal{N}^\bullet \right) &\simeq Rf_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, f^\dagger \mathcal{N}^\bullet), \\ Rf_* R\mathcal{H}om_{\mathcal{D}_X}(f^\star \mathcal{N}^\bullet, \mathcal{M}^\bullet) &\simeq R\langle om_{\mathcal{D}_Y} \left(\mathcal{N}^\bullet, \int_f \mathcal{M}^\bullet \right). \end{aligned}$$

Proof. We have

$$\begin{aligned}
& Rf_* R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}^\bullet, f^\dagger \mathcal{N}^\bullet) \\
& \simeq Rf_* \left((\Omega_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathbb{D}_X \mathcal{M}^\bullet) \otimes_{\mathcal{D}_X}^{\mathbb{L}} f^\dagger \mathcal{N}^\bullet \right) [-d_X] \\
& \simeq Rf_* \left((\Omega_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathbb{D}_X \mathcal{M}^\bullet) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{D}_{X \rightarrow Y} \otimes_{f^{-1} \mathcal{D}_Y}^{\mathbb{L}} f^{-1} \mathcal{N}^\bullet \right) [-d_Y] \\
& \simeq Rf_* \left((\Omega_X \otimes_{\mathcal{O}_X}^{\mathbb{L}} \mathbb{D}_X \mathcal{M}^\bullet) \otimes_{\mathcal{D}_X}^{\mathbb{L}} \mathcal{D}_{X \rightarrow Y} \right) \otimes_{\mathcal{D}_Y}^{\mathbb{L}} \mathcal{N}^\bullet [-d_Y] \\
& \simeq \left(\Omega_Y \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \int_f \mathbb{D}_X \mathcal{M}^\bullet \right) \otimes_{\mathcal{D}_Y}^{\mathbb{L}} \mathcal{N}^\bullet [-d_Y] \\
& \simeq \left(\Omega_Y \otimes_{\mathcal{O}_Y}^{\mathbb{L}} \mathbb{D}_Y \int_{f!} \mathcal{M}^\bullet \right) \otimes_{\mathcal{D}_Y}^{\mathbb{L}} \mathcal{N}^\bullet [-d_Y] \\
& \simeq R\mathcal{H}om_{\mathcal{D}_Y} \left(\int_{f!} \mathcal{M}^\bullet, \mathcal{N}^\bullet \right).
\end{aligned}$$

The first isomorphism comes from the lemma 4.12.6 and the definition of \mathbb{D}_X , the second isomorphism is the definition of f^\dagger , the third one is a general property (called the derived projection formula for \mathcal{D} -modules) of the inverse and direct image (cf. [32] proof of 1.5.21), the fourth one is the definition of \int_f , the fifth one comes from the theorem 4.14.1 and the last isomorphism is given by the isomorphism (84a). This establish the first isomorphism, the second follows from duality **why?**. \square

By applying $H^0(R\Gamma(Y, \bullet))$ to the above isomorphism we obtain the following

Corollary 4.14.10. *We have a natural isomorphisms*

$$\begin{aligned}
& Hom_{\mathbf{D}_h^b(\mathcal{D}_Y)} \left(\int_{f!} \mathcal{M}^\bullet, \mathcal{N}^\bullet \right) \simeq Hom_{\mathbf{D}_h^b(\mathcal{D}_X)}(\mathcal{M}^\bullet, f^\dagger \mathcal{N}^\bullet), \\
& Hom_{\mathbf{D}_h^b(\mathcal{D}_X)}(f^\star \mathcal{N}^\bullet, \mathcal{M}^\bullet) \simeq Hom_{\mathbf{D}_h^b(\mathcal{D}_Y)} \left(\mathcal{N}^\bullet, \int_f \mathcal{M}^\bullet \right).
\end{aligned}$$

Finally,

Theorem 4.14.11. *There exists a morphism of functors*

$$\int_{f!} \rightarrow \int_f : \mathbf{D}_h^b(\mathcal{D}_X) \rightarrow \mathbf{D}_h^b(\mathcal{D}_Y).$$

Moreover, if f is proper, then this morphism is an isomorphism.

Proof. By Hironaka's desingularization theorem ([12, Theorem A] or [31, Main theorem]), there exists a smooth completion \widetilde{X} of X . Since X is quasi projective, a desingularization \widetilde{X} of the Zariski closure \overline{X} of X in the projective space is such a completion and therefore f can be factored as

$$X \xrightarrow{i} X \times Y \xrightarrow{j} \widetilde{X} \times Y \xrightarrow{p} Y,$$

where i is the graph embedding and p is the projection. In this case i and p are proper and j is a open embedding. This reduces the problem to consider proper morphisms and open embeddings. For the case of a proper morphism f by theorem 4.13.1 we have an isomorphism

$$\int_{f!} = \mathbb{D}_Y \int_f \mathbb{D}_X \rightarrow \int_f.$$

On the other hand, if $f : X \rightarrow Y$ is an open embedding and $\mathcal{M}^\bullet \in \mathbf{D}_h^b(\mathcal{D}_X)$ then by the above corollary we have

$$\begin{aligned} \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_Y)} \left(\int_{j!} \mathcal{M}^\bullet, \int_j \mathcal{M}^\bullet \right) &\simeq \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)} \left(\mathcal{M}^\bullet, j^\dagger \int_j \mathcal{M}^\bullet \right) \\ &\simeq \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)}(\mathcal{M}^\bullet, \mathcal{M}^\bullet) \end{aligned}$$

and we obtain the desired morphism as the image of $id \in \mathrm{Hom}_{\mathbf{D}_h^b(\mathcal{D}_X)}(\mathcal{M}^\bullet, \mathcal{M}^\bullet)$. \square

4.15 Minimal extensions

We say that a \mathcal{D}_X -module \mathcal{M} is **simple** if it contains no coherent \mathcal{D}_X -submodules other than \mathcal{M} and 0.

Let us take $\mathcal{M} \in \mathbf{Mod}_h(\mathcal{D}_X)$. According to the proposition 4.8.1 the **total multiplicity** of \mathcal{M} defined by

$$m(\mathcal{M}) = \sum_{\mathcal{C} \in I(\mathrm{Ch}(\mathcal{M}))} m_{\mathcal{C}}(\widetilde{gr^F(\mathcal{M})})$$

is additive in the sense that we have $m(\mathcal{M}) = m(\mathcal{L}) + m(\mathcal{N})$ for any short exact sequence

$$0 \rightarrow \mathcal{L} \rightarrow \mathcal{M} \rightarrow \mathcal{N} \rightarrow 0$$

in $\mathbf{Mod}_h(\mathcal{D}_X)$. This implies that if we have a descending chain of non-zero holonomic sub-modules of \mathcal{M}

$$\mathcal{M} = \mathcal{M}_0 \supsetneq \mathcal{M}_1 \supsetneq \cdots \supsetneq \mathcal{M}_k,$$

then $m(\mathcal{M}_i) = m(\mathcal{M}_{i+1}) + m(\mathcal{M}_i/\mathcal{M}_{i+1})$ and therefore $m(\mathcal{M}) \geq k$. In particular, we see that \mathcal{M} can not have an infinite descending chain. The preceding reasoning tells us that $\mathbf{Mod}_h(\mathcal{D}_X)$ is an **artinian category**. With this property we can construct a **Jordan-Hölder series** of \mathcal{M} as follows. Let's suppose that we have constructed a chain $0 = \mathcal{M}_0 \subsetneq \mathcal{M}_1 \subsetneq \cdots \subsetneq \mathcal{M}_k$ of submodules of \mathcal{M} such that $\mathcal{M}_i/\mathcal{M}_{i-1}$ is simple. By the last reasoning we can take a submodule \mathcal{M}_{k+1} of \mathcal{M} such that $\mathcal{M}_{k+1}/\mathcal{M}_k$ is a minimal submodule of $\mathcal{M}/\mathcal{M}_k$ and therefore is simple. This process can not continue indefinitely because, by remark 4.2.7 and proposition 4.2.8 we know that \mathcal{M} is also locally noetherian module. So, we have a finite sequence

$$\mathcal{M} = \mathcal{M}_k \supsetneq \mathcal{M}_{k-1} \supsetneq \cdots \supsetneq \mathcal{M}_0 = 0$$

of holonomic \mathcal{D}_X -submodules such that $\mathcal{M}_i/\mathcal{M}_{i+1}$ is simple for each i .

On the other hand, let Y be a locally closed smooth subvariety of X . Let us assume that the inclusion map $i : Y \hookrightarrow X$ is affine. Then, by example 4.3.11 we know that $\mathcal{D}_{X \leftarrow Y}$ is locally free over \mathcal{D}_Y and the higher cohomology groups of i_* vanish. Therefore, for a holonomic \mathcal{D}_Y -module \mathcal{M} we have $H^l(\int_i \mathcal{M}) =$

$H^l(\int_{i^!} \mathcal{M}) = 0$ for all $l \neq 0$. Finally, by the theorems 4.14.1 and 4.14.11 we have a morphism

$$\int_{i^!} \mathcal{M} \rightarrow \int_i \mathcal{M} \quad (93)$$

in $\mathbf{Mod}_h(\mathcal{D}_X)$.

Definition 4.15.1. *We call the image $L(Y, \mathcal{M})$ of the morphism (93) the **minimal extension** of \mathcal{M} .*

Theorem 4.15.2. (i) *Let Y be a locally closed smooth connected subvariety of X such that $i : X \rightarrow Y$ is affine, and let \mathcal{M} be a simple holonomic \mathcal{D}_Y -module. Then the minimal extension $L(Y, \mathcal{M})$ is also simple, and it is characterized as the unique simple submodule (resp. unique simple quotient module) of $\int_i \mathcal{M}$ (resp. of $\int_{i^!} \mathcal{M}$).*

(ii) *Any simple holonomic \mathcal{D}_X -module is isomorphic to the minimal extension $L(Y, \mathcal{M})$ for some pair (Y, \mathcal{M}) , where Y is as in (i) and \mathcal{M} is a simple integrable connection on Y .*

(iii) *Let $L(Y, \mathcal{M})$ and $L(Y', \mathcal{M}')$ be as in (ii). We have $L(Y, \mathcal{M}) \simeq L(Y', \mathcal{M}')$ if and only if $\overline{Y} = \overline{Y'}$ and $\mathcal{M}|_U \simeq \mathcal{M}'|_U$ for an open dense subset U of $Y \cap Y'$.*

Proof. (i) Let us choose an open subset $U \subset X$ containing Y . We denote by $\kappa : Y \hookrightarrow U$ the closed embedding and by $j : U \rightarrow X$ the open embedding. We first show the following results.

- (a) For any $\mathcal{E} \in \mathbf{Mod}_{qc}^{\overline{Y}}(\mathcal{D}_X)$ we have $H^l i^\dagger(\mathcal{E}) = 0$ for $l \neq 0$. In other words $H^0 i^\dagger : \mathbf{Mod}_{qc}^{\overline{Y}}(\mathcal{D}_X) \rightarrow \mathbf{Mod}_{qc}(\mathcal{D}_Y)$ is an exact functor.
- (b) For any non-zero holonomic submodule \mathcal{N} of $\int_i \mathcal{M}$, we have $i^\dagger \mathcal{N} \simeq \mathcal{M}$.
- (c) $\int_i \mathcal{M}$ (resp. $\int_{i^!} \mathcal{M}$) has a unique simple holonomic submodule (resp. simple holonomic quotient module).
- (d) For a sequence $0 \neq \mathcal{N}_1 \subset \mathcal{N}_2 \subset \int_i \mathcal{M}$ of holonomic submodules of $\int_i \mathcal{M}$, we have $i^\dagger(\mathcal{N}_2/\mathcal{N}_1) = 0$.

We start by remarking that $j^\star = j^\dagger = j^{-1}$ because j is an open embedding. Now, given that κ is a closed embedding and for $\mathcal{E} \in \mathbf{Mod}_{qc}^{\overline{Y}}(\mathcal{D}_X)$ we have the relations $i^\dagger \mathcal{E} = \kappa^\dagger j^\dagger \mathcal{E} = \kappa^\dagger j^{-1} \mathcal{E}$ and $\mathbf{Supp}(j^{-1} \mathcal{E}) \subset \overline{Y} \cap U = Y$, then by Kashiwara's equivalence $H^l i^\dagger(\mathcal{E}) = H^l \kappa^\dagger(j^{-1} \mathcal{E}) = 0$ for all $l \neq 0$. This proves (a).

Now, by corollary 4.14.10

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}_X} \left(\mathcal{N}, \int_i \mathcal{M} \right) &= \mathrm{Hom}_{\mathcal{D}_X} \left(\mathcal{N}, \int_j \int_k \mathcal{M} \right) \\ &\simeq \mathrm{Hom}_{\mathcal{D}_U} \left(j^\star \mathcal{N}, \int_k \mathcal{M} \right). \end{aligned}$$

Since j is an open embedding, we have that $j^* = j^\dagger = j^{-1}$ is exact which implies that if we apply the functor j^\dagger to the inclusion $\mathcal{N} \hookrightarrow \int_i \mathcal{M}$ we get a non-zero morphism $\phi : j^\dagger \mathcal{N} \rightarrow \int_k \mathcal{M}$. Moreover, by theorem 4.14.1 we know that $\int_k \mathcal{M}$ is a simple holonomic \mathcal{D}_U -module, so by Kashiwara's equivalence, we can conclude that ϕ is surjective. Applying k^\dagger to ϕ we obtain a surjective morphism $i^\dagger \mathcal{N} \rightarrow k^\dagger \int_k \mathcal{M} = \mathcal{M}$. On the other hand, we have an injective morphism $i^\dagger \mathcal{N} \rightarrow i^\dagger \int_i \mathcal{M} \simeq \mathcal{M}$ because i^\dagger is exact by (a). Hence we must have $i^\dagger \mathcal{N} \simeq \mathcal{M}$. This proves (b).

To see (c) let's suppose that there exists two simple holonomic submodules $\mathcal{L} \neq \mathcal{L}'$ of $\int_i \mathcal{M}$. Set $\mathcal{N} = \mathcal{L} \oplus \mathcal{L}'$ (the sum is in fact a direct sum by simplicity). By (b) we have

$$\mathcal{M} \simeq i^\dagger \mathcal{N} \simeq i^\dagger \mathcal{L} \oplus i^\dagger \mathcal{L}' \simeq \mathcal{M} \oplus \mathcal{M}'$$

which is a contradiction. Finally, by exactness of i^\dagger we have $i^\dagger \mathcal{N}_2 / i^\dagger \mathcal{N}_1 \simeq i^\dagger(\mathcal{N}_2 / \mathcal{N}_1)$, and therefore from (b) we can conclude that $i^\dagger(\mathcal{N}_2 / \mathcal{N}_1) = 0$, since $i^\dagger \mathcal{N}_1 \subset i^\dagger \mathcal{N}_2 \subset i^\dagger \int_i \mathcal{M} = \mathcal{M}$.

Let's see the theorem. By (c) there exists a unique simple holonomic submodule \mathcal{L} of $\int_i \mathcal{M}$ and by corollary 4.14.10 there exist two isomorphisms

$$\begin{aligned} \text{Hom}_{\mathcal{D}_X} \left(\int_{i!} \mathcal{M}, \mathcal{L} \right) &\simeq \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, i^\dagger \mathcal{L}) \simeq \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{M}), \\ \text{Hom}_{\mathcal{D}_X} \left(\int_{i!} \mathcal{M}, \int_i \mathcal{M} \right) &\simeq \text{Hom}_{\mathcal{D}_Y} \left(\mathcal{M}, i^\dagger \int_i \mathcal{M} \right) \simeq \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{M}), \end{aligned}$$

from which we have a decomposition $\int_{i!} \mathcal{M} \rightarrow \mathcal{L} \hookrightarrow \int_i \mathcal{M}$ of the morphism $\int_{i!} \mathcal{M} \rightarrow \int_i \mathcal{M}$ (which is non-zero since it corresponds to the identity). Since \mathcal{L} is simple, the image of the latter morphism is \mathcal{L} and the proof of (i) is complete.

To see (ii), let's take \mathcal{L} a simple holonomic \mathcal{D}_X -module. By propositions 4.13.5 and 4.13.6 we can take an affine open dense subset Y of an irreducible component of $\mathbf{Supp}(\mathcal{L})$ such that if $i : Y \rightarrow X$ is the embedding, then $\mathcal{M} := i^\dagger \mathcal{L}$ is an integrable connection on Y and it is simple. By corollary 4.14.10 we have an isomorphism

$$\text{Hom}_{\mathcal{D}_X} \left(\int_{i!} \mathcal{M}, \mathcal{L} \right) = \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, i^\dagger \mathcal{L}) \simeq \text{Hom}_{\mathcal{D}_Y}(\mathcal{M}, \mathcal{M}) \neq 0$$

which implies, by simplicity, that there exists a surjective morphism $\int_{i!} \mathcal{M} \rightarrow \mathcal{L}$. Therefore, \mathcal{L} is a simple holonomic quotient module of $\int_{i!} \mathcal{M}$ and we obtain $\mathcal{L} = L(Y, \mathcal{M})$ by (i).

Let us finally prove (iii). To do that, we start by remarking that, under the hypothesis of (ii), we have that $\mathbf{Supp}(\mathcal{M}) = Y$ (we recall that \mathcal{M} is an integrable connexion on Y). This implies that $\mathbf{Supp}(\int_i \mathcal{M}) = \bar{Y}$ and therefore $Y \subseteq \mathbf{Supp}(L(Y, \mathcal{M})) \subseteq \bar{Y}$. But given that $\mathbf{Supp}(L(Y, \mathcal{M}))$ is a closed set we necessarily have $\mathbf{Supp}(L(Y, \mathcal{M})) = \bar{Y}$.

Now, if $L(Y, \mathcal{M}) \simeq L(Y', \mathcal{M}')$, then by the reasoning given in the previous paragraph we have $\bar{Y} = \bar{Y}'$. Moreover, given that Y, Y' are locally closed in X , we have that Y is open in \bar{Y} and Y' is open in \bar{Y}' . Let us take $U := \bar{Y} \setminus (Y_0 \cup Y'_0)$,

where $Y_0 := \overline{Y} \setminus Y$ is a closed subset of \overline{Y} (resp. Y'_0 is a closed subset of $\overline{Y'}$). It is clear that U is an open subset contained in \overline{Y} which is in fact dense in $Y \cap Y'$. By (b), we know that $i_1^\dagger L(Y, \mathcal{M}) = \mathcal{M}$ and $L(Y, \mathcal{M})_U \simeq L(Y', \mathcal{M}')_U$ which gives us $\mathcal{M}|_U \simeq \mathcal{M}'|_U$.

On the other hand, let us denote by $\mathcal{L} := L(Y, \mathcal{M})$ and $\mathcal{L}' := L(Y', \mathcal{M}')$. Let C be an irreducible component of $\mathbf{Supp}(\mathcal{L})$ and let us take $Y'' := U \cap C$. If $i_1 : Y'' \rightarrow \overline{Y}$ and $i_2 : Y'' \rightarrow \overline{Y'}$ are the closed embeddings (of course $i_1 = i_2$), then as in the proof of (ii) we have $\mathcal{L} = L(Y'', i_1^\dagger \mathcal{L})$. Similarly $\mathcal{L}' = L(Y'', i_2^\dagger \mathcal{L}')$. To end the proof of (iii), we only need to prove that $i_1^\dagger \mathcal{L} = i_2^\dagger \mathcal{L}'$. This follows from the fact that $i_1^\dagger \mathcal{L}$ is the inverse image of $\mathcal{M}|_U$ on Y'' and $i_2^\dagger \mathcal{L}'$ is the inverse image of $\mathcal{M}'|_U$ on Y'' . This ends the proof of the theorem. \square

4.15.1 Irreducible representations of the first Weyl algebra

In this subsection we will use the theory developed so far to classify the irreducible representations of the first Weyl algebra \mathbf{A}_1 . This classification has been achieved in [13] by algebraic methods.

Example 4.15.3. Let $X = \mathbb{A}_{\mathbb{C}}^1$ be the affine line. As in subsection 4.14, let us denote by $A_1(\mathbb{C}) := \mathcal{D}_{\mathbb{A}_{\mathbb{C}}^1}(\mathbb{A}_{\mathbb{C}}^1)$ the first Weyl algebra. If x is a fixed coordinate an explicit expression for $A_1(\mathbb{C})$ is

$$A_1(\mathbb{C}) = \frac{\mathbb{C}[x, \partial_x]}{([x, \partial_x] - 1)}.$$

Let $Y \subset X$ be a connected locally closed sub-variety of X , and $U \subseteq X$ be an open subset such that Y is closed in U . Let us consider the following cases.

- (i) $Y \subsetneq U$. Therefore $\dim(Y) < \dim(X) = 1$ and $Y = \{p\}$ is a single point. In this case, by \mathcal{D} -affinity, we have

$$L(\{p\}, \mathcal{O}_{\{p\}}) = \mathcal{D}_X \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{\{p\}}$$

which is simple.

- (ii) If $Y = U$, then Y contains the generic point η and $\overline{Y} = X$. By (iii) in theorem 4.15.2, if \mathcal{M} is a simple holonomic \mathcal{D}_U module and U' is another open subset such that \mathcal{M}' is a simple holonomic $\mathcal{D}_{U'}$, then $L(U, \mathcal{M}) \simeq L(U', \mathcal{M}')$ if and only if $\mathcal{M}|_\eta = \mathcal{M}'|_\eta$.

By (ii) in 4.15.2 we get that the simple holonomic \mathcal{D}_X -modules are, up to isomorphism, $L(\{p\}, \mathcal{O}_{\{p\}})$ and $L(\eta, \mathcal{M})$, where \mathcal{M} is a simple holonomic \mathcal{D}_η -module.

Now, $\Gamma(\eta, \mathcal{D}_\eta) = A_1(\mathbb{C})_{(0)} = \mathbb{C}(x)[\partial_x]$ and by \mathcal{D} -affinity, the global sections of simple \mathcal{D}_η -modules are simple $\Gamma(\eta, \mathcal{D}_\eta)$ -modules. Moreover, by [13, First part of theorem 4.4], the $A_1(\mathbb{C})$ -socle (sum of all simple $A_1(\mathbb{C})$ -sub-modules) of every simple $\Gamma(\eta, \mathcal{D}_\eta)$ -module is a simple $A_1(\mathbb{C})$ -sub-module. Therefore, by [13, Second part of theorem 4.4], the global sections of the minimal extension $L(\eta, \mathcal{D}_\eta)$ equals the $A_1(\mathbb{C})$ -socle of $\Gamma(\eta, \mathcal{D}_\eta)$.

Finally, every element P of $\Gamma(\eta, \mathcal{D}_\eta) = \mathbb{C}(x)[\partial_x]$ can be written as $P = \sum_{\alpha \in \mathbb{N}} Q_\alpha \partial_x^\alpha$, where $Q_\alpha \in \mathbb{C}(x)$. Then if we define

$$\text{ord}(P) := \{\alpha \in \mathbb{N} \mid Q_\alpha \neq 0\},$$

we can easily verify that $\Gamma(\eta, \mathcal{D}_\eta)$ is a non-commutative principal ideal domain ([22, Exercise 4.5]) and therefore by a classical argument, the representations of $\Gamma(\eta, \mathcal{D}_\eta)$ can be expressed in terms of factorization of elements of $\Gamma(\eta, \mathcal{D}_\eta)$. Moreover, a $\Gamma(\eta, \mathcal{D}_\eta)$ -module M is simple if and only if

$$M \simeq \Gamma(\eta, \mathcal{D}_\eta) / (\Gamma(\eta, \mathcal{D}_\eta) \cdot P)$$

for some irreducible element $P \in \Gamma(\eta, \mathcal{D}_\eta)$ ([40]). All in all, theorem 4.15.2 tells us that simple $A_1(\mathbb{C})$ -modules are parametrized, up to isomorphisms, by the closed points of X and the irreducible elements of $\Gamma(\eta, \mathcal{D}_\eta)$. This gives back the classification in [13].

5 Beilinson-Bernstein-Brylinski-Kashiwara localization theorem

In this section

5.1 Classification

In this subsection we will define a generalization of the sheaf of differential operators \mathcal{D}_X studied in the previous sections over a smooth complex algebraic variety. We recall for the reader that this sheaf is endowed with a filtration $\{F_d \mathcal{D}_X\}_{d \in \mathbb{N}}$ by the order of the differential operator (42). Moreover, we have the following canonical identifications

$$\mathcal{O}_X = F_0 \mathcal{D} \quad \text{and} \quad \begin{array}{ccc} F_1 \mathcal{D}_X / F_0 \mathcal{D}_X & = & \mathcal{T}_X \\ \xi & \mapsto & [\xi, \bullet] \end{array}$$

(the second identification follow from the third item in the proposition 4.2.1 and the example 4.2.3). In particular

$$\text{gr}_\bullet(\mathcal{D}_X) = \text{Sym}(\mathcal{T}_X).$$

The definition of the sheaves of **twisted differential operators** will be based in the previous identifications.

Definition 5.1.1. *Let X be a smooth complex algebraic variety. A sheaf of **twisted differential operators on X** is a triple $(\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i)$, such that \mathcal{D} is a filtered sheaf of associative \mathbb{C} -algebras with identity and $i : \mathcal{O}_X \rightarrow \mathcal{D}$ is an injective morphism of \mathbb{C} -algebras. Moreover, $(\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i)$ must satisfy the following properties:*

- (i) *The inclusion $i : \mathcal{O}_X \rightarrow \mathcal{D}$ gives rise to an isomorphism $\mathcal{O}_X \simeq F_0 \mathcal{D}$,*
- (ii) *the natural map $\text{Sym}(F_1 \mathcal{D} / F_0 \mathcal{D}) \rightarrow \text{gr}_\bullet(\mathcal{D})$ is an isomorphism of graded \mathcal{O}_X -algebras,*

(iii) the map

$$\begin{array}{ccc} F_1\mathcal{D}/F_0\mathcal{D} & \rightarrow & \mathcal{T}_X \\ \xi & \mapsto & [\xi, \bullet], \end{array}$$

is an isomorphism

(iv) there exists a covering of X by affine open subsets U such that $\mathcal{D}|_U \simeq \mathcal{D}_X|_U$.

Remark 5.1.2. (i) Commutativity of $\text{gr}_\bullet(\mathcal{D})$ ensures that if $f \in F_0\mathcal{D} \simeq \mathcal{O}_X$ and $\xi \in F_1\mathcal{D}/F_0\mathcal{D}$ then $[\xi, f] \in \mathcal{O}_X$, so condition (iii) makes sense.

(ii) Even if the fourth condition is a consequence of (i), (ii) and (iii) (cf. [42, Proposition 2.5.1]), in this notes we will accept this fact as part of the definition in order to simplify the parametrization of those sheaves (proposition 5.1.5 below).

The preceding sheaves have a natural parametrization. We will follow [50] to study this classification. We start with the following easy remark (cf. [32, Exercise 1.1.1]).

Remark 5.1.3. Let U be an affine open subset of X . The algebra $\mathcal{D}_X(U)$ is canonically isomorphic to the \mathbb{C} -algebra generated by the elements

$$\{f, \theta \mid f \in \mathcal{O}_X(U) \text{ and } \theta \in \mathcal{T}_X(U)\}$$

which satisfies the following fundamental relation:

$$[\theta, f] = \theta(f).$$

Lemma 5.1.4. Let $\varphi \in \text{Aut}(\mathcal{D}_X, \{F_d\mathcal{D}_X\}_{d \in \mathbb{N}}, i)$ such that $\varphi|_{\mathcal{O}_X} = \text{id}$. Then there exists a closed 1-form $\omega \in \Omega_X^1 = \text{Hom}_{\mathcal{O}_X}(\mathcal{T}_X, \mathcal{O}_X)$ such that

$$\varphi(\xi) = \xi - \omega(\xi)$$

for any vector field $\xi \in \mathcal{T}_X$. Moreover φ is completely determined by ω .

Proof. Let $f \in \mathcal{O}_X$ and $\xi \in \mathcal{T}_X$. Then

$$[\varphi(\xi), f] = [\varphi(\xi), \varphi(f)] = \varphi([\xi, f]) = \varphi(\xi(f)) = \xi(f),$$

where we have used the relation in the previous remark to establish the first and final equality. Evaluating this relation on $1 \in \mathcal{O}_X$, we have

$$\varphi(\xi)(f) = \xi(f) + f\varphi(\xi)(1).$$

Let us define $\omega \in \Omega_X^1$ by $\omega(\xi) := -\varphi(\xi)(1)$. We immediately have

$$\omega([\xi, \eta]) = -\varphi([\xi, \eta])(1) = -(\varphi(\xi)\varphi(\eta) - \varphi(\eta)\varphi(\xi))(1) = \xi(\omega(\eta)) - \eta(\omega(\xi)),$$

for $\eta, \xi \in \mathcal{T}$ local sections, which implies

$$d\omega(\xi \wedge \eta) = \xi(\omega(\eta)) - \eta(\omega(\xi)) - \omega([\xi, \eta]) = 0$$

and ω is a closed 1-form. We finally remark that φ is completely determined by ω . The final statement follows from the fact that, by hypothesis, φ preserves the filtration and the induced endomorphism $\text{gr}_\bullet(\varphi)$ of $\text{gr}_\bullet(\mathcal{D}_X)$ is the identity morphism. Hence φ is an automorphism. \square

The previous lemma implies that we have an injective map from the group of automorphisms $\text{Aut}(\mathcal{D}_X, \{F_d \mathcal{D}_X\}_{d \in \mathbb{N}}, i)$ into the additive group of closed 1-forms $\mathcal{Z}^1(X)$. This map clearly respects the groups structures. We have the following lemma [50, Lemma 1.2].

Proposition 5.1.5. *The natural morphism of $\text{Aut}(\mathcal{D}_X, \{F_d \mathcal{D}_X\}_{d \in \mathbb{N}}, i)$ into $\mathcal{Z}^1(X)$ is an isomorphism of groups.*

Proof. We need to prove surjectivity. Let us take ω a closed 1-form, and let us consider the endomorphism φ of \mathcal{T}_X defined by $\varphi(\xi) = \xi - \omega(\xi)$. If we prove that φ preserves the natural bracket in \mathcal{T}_X , then it extends in a unique way to the whole \mathcal{D}_X . This follows from the next relations:

$$\begin{aligned} \varphi([\xi, \eta]) &= [\eta, \xi] - \omega([\xi, \eta]) \\ &= [\xi, \eta] - \xi(\omega(\eta)) + \eta(\omega(\xi)) \\ &= [\xi - \omega(\xi), \eta - \omega(\eta)] \\ &= [\varphi(\xi), \varphi(\eta)]. \end{aligned}$$

To prove that φ is actually an automorphism we can pass through $\text{gr}_\bullet(\varphi) = id$ as before. \square

We can finally give the parametrisation of the sheaves of twisted differential operators. We follow word by word the reasoning given in [50, Page 2 and theorem 1.3].

Theorem 5.1.6. *The isomorphism classes of twisted differential operators on X are in bijection with the elements of $H^1(X, \mathcal{Z}_X^1)$.*

Proof. Let us start the proof by constructing a map

$$\iota : \mathbf{Isoclass} \rightarrow H^1(X, \mathcal{Z}_X^1).$$

where $\mathbf{Isoclass}$ denotes the set of isomorphic classes of twisted differential operators. Let $(\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i) \in \mathbf{Isoclass}$. By definition, there exists an affine covering $\mathcal{U} = \{U_j\}_{1 \leq j \leq n}$ such that for all $1 \leq j \leq n$ there exists an isomorphism $\psi_j : \mathcal{D}|_{U_j} \rightarrow \mathcal{D}_X|_{U_j} = \mathcal{D}_{U_j}$. Moreover, for every $1 \leq i, k \leq n$ there exists an automorphism ϕ_{jk} of $(\mathcal{D}_{U_i \cap U_k}, F_d \mathcal{D}_{U_i \cap U_k}, i_{U_i \cap U_k})$ such that the diagram

$$\begin{array}{ccc} & \mathcal{D}|_{U_j \cap U_k} & \\ \swarrow \psi_k & & \searrow \psi_j \\ \mathcal{D}_{U_j \cap U_k} & \xrightarrow{\phi_{jk}} & \mathcal{D}_{U_j \cap U_k} \end{array}$$

By the preceding lemma, there exists a closed 1-form ω_{jk} on $U_j \cap U_k$ which determines ϕ_{jk} . Moreover, if $U_j \cap U_k \cap U_l \neq \emptyset$, then we have the commutative diagram

$$\begin{array}{ccccc} & \mathcal{D}|_{U_j \cap U_k \cap U_l} & & & \\ \swarrow \psi_l & \downarrow \psi_j & \searrow \psi_k & & \\ \mathcal{D}_{U_j \cap U_k \cap U_l} & \xrightarrow{\phi_{kl}} & \mathcal{D}_{U_j \cap U_k \cap U_l} & \xrightarrow{\phi_{jk}} & \mathcal{D}_{U_j \cap U_k \cap U_l} \end{array}$$

which implies that $\phi_{jl} = \phi_{jk} \circ \phi_{kl}$ on $U_j \cap U_k \cap U_l$. From this, we can conclude that

$$\phi_{jl}(\xi) = \xi - \omega_{jl}(\xi) = (\phi_{jk} \circ \phi_{kl})(\xi) = \phi_{jk}(\xi - \omega_{kl}(\xi)) = \xi - \omega_{kl}(\xi) - \omega_{jk}(\xi)$$

for $\xi \in \mathcal{T}_{U_j \cap U_k \cap U_l}$. In other words

$$\omega_{jl} = \omega_{jk} + \omega_{kl}.$$

The preceding proves that if $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{Z}_X^1)$ denotes the Čech complex of \mathcal{Z}_X^1 corresponding to the cover \mathcal{U} , then $\omega = (\omega_{jk})_{1 \leq j < k \leq n}$ is an element of $Z^1(\mathcal{U}, \mathcal{Z}_X^1)$, because $d\omega = 0$.

On the other hand, let us suppose that we take another set of local isomorphisms $\psi'_j : \mathcal{D}|_{U_j} \rightarrow \mathcal{D}_{U_j}$. This gives another set of automorphisms $(\phi'_{jk})_{1 \leq j < k \leq n}$ and another $\omega' \in Z^1(\mathcal{U}, \mathcal{Z}_X^1)$. By applying the preceding lemma, we get automorphisms σ_j of (\mathcal{D}_{U_j}) such that $\psi'_j = \sigma_j \circ \psi_j$

$$\begin{array}{ccc} \mathcal{D}_{U_j} & \xrightarrow{\psi_j^{-1}} & \mathcal{D}|_{U_j} \\ & \searrow \sigma_j & \downarrow \psi'_j \\ & & \mathcal{D}_{U_j}, \end{array}$$

and closed 1-forms ρ_j associated to them, which are evidently elements of $\mathcal{C}^0(\mathcal{U}, \mathcal{Z}_X^1)$. Now, using commutativity in the first diagram, we see that

$$\sigma_j \circ \phi_{jk} \circ \psi_k = \sigma_j \circ \psi_j = \psi'_j = \phi'_{jk} \circ \psi'_k = \phi'_{jk} \circ \sigma_k \circ \psi_k$$

on $U_j \cap U_k$. This implies that $\sigma_j \circ \phi_{jk} = \phi'_{jk} \circ \sigma_k$, and therefore

$$\rho_j + \omega_{jk} = \omega'_{jk} + \rho_k.$$

It follows that $\omega' - \omega = d\rho$, and we can define $\iota((\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i))$ by the classe of ω in $H^1(\mathcal{U}, \mathcal{Z}_X^1)$. Let us prove that this is a bijection.

Injectivity: Let us take $(\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i)$ and $(\mathcal{D}', \{F_d \mathcal{D}'\}_{d \in \mathbb{N}}, i')$ two sheaves of twisted differential operators, such that $\iota((\mathcal{D}, \{F_d \mathcal{D}\}_{d \in \mathbb{N}}, i)) = \iota((\mathcal{D}', \{F_d \mathcal{D}'\}_{d \in \mathbb{N}}, i'))$. Both sheaves determine an affine open covering \mathcal{U} and $\omega, \omega' \in Z^1(\mathcal{U}, \mathcal{Z}_X^1)$, such that they define the same element in $H^1(X, \mathcal{Z}_X^1)$. Moreover, following the same reasoning given in the first part of the proof, and taking possibly a refinement of \mathcal{U} , we may assume that we have locally isomorphisms $\psi_j : \mathcal{D}|_{U_j} \rightarrow \mathcal{D}_{U_j}$ and $\psi'_j : \mathcal{D}'|_{U_j} \rightarrow \mathcal{D}_{U_j}$, and also that there exists $\rho = (\rho_j) \in \mathcal{C}^0(\mathcal{U}, \mathcal{Z}_X^1)$, such that $\omega - \omega' = d\rho$. As we already know, the map ρ determines automorphisms $\sigma_j : \mathcal{D}_{U_j} \rightarrow \mathcal{D}_{U_j}$, and therefore $\psi''_j := \sigma_j \circ \psi'_j : \mathcal{D}'|_{U_j} \rightarrow \mathcal{D}_{U_j}$ is a family of local isomorphism, which determines a family of local isomorphisms $\{\phi''_{jk}\}$ and a 1-form ω'' as we have explained before. We have the following relations

$$\sigma_j \circ \phi'_{jk} \circ \psi'_k = \sigma_j \circ \psi'_j = \psi''_j = \phi''_{jk} \circ \psi''_k = \psi''_{jk} \circ \sigma_k \circ \psi'_k$$

which imply that $\sigma_j \circ \phi'_{jk} = \psi''_{jk} \circ \sigma_k$, in other words $\phi''_{jk} = \sigma_j \circ \phi'_{jk} \circ \sigma_k^{-1}$ on $U_j \cap U_k$. This tells us that $\omega''_{jk} = \omega'_{jk} + \rho_j - \rho_k$ on $U_j \cap U_k$. This is $\omega'' = \omega' + d\rho = \omega$, and therefore $\phi_{jk} = \phi''_{jk}$ on $U_j \cap U_k$. With this information

we can define local isomorphisms $\chi_j := (\psi''_j)^{-1} \circ \psi_j : \mathcal{D}|_{U_j} \rightarrow \mathcal{D}'|_{U_j}$, such that on $U_j \cap U_k$ we have

$$\chi_j = (\psi''_j)^{-1} \circ \psi_j = (\phi_{jk} \circ \psi''_k)^{-1} \circ \phi_{jk} \circ \psi_k = (\psi''_k)^{-1} \circ \psi_k = \chi_k,$$

so they extend to a global isomorphism χ of \mathcal{D} onto \mathcal{D}' .

Surjectivity: Let $\bar{\omega} \in H^1(X, \mathcal{Z}_X^1)$. Then it determines a closed 1-form $\omega \in Z(\mathcal{U}, \mathcal{Z}_X^1)$ with $d\omega = 0$. By lemma 5.1.4 every ω_{ij} corresponds to an isomorphism $\phi_{ij} : \mathcal{D}_{U_j \cap U_k} \rightarrow \mathcal{D}_{U_j \cap U_k}$. Moreover $\phi_{il} = \phi_{jk} \circ \phi_{kl}$ because ω is closed. With this information we can conclude that $(\mathcal{D}_{U_j}, \phi_{ij})$ is a glueing data for a sheaf of \mathbb{C} -algebras with respect to the covering \mathcal{U} . Hence, there exists a sheaf \mathcal{D} of \mathbb{C} -algebras with isomorphisms $\psi_j : \mathcal{D}|_{U_j} \rightarrow \mathcal{D}_{U_j}$. It is an easy exercise to prove that \mathcal{D} is a sheaf of twisted differential operators in the sense of the definition 5.1.1, and by construction $\iota(\mathcal{D}) = \bar{\omega}$. \square

5.2 Twisted differential operators on flag varieties

In this section we will introduce a family of twisted differential operators parametrized by \mathfrak{t}^\vee . We will propose two, a priori different, definitions and at the end of 5.2.25, we will prove that they are equivalent. Most of the material of this subsection follows almost word by word the excellent lecture notes of Yi Sun [77] and J. Simental [73].

Let \mathbb{G} be a simply connected semisimple complex algebraic group with Lie algebra \mathfrak{g} . We fix one for all a Borel subgroup $\mathbb{B} \subseteq \mathbb{G}$ which contains a maximal torus $\mathbb{T} \subseteq \mathbb{B}$ of \mathbb{G} . We will also denote by \mathbb{N} the unipotent radical of \mathbb{B} and by \mathbb{N}^- the unipotent radical of the opposite Borel subgroup \mathbb{B}^- of \mathbb{B} . We recall for the reader that we have a canonical isomorphism $\mathbb{T} = \mathbb{B}/\mathbb{N}$. *Let us recall the following facts coming from the first sections of this work. First of all, given that \mathbb{G} is semi-simple, the Lie algebra \mathfrak{g} is semi-simple. Moreover, because we assume \mathbb{G} simply-connected, we have the Lie group Lie algebra correspondence. In particular, under this correspondence $\mathfrak{t} := \text{Lie}(\mathbb{T})$ is a maximal torus of \mathfrak{g} , $\mathfrak{b} := \text{Lie}(\mathbb{B})$ is a Borel subalgebra such that $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$, where $\mathfrak{n} := \text{Lie}(\mathbb{N})$ contains the information about the positive roots.*

Throughout this subsection and until the end of this chapter, we will always denote by $X = \mathbb{G}/\mathbb{B}$ the complex flag variety. We recall for the reader that this variety is independent of the choice of the Borel subgroup, and that fixing such a group we define a system of positive roots for the Lie algebra \mathfrak{g} .

Finally, we recall for the reader that the \mathbb{G} -equivariant vector bundles on the flag variety X are in 1-1 correspondence with representations of the Borel subgroup \mathbb{B} . This is obtained as follows. First of all, given a representation V of \mathbb{B} , we can construct a \mathbb{G} -equivariant vector bundle on X by taking the quotient of the \mathbb{B} -action on the trivial bundle $\mathbb{G} \times V$ defined by

$$b \cdot (g, v) := (gb^{-1}, b \cdot v)$$

(here $b \cdot v$ denotes the induced \mathbb{B} -action on V). We denote this space by $\mathbb{G} \times_{\mathbb{B}} V$. Now, if $\pi : \mathbb{G} \rightarrow X$ denotes the projection then the canonical map $\mathbb{G} \times V \rightarrow X$

defined by $(g, v) \mapsto \pi(v)$ is clearly constant on the \mathbb{B} -orbits, and therefore it induces a morphism

$$\pi_V : \mathbb{G} \times_{\mathbb{B}} V \rightarrow X.$$

We remark for the reader that π_V defines a vector bundle because π is locally trivial [39, Part II, 1.10 (2)] (I should probably remark that this follows from the presentation of Filippo after explaining this somewhere in the second section). Furthermore, the left \mathbb{G} -action on the trivial bundle $\mathbb{G} \times V$ defined by

$$g' \cdot (g, v) = (g'g, v)$$

induces a \mathbb{G} -action on $\mathbb{G} \times_{\mathbb{B}} V$ such that we have the following commutative diagram

$$\begin{array}{ccc} \mathbb{G} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{G} \times_{\mathbb{B}} V & \xrightarrow{\mathbb{G}\text{-action}} & \mathbb{G} \times_{\mathbb{B}} V \\ \downarrow \mathrm{id}_{\mathbb{G}} \times \pi_V & & \downarrow \pi_V \\ \mathbb{G} \times_{\mathrm{Spec}(\mathbb{C})} X & \xrightarrow{\mathbb{G}\text{-action}} & X. \end{array}$$

On the other hand, if $\pi_E : E \rightarrow X$ is a \mathbb{G} -equivariant vector bundle, then the fiber $E_{e_{\mathbb{B}}}$ (at the origin of the flag variety) is clearly a \mathbb{B} -representation. Our correspondence is

$$\begin{array}{ccc} \{\mathbb{G}\text{-equivariant vector bundles}\} & \leftrightarrow & \{\mathbb{B}\text{-representations}\} \\ E & \mapsto & E_{e_{\mathbb{B}}} \\ \mathbb{G} \times_{\mathbb{B}} V & \longleftarrow & V \end{array}$$

Let us translate the preceding geometric construction in an algebraic setting. Let $\pi_E : E \rightarrow X$ be a (geometric) \mathbb{G} -equivariant vector bundle. The presheaf \mathcal{L}_E defined by

$$U \subseteq X \mapsto \mathcal{L}_E(U) := \{s : U \rightarrow E \mid s \text{ is a section over } U\}$$

is in fact sheaf, and it is called the **sheaf of sections of the line bundle E** . This is a locally free sheaf of rank equals to the rank of E . The \mathbb{G} -equivariance of π_E translates into the following fact. Let $p_2 : \mathbb{G} \times_{\mathrm{Spec}(\mathbb{C})} E \rightarrow E$ be the second projection. The \mathbb{G} -action on E induces an isomorphism

$$\Psi : (\mathbb{G}\text{-act})^*(\mathcal{L}_E) \rightarrow p_2^*(\mathcal{L}_E)$$

which endows the sheaf \mathcal{L}_E of a \mathbb{G} -equivariant structure in the sense of [32, Definition 9.10.2] (for further explanations the reader can take a look to [61, Proposition 3.4]).

Finally, we remark for the reader that given a locally free sheaf \mathcal{L} of finite rank, then there is a canonical way to construct a vector bundle E on X such that $\mathcal{L}_E = \mathcal{L}$ (cf. [61, Page 15]). From now on, we will say that E is a **geometric vector bundle** and that \mathcal{L} is an **algebraic vector bundle**.

Remark 5.2.1. ([32, Page 256]) From the precedence 1-1 correspondence we can conclude that \mathbb{G} -equivariant line bundles on X corresponds to a one-dimensional \mathbb{B} -module. Moreover, the Jordan decomposition is preserved by homomorphisms of algebraic groups, and therefore the action of the unipotent radical \mathbf{N} of \mathbb{B} on one-dimensional \mathbb{B} -modules is trivial. This proves that \mathbb{G} -equivariant line bundles on X corresponds to (**algebraic**) characters λ of $\mathbb{T} = \mathbb{B}/\mathbf{N}$. From now on, we will denote by $\mathcal{L}(\lambda)$ the (algebraic) line bundle induced by a character $\lambda \in \mathrm{Hom}_{\mathrm{grps}}(\mathbb{T}, \mathbb{G}_m)$.

5.2.1 Actions of Lie algebras on \mathbb{G} -equivariant sheaves

Let \mathcal{L} be a \mathbb{G} -equivariant (algebraic) vector bundle. The goal of this subsection is to construct a structure of \mathfrak{g} -module over $\text{End}_{\mathbb{C}}(\mathcal{L})$.

To start with, let us recall that in the subsection 3.4 we have introduced the functor $\text{Lie}(\bullet)$ from the category $\mathbf{AlgGps}_{/\mathbb{C}}$ of complex algebraic groups to the category $\mathcal{L}ie_{\mathbb{C}}$ of finite-dimensional complex Lie algebras (theorem 3.4.2). This functor can be generalized as follows. Let \mathbb{G} be a complex algebraic group and $\mathbb{C}[\varepsilon] := \mathbb{C}[t]/(t^2)$ be the ring of dual numbers. Writing $S_{\varepsilon} := S \times_{\text{Spec}(\mathbb{C})} \text{Spec}(\mathbb{C}[\varepsilon])$, we can define the functor $\text{Lie}(\mathbb{G})$ from the category $\mathbf{Sch}_{/\mathbb{C}}$ to the category \mathbf{Grp} , via the short exact sequence

$$1 \rightarrow \text{Lie}(\mathbb{G})(S) \rightarrow \mathbb{G}(S_{\varepsilon}) \rightarrow \mathbb{G}(S) \rightarrow 1.$$

It is clear that $\mathfrak{g} = \text{Lie}(\mathbb{G})(k)$ and by proposition 3.3.2, the previous sequence allows to identify $\mathfrak{g} = \text{Der}_{\mathbb{C}}(\mathbb{C}[\mathbb{G}], \mathbb{C})$.

On the other hand, the left \mathbb{G} -action on X defines a map of functors $\mathbb{G} \rightarrow \text{Aut}(X)$, where $\text{Aut}(X)$ is the complex algebraic group (reference?) defined by

$$\text{Aut}(X)(R) := \text{Aut}_{\text{Sch}_{/\mathbb{C}}}(X \times_{\text{Spec}(\mathbb{C})} \text{Spec}(R), X \times_{\text{Spec}(\mathbb{C})} \text{Spec}(R)).$$

An important object to understand is the one defined by the \mathbb{C} -points of $\text{Lie}(\text{Aut}(X))$. This can be done as follows. By definition we have an exact sequence

$$0 \rightarrow \text{Lie}(\text{Aut}(X))(\mathbb{C}) \rightarrow \text{Aut}(X)(\mathbb{C}[\varepsilon]) \rightarrow \text{Aut}(X)(\mathbb{C}) \rightarrow 0$$

which helps us to compute $\text{Lie}(\text{Aut}(X))(\mathbb{C})$ via the relations

$$\begin{aligned} \text{Lie}(\text{Aut}(X))(\mathbb{C}) &= \{\phi \in \text{Aut}(X_{\varepsilon}) \mid \phi|_X = \text{id}\} \\ &= \{\phi \in \text{Hom}(\mathcal{O}_X[\varepsilon], \mathcal{O}_X[\varepsilon]) \mid \phi|_{\mathcal{O}_X} = \text{id}\} \\ &= \text{Der}_{\mathbb{C}}(\mathcal{O}_X). \end{aligned}$$

In other words, we have a canonical map

$$\tau : \mathfrak{g} \rightarrow \text{Der}_{\mathbb{C}}(\mathcal{O}_X). \tag{94}$$

Remark 5.2.2. In the literature it is to find that the preceding morphism is the result of differentiating the \mathbb{G} -action on \mathcal{O}_X . We also remark fro the reader that under the identification $\mathfrak{g} = \text{Der}_{\mathbb{C}}(\mathbb{C}[\mathbb{G}], \mathbb{C})$, we can see that $\tau(\eta)$ is defined via the composition

$$\mathcal{O}_X \xrightarrow{\text{act}^*} \mathbb{C}[\mathbb{G}] \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\eta} \mathcal{O}_X.$$

What is hidden behind the previous construction is the fact that the \mathbb{G} -action on X induces a \mathbb{G} -equivariant structure on \mathcal{O}_X . The following proposition shows that we can define an analogous map to Ψ , for any \mathbb{G} -equivariant algebraic vector bundle on X .³⁷

³⁷In fact, for any \mathbb{G} -equivariant quasi-coherent sheaf on X .

Proposition 5.2.3. *Let \mathcal{L} be a \mathbb{G} -equivariant algebraic vector bundle on X . There exists a canonical map*

$$\Psi_{\mathcal{L}} : \mathfrak{g} \rightarrow \text{End}_{\mathbb{C}}(\mathcal{L}),$$

such that for every $\eta \in \mathfrak{g}$, $f \in \mathcal{O}_X$ and $s \in \mathcal{L}$, we have

$$\Psi_{\mathcal{L}}(\eta)(fs) = f \cdot \Psi_{\mathcal{L}}(\eta)(s) + \tau(\eta)(f) \cdot s. \quad (95)$$

Proof. Let $\Phi : \text{act}^*\mathcal{L} \rightarrow p_2^*\mathcal{L}$ be the isomorphism inducing the \mathbb{G} -equivariant structure on \mathcal{L} . The morphism $\Psi_{\mathcal{L}}$ is defined as follows. Let $U \subseteq X$ be an affine open subset and $\eta \in \mathfrak{g} = \text{Der}_{\mathbb{C}}(\mathbb{C}[\mathbb{G}], \mathbb{C})$. Then $\Psi_{\mathcal{L}}(\eta)$ is given by the composition

$$\begin{aligned} \Psi_{\mathcal{L}}(\eta) : \Gamma(U, \mathcal{L}) &\rightarrow \Gamma(\mathbb{G} \times X, \text{act}^*\mathcal{L}) \xrightarrow{\Psi} \Gamma(\mathbb{G} \times X, p_2^*\mathcal{F}) \\ &\simeq \mathbb{C}[\mathbb{G}] \otimes_{\mathbb{C}} \Gamma(X, \mathcal{L}) \\ &\xrightarrow{id \otimes \text{res}_{U, X}} \mathbb{C}[\mathbb{G}] \otimes_{\mathbb{C}} \Gamma(U, \mathcal{L}) \\ &\xrightarrow{\eta} \Gamma(U, \mathcal{L}) \end{aligned}$$

The first morphism is induced by the canonical morphism $\mathcal{L} \rightarrow \text{act}_*\text{act}^*\mathcal{L}$ (let us note that $\text{act} : \mathbb{G} \times U \rightarrow X$ because \mathbb{G} acts transitively on X), and the isomorphism is given by the Künneth formula [27, Théorème 6.7.8].

Let us finally prove the formula (95). To do that, let us denote by $\epsilon : \{e\} \rightarrow \mathbb{G}$ the closed embedding of the identity into \mathbb{G} . By (19) we have

$$\begin{aligned} \Psi_{\mathcal{L}}(\eta)(f \cdot s) &= \eta(\Phi(\text{act}^*(f \cdot s))) \\ &= \eta(\text{act}^*(f) \cdot \Phi(\text{act}^*(s))) \\ &= \epsilon^*(\text{act}^*(f)) \cdot \eta(\Phi(\text{act}^*(s))) + \eta(\text{act}^*(f)) \cdot \epsilon^*(\Phi(\text{act}^*(s))) \\ &= f \cdot \Psi_{\mathcal{L}}(s) + \tau(\eta)(f) \cdot s \end{aligned}$$

□

5.2.2 Twisted differential operators (algebraic case)

Let $\lambda \in \text{Hom}_{\text{gps}}(\mathbb{T}, \mathbb{G}_m)$ be an **algebraic character** and $\mathcal{L}(\lambda)$ the algebraic line bundle on X induced by λ . We define the sheaf $\mathcal{D}_X^{\mathcal{L}(\lambda)} \subset \mathcal{E}nd_{\mathbb{C}}(\mathcal{L}(\lambda))$ of **differential operators acting on $\mathcal{L}(\lambda)$** by ($f \in \mathcal{O}_X$)

- $F_d \mathcal{D}_X^{\mathcal{L}(\lambda)} = 0$ if $d < 0$,
- $F_d \mathcal{D}_X^{\mathcal{L}(\lambda)} = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{L}(\lambda)) \mid Pf - fP \in F_{d-1} \mathcal{D}_X^{\mathcal{L}(\lambda)}\}$ if $d \geq 0$.

The sheaf $\mathcal{D}_X^{\mathcal{L}(\lambda)}$ is a sheaf of rings which contains $\mathcal{O}_X = F_0(\mathcal{D}_X^{\mathcal{L}(\lambda)})$ as a subring. Let us explain why this sheaf is in fact a sheaf of twisted differential operators. To start with, we note that we have the following isomorphism of sheaves of rings

$$\begin{aligned} \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^{\vee} &\simeq \mathcal{D}_X^{\mathcal{L}(\lambda)} \\ (s \otimes P \otimes s^{\vee}) &\mapsto P((s^{\vee}, \bullet))s. \end{aligned}$$

Here \mathcal{D}_X is the usual sheaf of differential operators on X . Moreover, if we endow the left-hand side with the filtration

$$\mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} F_d \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee$$

then the previous isomorphism is in fact an isomorphism of filtered \mathbb{C} -algebras. This proves that

$$\mathrm{gr}_\bullet(\mathcal{D}_X^{\mathcal{L}(\lambda)}) \simeq \mathcal{L}(\lambda) \otimes_{\mathcal{O}_X} \mathrm{gr}(\mathcal{D}_X) \otimes_{\mathcal{O}_X} \mathcal{L}(\lambda)^\vee \simeq \mathrm{gr}(\mathcal{D}_X) \simeq \mathrm{Sym}(\mathcal{T}_X).$$

Finally, to prove the fourth condition in the definition 5.1.1, we may take an affine open covering \mathcal{U} of X which trivialises the geometric line bundle $\mathbf{L}(\lambda) = \mathbb{G} \times_{\mathbb{B}} \mathbb{C}_\lambda$, then for every $U \in \mathcal{U}$ we have $\mathcal{L}(\lambda)|_U \simeq \mathcal{O}_U$ and therefore $\mathcal{D}_X^{\mathcal{L}(\lambda)}|_U \simeq \mathcal{D}_U$.³⁸

On the other hand, the \mathbb{G} -equivariant structure of the sheaf \mathcal{L} and the proposition 5.2.3 give us a map

$$\Psi_\lambda : \mathfrak{g} \rightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{L}(\lambda)) \quad (96)$$

which extends to a unique morphism of graded \mathbb{C} -algebra $\mathcal{U}(\mathfrak{g}) \rightarrow \mathrm{End}_{\mathbb{C}}(\mathcal{L}(\lambda))$.

Proposition 5.2.4. *The morphism (96) induces a canonical morphism of filtered \mathbb{C} -algebras*

$$\Phi_\lambda : \mathcal{U}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^{\mathcal{L}(\lambda)})$$

Proof. It is enough to prove that the image of Ψ_λ is contained in $\Gamma(X, \mathcal{D}_X^{\mathcal{L}(\lambda)})$. In other words, we need to prove that if $f, g \in \Gamma(X, \mathcal{O}_X)$, then

$$[f, [g, \Psi_\lambda(\eta)]] = 0.$$

This is an easy application of (95), because for any local section $s \in \mathcal{L}(\lambda)$ we have

$$[f, [g, \Psi_\lambda(\eta)]](s) = fg\Psi_\lambda(\eta)(s) - f\Psi_\lambda(\eta)(gs) - g\Psi_\lambda(\eta)(fs) + \Psi_\lambda(\eta)(fgs) = 0.$$

□

Remark 5.2.5. ([50, Proposition 1.4]) Let \mathcal{O}_X^* be the subsheaf of invertible elements of \mathcal{O}_X . It is known that the first group of cohomology $H^1(X, \mathcal{O}_X^*)$ equals the picard group $\mathrm{Pic}(X)$. Furthermore, the **logarithm derivative**

$$\begin{aligned} d\log : \mathcal{O}_X^* &\rightarrow \mathcal{Z}_X^1 \\ f &\mapsto f^{-1}df \end{aligned}$$

is homomorphism of sheaves of abelian groups, which induces a morphism in cohomology $H^1(d\log) : H^1(X, \mathcal{O}_X^*) \rightarrow H^1(X, \mathcal{Z}_X^1)$. If $\lambda \in \mathrm{Hom}_{\mathrm{gps}}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character and $\mathcal{L}(\lambda)$ is considered as an element of $\mathrm{Pic}(X)$, then

$$\iota(\mathcal{D}_X^{\mathcal{L}(\lambda)}) = H^1(d\log)(\mathcal{L}(\lambda)).$$

where ι is the bijection in theorem 5.1.6. This tells us that the group of algebraic characters $X^*(\mathbb{T}) = \mathrm{Hom}_{\mathrm{gps}}(\mathbb{T}, \mathbb{G}_m)$ can be considered as a subgroup of $H^1(X, \mathcal{Z}_X^1)$. In the next subsections we will explain how to extend this identification to the whole \mathfrak{t}^\vee .

³⁸We remark for the reader that this isomorphism is non-canonical because it depends of the local section $s \in \mathcal{L}(\lambda)|_U$ giving rise to the isomorphism $\mathcal{L}(\lambda)|_U = s \cdot \mathcal{O}_U = \mathcal{O}_U$.

5.2.3 General construction

In this subsection we extend the preceding constructions of Twisted differential operators attached to an algebraic character. The goal of this subsection is then to consider the case when $\lambda \in \mathfrak{t}^\vee$. Difficulties arrive immediately because we do not dispose any more of the algebraic line bundle $\mathcal{L}(\lambda)$. We will follow [7], [8], [15], [16], [11] and [50].

We start by recalling the parametrization of the **central characters** χ via characters of \mathfrak{t} . Let \mathfrak{z} be the center of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. By Schur's lemma, the center \mathfrak{z} acts on any finite-dimensional irreducible $\mathcal{U}(\mathfrak{g})$ -module via a map of \mathbb{C} -algebras $\chi : \mathfrak{z} \rightarrow \mathbb{C}$, which are called **central characters**. For those characters, it is clear that $\mathcal{U}(\mathfrak{g}) \cdot \ker(\chi)$ is a two-sided ideal of $\mathcal{U}(\mathfrak{g})$, and we can define the **central redaction**

$$\mathcal{U}(\mathfrak{g})_\chi := \mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \cdot \ker(\chi).$$

We remark for the reader that \mathfrak{z} acts on $\mathcal{U}(\mathfrak{g})_\chi$ via χ .

On the other hand, we have proved that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ splits as a direct sum

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{t}) \oplus (\mathfrak{n}^- \cdot \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g}) \cdot \mathfrak{n})$$

and we can consider the projection on the first factor

$$\pi_{\mathfrak{b}} : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{t}) \tag{97}$$

which is in fact a morphism of \mathbb{C} -algebras (cf. [77, Lemma 3.14]). Given that any character $\lambda \in \mathfrak{t}^\vee$ gives rise to a morphism of \mathbb{C} -algebras $\lambda : \mathcal{U}(\mathfrak{t}) \rightarrow \mathbb{C}$, the map $\pi_{\mathfrak{b}}$ can be reinterpreted as a map

$$\mathfrak{t}^* \rightarrow \text{MaxSpec}(\mathfrak{z})$$

Now, the previous map has the particularity that it depends of the Borel algebra $\mathfrak{b} \subseteq \mathfrak{g}$ choosing to fix a system of positive roots in the root decomposition of \mathfrak{g} . In order to avoid this dependency it is necessary to modify the projection $\pi_{\mathfrak{b}}$ to obtain a map $\psi : \mathfrak{z} \rightarrow \mathcal{U}(\mathfrak{t})$ independent of the choice of \mathfrak{t} and \mathfrak{b} . To do that, let us consider the Weyl-character

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

which is also the longest positive root relative to \mathfrak{b} . The map ψ is defined as the composite

$$\mathfrak{z} \xrightarrow{\pi_{\mathfrak{b}}} \mathcal{U}(\mathfrak{t}) \xrightarrow{\eta \mapsto \rho(\eta)1} \mathcal{U}(\mathfrak{t}).$$

We recall that this map is called the **Harish-Chandra isomorphism**. It is independent of \mathfrak{b} , or in other words of the Borel subalgebra \mathfrak{b} , and it restricts to an isomorphism

$$\mathfrak{z} \simeq \mathcal{U}(\mathfrak{t})^W$$

of W -invariants under the action of the Weyl group on $\mathcal{U}(\mathfrak{t})$. This gives another map

$$\begin{aligned} \mathfrak{t}^* &\rightarrow \text{MaxSpec}(\mathfrak{z}) \\ \lambda &\mapsto \mathfrak{m}_\lambda := \ker(\chi_\lambda : \mathfrak{z} \xrightarrow{\psi} \mathcal{U}(\mathfrak{t}) \xrightarrow{\lambda} \mathbb{C}) \end{aligned} \quad (98)$$

With the following properties

Lemma 5.2.6. *The map (98) satisfies the following properties:*

- (i) *Every central character χ lies in the image of the map (98).*
- (ii) *$\chi_\lambda = \chi_\mu$ if and only if there is some $w \in W$ such that $\lambda = w(\mu)$.*

Proof. The lemma is an immediate consequence of the following geometric fact about quotients of affine varieties by finite groups [28, Leture 10]: Let Y be a complex affine variety equipped with an action of a finite group H . Then the map $\text{MaxSpec}(\mathcal{O}(Y)) \rightarrow \text{MaxSpec}(\mathcal{O}(Y)^H)$ is the quotient map for the H -action on $\text{MaxSpec}(\mathcal{O}(Y))$. \square

Once that we have introduced the objects in the algebraic side, let us analyse the geometric side. To start with, let us recall that we have construct a canonical map in (94)

$$\mathfrak{g} \rightarrow \mathcal{T}_X.$$

This map allows to define the **Lie algebroid** \mathfrak{g}° of the Lie algebra \mathfrak{g} as follows. As an \mathcal{O}_X -module $\mathfrak{g}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g}$ and we equip it with a Lie bracket $[\bullet, \bullet] : \mathfrak{g}^\circ \times \mathfrak{g}^\circ \rightarrow \mathfrak{g}^\circ$ extending the lie bracket of \mathfrak{g} and

$$[f \otimes \xi, g \otimes \eta] = f\tau(\xi)(g) \otimes \eta - g\tau(\eta)(f) \otimes \xi + fg \otimes [\xi, \eta]$$

for $f, g \in \mathcal{O}_X$ local sections and $\eta, \xi \in \mathfrak{g}$. With this information, we can see that τ defines a homomorphism of sheaves of Lie algebras from \mathfrak{g}° into the tangent sheaf \mathcal{T}_X , which we denote by τ° . Furthermore, we can define the **universal enveloping algebra** \mathcal{U}° of the Lie algebroid \mathfrak{g}° as the sheaf of algebras endowed with a canonical map $\nu : \mathcal{O}_X \oplus \mathfrak{g}^\circ \rightarrow \mathcal{U}^\circ$ such that $\text{Im}(\nu)$ generates \mathcal{U}° subjects to the following relations

- $\nu(fg) = \nu(f) \cdot \nu(g)$.
- $\nu([\eta, \xi]) = \nu(\eta) \cdot \nu(\xi) - \nu(\xi) \nu(\eta)$.
- $\nu(f\eta) = \nu(f) \cdot \nu(\eta)$.
- $\nu(\tau^\circ(\eta)(f)) = \nu(\eta) \cdot \nu(f) - \nu(f) \cdot \nu(\eta)$.

Where $f, g \in \mathcal{O}_X$ and $\eta, \xi \in \mathfrak{g}^\circ$. From this relations, we see that $\mathcal{U}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{g})$ as sheaves of complex vector spaces and the product is given by

$$(f \otimes \xi) \cdot (g \otimes \eta) = f\tau(\xi)(g) \otimes \eta + fg \otimes \xi\eta.$$

Moreover, \mathcal{U}° carries a natural filtration coming from the structure of filtered \mathbb{C} -algebra on $\mathcal{U}(\mathfrak{g})$. In other words,

$$F_a \mathcal{U}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} F_a \mathcal{U}(\mathfrak{g})$$

for any $d \in \mathbb{Z}_{>0}$. In particular,

$$F_0\mathcal{U}^\circ = \mathcal{O}_X \quad \text{and} \quad F_1\mathcal{U}^\circ = \mathcal{O}_X \oplus \mathfrak{g}^\circ.$$

This tells us that $F_1\mathcal{U}^\circ$ generates \mathcal{U}° as a sheaf of algebras if we defines the right multiplicative structure on \mathcal{U}° .

Finally, we remark that \mathcal{U}° has both a structure of $\mathcal{U}(\mathfrak{g})$ -modules defined by

$$\xi(f \otimes \eta) = \tau(\xi)(f) \otimes \eta + f \otimes \xi\eta,$$

where $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $f \in \mathcal{O}_X$, the \mathcal{O}_X -structure is just multiplication on the first factor. Following the preceding definition, the reader can clearly see that if $\xi \in \mathfrak{g}$, $\eta \in \mathcal{U}(\mathfrak{g})$ and $f, g \in \mathcal{O}_X$, then

$$[\xi, g](f \otimes \eta) = [\tau(\xi), g]f \otimes \eta,$$

where the bracket on left occurs in the Lie algebroid \mathfrak{g}° , and the bracket on the right in \mathcal{T}_X .

Remark 5.2.7. By construction, the morphism τ° extends to a unique morphism of sheaves of \mathcal{O}_X -algebras

$$\Phi_0^\circ : \mathcal{U}^\circ \rightarrow \mathcal{D}_X,$$

such that at the level of global sections $\Phi_0 : \mathcal{U}(\mathfrak{g}) \rightarrow H^0(X, \mathcal{D}_X)$ is the map defined in the proposition 5.2.4³⁹ taking $\mathcal{L} = \mathcal{O}_X$. Moreover if $\lambda \in \text{Hom}_{\text{gps}}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character, then the preceding reasoning gives a canonical morphism of sheaves of filtered rings

$$\Phi_\lambda^\circ : \mathcal{U}^\circ \rightarrow \mathcal{D}_X^{\mathcal{L}(\lambda)}$$

such that $H^0(\Phi_\lambda^\circ) = \Phi_\lambda$ in the same proposition.

In the proof of the next lemma we follow word by word the argument given in [37].

Lemma 5.2.8. *The morphism $\tau^\circ : \mathfrak{g}^\circ \rightarrow \mathcal{T}_X$ is an epimorphism.*

Proof. [to give the proof.](#) □

Now, let us define

$$\mathfrak{b}^\circ := \ker(\tau^\circ : \mathfrak{g}^\circ \rightarrow \mathcal{T}_X),$$

and let us note that given that τ is induced via the left \mathbb{G} -action on X and the left \mathbb{B} -action is clearly trivial, then we can describe \mathfrak{b}° as follows

$$\mathfrak{b}^\circ = \{\eta \in \mathfrak{g}^\circ \mid \eta(x) \in \mathfrak{b}_x \text{ for all } x \in X\},$$

where $\mathfrak{b}_x \subseteq \mathfrak{g}$ is the Borel subalgebra corresponding to (the Borel subgroup) $x \in X$. In particular, if $\mathfrak{n}^\circ := [\mathfrak{b}^\circ, \mathfrak{b}^\circ]$, then \mathfrak{b}° and \mathfrak{n}° are subsheaves of \mathfrak{g}° and

$$\mathfrak{n}^\circ = \{\eta \in \mathfrak{g}^\circ \mid \eta(x) \in \mathfrak{n}_x \text{ for all } x \in X\}.$$

³⁹Given that X is a projective variety $H^0(X, \mathcal{O}_X) = \mathbb{C}$.

Here \mathfrak{n}_x is the Lie algebra associated to the unipotent radical of (the Borel subgroup) $x \in X$. Moreover, $\mathfrak{b}^\circ/\mathfrak{n}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{t}$.⁴⁰ This meaningful identification allows us to extend any arbitrary weight $\lambda \in \mathfrak{t}^\vee$, to a morphism of \mathcal{O}_X -modules

$$\lambda^\circ : \mathfrak{b}^\circ \rightarrow \mathfrak{b}^\circ/\mathfrak{n}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{t} \xrightarrow{\lambda} \mathcal{O}_X, \quad (99)$$

and we can define a right ideal $\mathcal{I}^\lambda(\mathfrak{g}^\circ) \subset \mathcal{U}^\circ$ by⁴¹

$$\mathcal{I}^\lambda(\mathfrak{g}^\circ) := \sum_{\eta \in \mathfrak{b}^\circ} (\eta - (\rho + \lambda)^\circ(\eta)) \mathcal{U}^\circ.$$

We have the following important property.

Lemma 5.2.9. *For any arbitrary weight $\lambda \in \mathfrak{t}^\vee$, the right ideal $\mathcal{I}^\lambda(\mathfrak{g}^\circ)$ is a two-sided ideal.*

Proof. It is enough to prove that $[\mathfrak{b}^\circ, \mathfrak{g}^\circ] \subseteq \mathfrak{b}^\circ$, and this follows from the fact that the morphism τ° is a morphism of \mathcal{O}_X -algebras. Hence for any $\xi \in \mathfrak{b}^\circ$ and $\eta \in \mathfrak{g}^\circ$ we have

$$\tau^\circ([\xi, \eta]) = \tau^\circ(\xi)\tau^\circ(\eta) - \tau^\circ(\eta)\tau^\circ(\xi) = 0, \quad (100)$$

which implies that $[\mathfrak{b}^\circ, \mathfrak{g}^\circ] \subseteq \mathfrak{b}^\circ$. \square

Remark 5.2.10. (*[77, Remark below lemma 4.4]*) We remark for the reader that $\mathcal{I}^\lambda(\mathfrak{g}) := \sum_{\eta \in \mathfrak{b}} (\eta - (\rho + \lambda)(\eta)) \mathcal{U}(\mathfrak{g})$ is not a two-sided ideal of $\mathcal{U}(\mathfrak{g})$. This because \mathfrak{b} is not the kernel of the map $\tau : \mathfrak{g} \rightarrow \Gamma(X, \mathcal{T}_X)$. This clarifies one of the fundamental roles of the sheaves theory.

Definition 5.2.11. *Let $\lambda \in \mathfrak{t}^\vee$ be an arbitrary character. We have a sheaf of \mathcal{O}_X -algebras*

$$\mathcal{D}_X^\lambda := \mathcal{U}^\circ / \mathcal{I}^\lambda(\mathfrak{g}^\circ).$$

The sheaf \mathcal{D}_X^λ comes equipped with the **quotient filtration** and the two-sided ideal $\mathcal{I}^\lambda(\mathfrak{g}^\circ)$ can be endowed with the induced filtration. Moreover, we have a canonical projection (which is clearly a map of filtered sheaves)

$$\Phi_\lambda^\circ : \mathcal{U}^\circ \rightarrow \mathcal{D}_X^\lambda,$$

and taking global sections a map of filtered \mathbb{C} -algebras

$$\Phi_\lambda : \mathcal{U}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^\lambda). \quad (101)$$

Proposition 5.2.12. *For any character $\lambda \in \mathfrak{t}^\vee$, the sheaf \mathcal{D}_X^λ is a sheaf of twisted differential operators on X .*

We will check the first three condition in the definition 5.1.1 and in the next subsection we will give a different description of the sheaves \mathcal{D}_X^λ that will allow us to verify the fourth condition.⁴²

⁴⁰In fact, $\mathfrak{b}^\circ/\mathfrak{n}^\circ = \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{h}$, where \mathfrak{h} is the Lie algebra of the **abstract Cartan subgroup** \mathbb{H} which is canonically isomorphic to \mathbb{T} .

⁴¹Let us remark that this is a bit different from [8]. The reason is that in this work $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. However in [8], they take the positive roots Δ^+ as those roots which are non-contained in \mathfrak{n} . This forces to consider a shift of ρ in the Harish-Chandra morphism, and not of $-\rho$ as we have done.

⁴²This can be prove directly as is done in [50, Page 13].

Proof. The first condition follows from the fact that $F_0\mathcal{U}^\circ = \mathcal{O}_X$. This also implies that $\mathrm{gr}_1(\mathcal{U}^\circ) = \mathfrak{g}^\circ$, and therefore $\mathrm{gr}_1(\mathcal{I}^\lambda(\mathfrak{g}^\circ)) = \mathfrak{b}^\circ$. From these facts and the lemma 5.2.8 we see that $\mathrm{gr}_1(\mathcal{D}_X^\lambda) = \mathfrak{g}^\circ/\mathfrak{b}^\circ = \mathcal{T}_X$. This proves the third condition. To prove (ii) we use the fact that $\mathrm{gr}_i(\mathcal{I}^\lambda(\mathfrak{g}^\circ)) = \mathfrak{b}^\circ \cdot \mathrm{Sym}^i(\mathfrak{g}^\circ) \subseteq \mathrm{Sym}^i(\mathfrak{g}^\circ)$, which implies that

$$\mathrm{gr}_i(\mathcal{U}^\circ/\mathcal{I}^\lambda(\mathfrak{g}^\circ)) \simeq \mathrm{Sym}^i(\mathfrak{g}^\circ/\mathfrak{b}^\circ) \simeq \mathrm{Sym}^i(\mathcal{T}_X).$$

□

5.2.4 Torsors and relative enveloping algebras

In this subsection we will give another approach to define sheaves of differential operators by using the theory of torsors ([51]). We will follow [3], [15] and [60].

\mathbb{T} -torsors 5.2.13. Let us suppose that \widetilde{Y} and Y are smooth separated complex algebraic varieties, such that \widetilde{Y} is endowed with a right \mathbb{T} -action $\sigma : \widetilde{Y} \times \mathbb{T} \rightarrow \widetilde{Y}$. We will also assume that \mathbb{T} acts trivially on Y . For instance, the reader can suppose that $Y = X$ is the complex flag variety.

We say that a morphism $\xi : \widetilde{Y} \rightarrow Y$ is a \mathbb{T} -torsor for the Zariski topology, if ξ is a faithfully flat morphism such that the diagram

$$\begin{array}{ccc} \widetilde{Y} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T} & \xrightarrow{\sigma} & \widetilde{Y} \\ \downarrow p_1 & & \downarrow \xi \\ \widetilde{Y} & \xrightarrow{\xi} & Y \end{array}$$

is commutative and the morphism (induced by the previous diagram)

$$\begin{array}{ccc} \widetilde{Y} \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T} & \rightarrow & \widetilde{Y} \times_Y \widetilde{Y} \\ (x, h) & \mapsto & (x, xh) \end{array}$$

is an isomorphism.

Let $U \subseteq Y$ be an affine open subset and $p_1 : U \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T} \rightarrow U$ be the first projection. We say that U **trivializes** the torsor ξ if there is a \mathbb{T} -equivariant isomorphism $\alpha_U : U \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T} \rightarrow \xi^{-1}(U)$, where \mathbb{T} acts on $U \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T}$ by right translations on the second factor, and such that the diagram

$$\begin{array}{ccc} U \times_{\mathrm{Spec}(\mathbb{C})} \mathbb{T} & \xrightarrow{\alpha_U} & \xi^{-1}(U) \\ \searrow p_1 & & \swarrow \xi \\ & U & \end{array}$$

is commutative. In particular, $\xi^{-1}(U)$ is a \mathbb{T} -invariant affine open subset of \widetilde{Y}

Remark 5.2.14. Given that Y is separated, the set \mathcal{S} of affine open subsets U of Y that trivializes the torsor and such that $\mathcal{O}_Y(U)$ is a finitely generated \mathbb{C} -algebra, it is stable under intersections. Moreover, if $U \in \mathcal{S}$ and W is an affine open subscheme of U , then $W \in \mathcal{S}$.

Definition 5.2.15. We say that $\xi : \widetilde{Y} \rightarrow Y$ is *locally trivial for the Zariski topology* if X can be covered by opens in \mathcal{S} .

Lemma 5.2.16. ([60, Lemma 3.1.3]) Let $\xi : \widetilde{Y} \rightarrow Y$ be a locally trivial \mathbb{T} -torsor and let \mathcal{M} be a quasi-coherent $\mathcal{O}_{\widetilde{Y}}$ -module. Then $R^1\xi_*\mathcal{M} = 0$.

Proof. We recall for the reader that $R^1\xi_*\mathcal{M}$ is the sheaf associated to the presheaf ([29, Chapter III, proposition 8.1])

$$U \subseteq Y \mapsto H^1(\xi^{-1}(U), \mathcal{M}).$$

Given that ξ is locally trivial, the set \mathcal{S} of affine open subsets of Y that trivializes the torsor is a base for the Zariski topology of Y . Moreover, if $U \in \mathcal{S}$ then by definition $\xi^{-1}(U)$ is an affine open subset of X and given that \mathcal{M} is quasi-coherent, by Serre's vanish theorem we can conclude that $H^1(\xi^{-1}(U), \mathcal{M}) = 0$. \square

Sheaves of \mathbb{T} -invariant sections 5.2.17. Let $\Psi : \sigma^*\mathcal{M} \rightarrow p_1^*\mathcal{M}$ be a \mathbb{T} -equivariant quasi-coherent $\mathcal{O}_{\widetilde{Y}}$ -module. By the Künneth formula [27, Theorem 6.7.8] we have a canonical isomorphism

$$\Gamma(\widetilde{Y} \times_{\text{Spec}(\mathbb{C})} \mathbb{T}, p_1^*\mathcal{M}) = \Gamma(\widetilde{Y}, \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}]$$

which composing with the application

$$\Gamma(\widetilde{Y}, \mathcal{M}) \rightarrow \Gamma(\widetilde{Y} \times_{\text{Spec}(\mathbb{C})} \mathbb{T}, \sigma^*\mathcal{M}) \xrightarrow{\Gamma(\Psi)} \Gamma(\widetilde{Y} \times_{\text{Spec}(\mathbb{C})} \mathbb{T}, p_1^*\mathcal{M})$$

(the first morphism is induced via the canonical morphism $\mathcal{M} \rightarrow \sigma_*\sigma^*\mathcal{M}$) gives us a morphism

$$\vartheta : \Gamma(\widetilde{Y}, \mathcal{M}) \rightarrow \Gamma(\widetilde{Y}, \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}]$$

defining a structure of \mathbb{T} -module on $\Gamma(\widetilde{Y}, \mathcal{M})$ [52, Chapter 0, definition 1.6].

Definition 5.2.18. The *\mathbb{T} -invariant elements* of $\Gamma(\widetilde{Y}, \mathcal{M})$ are the elements $P \in \Gamma(\widetilde{Y}, \mathcal{M})$ such that $\vartheta(P) = P \otimes 1$. This subspace will be denoted by $\Gamma(\widetilde{Y}, \mathcal{M})^{\mathbb{T}}$.

Now, let us suppose that $\widetilde{\mathcal{S}}$ is a basis for the Zariski topology of \widetilde{Y} , consisting of affine open subsets which are invariant under the right \mathbb{T} -action. This means that for every $\widetilde{U} \in \widetilde{\mathcal{S}}$ the morphism σ , inducing the right \mathbb{T} -action on \widetilde{Y} , gives rise to a right \mathbb{T} -action $\sigma_{\widetilde{U}} : \widetilde{U} \times_{\mathbb{C}} \mathbb{T} \rightarrow \widetilde{U}$ on \widetilde{U} . By pulling back Ψ under the inclusion $\widetilde{U} \times_{\mathbb{C}} \mathbb{T} \hookrightarrow \widetilde{Y} \times_{\mathbb{C}} \mathbb{T}$ we get an isomorphism $\Psi_{\widetilde{U}} : \sigma_{\widetilde{U}}^*\mathcal{M}_{\widetilde{U}} \rightarrow p_1^*\mathcal{M}_{\widetilde{U}}$ which satisfies the respective cocycle condition ([32, (9.10.10)]), and, as before, we obtain a comodule morphism

$$\vartheta_{\widetilde{U}} : \Gamma(\widetilde{U}, \mathcal{M}) \rightarrow \Gamma(\widetilde{U}, \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}].$$

As in definition 5.2.18, we can define the vector subspace of **\mathbb{T} -invariant sections** on \widetilde{U} by

$$\Gamma(\widetilde{U}, \mathcal{M})^{\mathbb{T}} := \{P \in \Gamma(\widetilde{U}, \mathcal{M}) \mid \Delta_{\widetilde{U}}(P) = P \otimes 1\}. \quad (102)$$

Now, if $\tilde{V}, \tilde{U} \in \tilde{\mathcal{S}}$ satisfy $\tilde{V} \subseteq \tilde{U}$, then by functoriality we have a commutative diagram

$$\begin{array}{ccc} \Gamma(\tilde{U}, \mathcal{M}) & \xrightarrow{\vartheta_{\tilde{U}}} & \Gamma(\tilde{U}, \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}] \\ \downarrow \text{rest}_{\tilde{V}}^{\tilde{U}} & & \downarrow \text{rest}_{\tilde{V}}^{\tilde{U}} \otimes id \\ \Gamma(\tilde{V}, \mathcal{M}) & \xrightarrow{\vartheta_{\tilde{V}}} & \Gamma(\tilde{V}, \mathcal{M}) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}] \end{array}$$

and therefore the restriction map $\text{rest}_{\tilde{V}}^{\tilde{U}} : \Gamma(\tilde{U}, \mathcal{M})^{\mathbb{T}} \rightarrow \Gamma(\tilde{V}, \mathcal{M})^{\mathbb{T}}$ is well-defined. In order to prove that the preceding tools define an $\tilde{\mathcal{S}}$ -sheaf we need to verify the glueing condition, but this is clear because \mathcal{M} is already a sheaf and the restriction maps are \mathbb{T} -equivariant. This construction induces a sheaf $(\mathcal{M})^{\mathbb{T}}$ over \tilde{Y} . We sum up the preceding construction in the next definition.

Definition 5.2.19. *Let \tilde{Y} be a smooth separated complex algebraic variety endowed with a right \mathbb{T} -action, and $\tilde{\mathcal{S}}$ a basis for the Zariski topology stable under the right \mathbb{T} -action. For every \mathbb{T} -equivariant $\mathcal{O}_{\tilde{Y}}$ -module \mathcal{M} , the subsheaf $(\mathcal{M})^{\mathbb{T}}$ is called the **subsheaf of \mathbb{T} -invariant sections** of \mathcal{M} .*

As an application of the preceding construction let us point out that if $\xi : \tilde{Y} \rightarrow Y$ is a locally trivial \mathbb{T} -torsor, then we actually dispose of a subsheaf of \mathbb{T} -invariant sections of the direct image sheaf $\xi_*\mathcal{M}$, with \mathcal{M} a \mathbb{T} -equivariant $\mathcal{O}_{\tilde{Y}}$ -module. In fact, if \mathcal{S} denotes the collection of all affine open subsets that trivialises the torsor ξ , then for every $U \in \mathcal{S}$ we know that $\xi^{-1}(U)$ is stable under the right \mathbb{T} -action and, as in (102), we can define

$$\Gamma(U, \xi_*\mathcal{M})^{\mathbb{T}} := \{P \in \Gamma(U, \xi_*\mathcal{M}) \mid \vartheta_U(P) = P \otimes 1\}.$$

As before, this process defines an \mathcal{S} -sheaf and therefore we get a subsheaf of **\mathbb{T} -invariant sections**

$$(\xi_*\mathcal{M})^{\mathbb{T}} \subseteq \xi_*\mathcal{M}. \quad (103)$$

For the rest of this subsection we will always suppose that $\xi : \tilde{Y} \rightarrow Y$ is a locally trivial \mathbb{T} -torsor.

Lemma 5.2.20. *([3, Lemma 4.3]) The morphism $\xi : \tilde{Y} \rightarrow Y$ induces an isomorphism $\xi^{\natural} : \mathcal{O}_Y \rightarrow (\xi_*\mathcal{O}_{\tilde{Y}})^{\mathbb{T}}$.*

Proof. As this is a local problem, we can take $U \in \mathcal{S}$ and suppose that $\xi : \xi^{-1}(U) = U \times_{\text{Spec}(\mathbb{C})} \mathbb{T} \rightarrow U$ is the first projection. Since rational cohomology commutes with direct limits [39, Part I, Lemma 4.17] and $\mathcal{O}_Y(U)$, we can conclude that $(\xi_*\mathcal{O}_{\tilde{Y}})^{\mathbb{T}}(U) = (\mathcal{O}_X(U) \otimes_{\mathbb{C}} \mathbb{C}[\mathbb{T}])^{\mathbb{T}} = \mathcal{O}_X(U)$. \square

Lemma 5.2.21. *Let \mathcal{M} be a \mathbb{T} -equivariant quasi-coherent $\mathcal{O}_{\tilde{Y}}$ -module, then $(\xi_*\mathcal{M})^{\mathbb{T}}$ is a quasi-coherent \mathcal{O}_Y -module.*

Proof. By definition and the preceding lemma, we only need to show that the \mathbb{T} -action respects the $\xi_*\mathcal{O}_{\tilde{Y}}$ -structure of $\xi_*\mathcal{M}$. This can be proved locally over an affine open subset $U \in \mathcal{S}$. In this case, we know that $\xi^{-1}(U)$ is endowed with a \mathbb{T} -action and therefore $\Gamma(U, \xi_*\mathcal{M})^{\mathbb{T}}$ is a $\Gamma(U, \xi_*\mathcal{O}_{\tilde{Y}})$ -module by [80, 1.4]. \square

It is well known that the tangent sheaf $\mathbb{T}_{\mathbb{T}}$ is a \mathbb{T} -equivariant quasi-coherent $\mathcal{O}_{\tilde{Y}}$ -module, hence we can consider the subsheaf

$$\tilde{\mathcal{T}} := (\xi_* \mathcal{T}_{\tilde{Y}})^{\mathbb{T}}$$

of $\xi_* \mathcal{T}_{\tilde{Y}}$. Moreover, the \mathbb{T} -equivariant structure of $\mathcal{O}_{\tilde{Y}}$ and $\mathcal{T}_{\tilde{Y}}$ induce a \mathbb{T} -equivariant structure on $F_d \mathcal{D}_{\tilde{Y}}$, such that $\mathcal{D}_{\tilde{Y}}$ becomes a sheaf of \mathbb{C} -algebras endowed with a \mathbb{T} -invariant structure.

Definition 5.2.22. ([16, Page 18]) *Let $\xi : \tilde{Y} \rightarrow Y$ be a locally trivial \mathbb{T} -torsor. We define the **relative enveloping algebra** of the torsor to be the sheaf of \mathbb{T} -invariant of $\xi_* \mathcal{D}_{\tilde{Y}}$ by*

$$\tilde{\mathcal{D}} := (\xi_* \mathcal{D}_{\tilde{Y}})^{\mathbb{T}}.$$

This sheaf is equipped with a natural filtration

$$F_d \tilde{\mathcal{D}} := (\xi_* F_d \mathcal{D}_{\tilde{Y}})^{\mathbb{T}}$$

defined via the order filtration on $\mathcal{D}_{\tilde{Y}}$.

In the proof of the next proposition we will need the following facts. The reader can find very explicit proofs in [3, Lemma 4.4 and 4.5].

Remark 5.2.23. (i) There exists a canonical isomorphism

$$\mathcal{U}(\mathfrak{t}) \rightarrow \Gamma(\mathbb{T}, \mathcal{D}_{\mathbb{T}})^{\mathbb{T}}.$$

(ii) Differentiating the \mathbb{T} -action on \tilde{Y} we obtain a Lie homomorphism

$$j : \mathfrak{t} \rightarrow \mathcal{T}_{\tilde{Y}}.$$

Furthermore, if ξ is locally trivial and $\theta \in \tilde{\mathcal{T}}(U)$ for some affine open subset $U \subseteq Y$, then θ is a \mathbb{T} -invariant vector field on $\xi^{-1}(U)$. In particular, it is a \mathbb{T} -linear endomorphism of $\mathcal{O}_{\tilde{Y}}(\xi^{-1}(U))$. In other words, θ preserves $\mathcal{O}_{\tilde{Y}}(\xi^{-1}(U))^{\mathbb{T}} = \mathcal{O}_Y(U)$ and it induces a vector field $\nu(\theta) \in \mathcal{T}_Y(U)$.

The preceding constructions fit into complex of \mathcal{O}_Y -modules

$$0 \rightarrow \mathfrak{t} \otimes_{\mathbb{C}} \mathcal{O}_Y \xrightarrow{j} \tilde{\mathcal{T}} \xrightarrow{\nu} \mathcal{T}_Y \rightarrow 0 \quad (104)$$

which is functorial in \tilde{T} . Moreover, if $U \in \mathcal{S}$ then the sequence is split exact, which implies that this is exact and $\tilde{\mathcal{T}}$ is locally free.

The next section will give us a precise description of the local behaviour of the sheaf $\tilde{\mathcal{D}}$. It will also indicate us the term to be corrected in order to obtain a sheaf of twisted differential operators. We will follow word by word the reasoning given in [3, Proposition 4.6].

Proposition 5.2.24. *For any $U \in \mathcal{S}$, there exists an isomorphism of sheaves of filtered \mathbb{C} -algebras*

$$\alpha(U) : \tilde{\mathcal{D}}|_U \xrightarrow{\cong} \mathcal{D}|_U \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t}).$$

If ξ is locally trivial, then there exists a canonical isomorphism

$$\text{Sym}(\tilde{\mathcal{T}}) = \text{gr}_{\bullet}(\tilde{\mathcal{D}}). \quad (105)$$

Proof. Let $U \in \mathcal{S}$. We have the following isomorphisms of filtered \mathbb{C} -algebras

$$\mathcal{D}_Y(U) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t}) \simeq (\mathcal{D}_Y(U) \otimes_{\mathbb{C}} \mathcal{D}_{\mathbb{T}}(\mathbb{T}))^{\mathbb{T}} \simeq \mathcal{D}_{U \times \mathbb{T}}(U \times \mathbb{T})^{\mathbb{T}} \simeq \widetilde{\mathcal{D}}(U).$$

This isomorphisms are clearly compatible with restriction to Zariski open subsets contained in V , and we get an isomorphism of sheaves of filtered \mathbb{C} -algebras

$$\mathcal{D}_U \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t}) \simeq \widetilde{\mathcal{D}}|_U.$$

On the other hand, the natural inclusion $\mathcal{T}_{\widetilde{Y}} \rightarrow F_1 \mathcal{D}_{\widetilde{Y}}$ induces a morphism of \mathcal{O}_Y -modules $\widetilde{\mathcal{T}} \rightarrow F_1 \widetilde{\mathcal{D}}/F_0 \widetilde{\mathcal{D}}$, which induces by universal property a morphism of graded \mathcal{O}_Y -algebras

$$\widetilde{\varphi} : \text{Sym}(\widetilde{\mathcal{T}}) \rightarrow \text{gr}_{\bullet}(\widetilde{\mathcal{D}}).$$

Furthermore, according to the preceding remark, we have an isomorphism of \mathcal{O}_U -modules

$$\zeta : \widetilde{\mathcal{T}} \xrightarrow{\simeq} \mathcal{T}_U \oplus \mathcal{O}_U \otimes_{\mathbb{C}} \mathfrak{t}.$$

All in all, we can consider the following commutative diagram

$$\begin{array}{ccc} \text{Sym}(\widetilde{\mathcal{T}}(U)) & \xrightarrow{\widetilde{\varphi}(U)} & \text{gr}_{\bullet}(\widetilde{\mathcal{D}}(U)) \\ \text{Sym}(\zeta(U)) \uparrow & & \text{gr}_{\bullet}(\alpha(U)) \uparrow \\ \text{Sym}(\mathcal{T}_U(U) \oplus \mathcal{O}_U(U) \otimes_{\mathbb{C}} \mathfrak{t}) & & \text{gr}_{\bullet}(\mathcal{D}_U(U) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t})) \\ \uparrow & & \uparrow \\ \text{Sym}(\mathcal{T}_U(U)) \otimes_{\mathbb{C}} \mathbb{S}(\mathfrak{t}) & \longrightarrow & \text{gr}_{\bullet}(\mathcal{D}_U(U)) \otimes \text{gr}_{\bullet}(\mathcal{U}(\mathfrak{t})), \end{array}$$

when we are possible shrinking U to an open affine subset endowed with local coordinates given the isomorphism at the bottom. This proves that the map at the top is an isomorphism, and therefore if ξ is locally trivial we can conclude that (105) is an isomorphism as well. \square

Twisted differential operators on homogeneous spaces 5.2.25. In this section we will apply the preceding construction to the following homogeneous spaces. Let \mathbf{N} be the unipotent radical of the Borel subgroup $\mathbb{B} \subseteq \mathbb{G}$. We recall for the reader that \mathbb{T} normalizes \mathbf{N} and we have a canonical isomorphism $\mathbb{T} = \mathbb{B}/\mathbf{N}$. With this notation, we can consider the homogeneous space $\widetilde{X} := \mathbb{G}/\mathbf{N}$, which is endowed with a right \mathbb{T} -action

$$g\mathbf{N} \cdot b\mathbf{N} = gb\mathbf{N}$$

$g \in \mathbb{G}$ and $b \in \mathbf{N}$. The space \widetilde{X} is smooth separated complex algebraic variety known as the **basic affine scheme**. Moreover, the canonical projection

$$\xi : \widetilde{X} \rightarrow X$$

is a locally trivial \mathbb{T} -torsor [3, Lemma 4.7].

Remark 5.2.26. (i) Given that \widetilde{X} is endowed with a natural left \mathbb{G} -action, we have a \mathbb{C} -linear Lie homomorphism

$$\widetilde{\tau} : \mathfrak{g} \rightarrow \mathcal{T}_{\widetilde{X}}.$$

Moreover, given that the \mathbb{G} -action clearly commutes with the \mathbb{T} -action, the preceding maps descends to a \mathbb{C} -linear Lie homomorphism

$$\tilde{\tau} : \mathcal{O}_X \otimes_{\mathbb{C}} \mathfrak{g} \rightarrow \tilde{\mathcal{T}}$$

of locally free sheaves on X .

(ii) ([3, Proposition 4.8 (ii)]) Since \tilde{X} is a homogeneous space and the geometric fibers of $\tilde{\tau}$ are surjective (proposition 3.5.11), then the morphism $\tilde{\tau}$ is surjective (this is the same reasoning given in the lemma 5.2.8).

(iii) ([3, Lemma 4.9]) The map $\tilde{\tau}$ extends to a filtered \mathbb{C} -algebra morphism

$$\tilde{\Psi}^\circ : \mathcal{U}^\circ \rightarrow \tilde{\mathcal{D}}$$

keeping the \mathfrak{g} -action on \tilde{X} by \mathbb{T} -invariant vector fields. Moreover, by (105) we know that $\text{gr}_\bullet(\tilde{\mathcal{D}}) = \text{Sym}(\tilde{\mathcal{T}})$, and the restriction of $\text{gr}(\tilde{\Psi}^\circ)$ to \mathfrak{g} equals the morphism $\tilde{\tau}$ by construction. This implies that $\text{gr}(\tilde{\Psi}^\circ) = \text{Sym}(\tilde{\tau})$ and therefore $\tilde{\Psi}^\circ$ is surjective.

Now, let us recall that we have a Lie algebra morphism $j : \mathfrak{t} \rightarrow \tilde{\mathcal{T}}$, coming from the right \mathbb{T} -action on \tilde{X} . This maps extends to a \mathbb{C} -algebra morphism $\mathcal{U}(\mathfrak{h}) \rightarrow \tilde{\mathcal{D}}$, which factors through the center of $\tilde{\mathcal{D}}$ because \mathbb{T} is commutative.

Definition 5.2.27. *Let $\lambda \in \mathfrak{t}^\vee$ be an arbitrary character of \mathfrak{t} , and let us denote by \mathbb{C}_λ the one-dimensional $\mathcal{U}(\mathfrak{t})$ -module defined by λ . We define a sheaf $\mathcal{D}_{X,\lambda}$ of filtered \mathbb{C} -algebras by*

$$\mathcal{D}_{X,\lambda} := \tilde{\mathcal{D}} \otimes_{\mathcal{U}(\mathfrak{t})} \mathbb{C}_\lambda.$$

Remark 5.2.28. If we endow \mathbb{C}_λ with the trivial filtration $F_{-i}\mathbb{C}_\lambda = F_0\mathbb{C}_\lambda = 0$ and $F_i\mathbb{C}_\lambda = \mathbb{C}_\lambda$, for every $i \in \mathbb{N}$, then \mathbb{C}_λ becomes a filtered $\mathcal{U}(\mathfrak{t})$ -module and we may endow $\mathcal{D}_{X,\lambda}$ with a structure of sheaf of filtered \mathbb{C} -algebras via the tensor filtration ([60, Page 20]).

In the proof of the next proposition we will follow word by word the lines of reasoning given in [3, Lemma 6.4].

Proposition 5.2.29. (i) *Let $U \in \mathcal{S}$ be an affine open subset that trivializes the torsor $\xi : \tilde{X} \rightarrow X$. Then $\mathcal{D}_{X,\lambda}$ is isomorphic to \mathcal{D}_U as sheaves of filtered \mathbb{C} -algebras.*

(ii) *There exists a canonical isomorphism of graded sheaves $\text{gr}_\bullet(\mathcal{D}_{X,\lambda}) = \text{Sym}(\mathcal{T}_X)$.*

Proof. (i) By proposition 5.2.24, we have an isomorphism of filtered sheaves

$$\tilde{\mathcal{D}}|_U \simeq \mathcal{D}_U \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t}).$$

This isomorphism induces by construction, an isomorphism of sheaves of filtered \mathbb{C} -algebras $\mathcal{D}_{X,\lambda}|_U \simeq \mathcal{D}_U$.

(ii) By [60, page 20], there exists a canonical morphism of sheaves

$$\text{gr}_\bullet(\tilde{\mathcal{D}}) \otimes_{\text{gr}_\bullet(\mathcal{U}(\mathfrak{t}))} \text{gr}_\bullet(\mathbb{C}_\lambda) \rightarrow \text{gr}_\bullet(\mathcal{D}_{X,\lambda}).$$

Now, by (105) we know that $\mathrm{gr}_\bullet(\widetilde{\mathcal{D}}) = \mathrm{Sym}(\widetilde{\mathcal{T}})$. Moreover, it is known that $\mathrm{gr}_\bullet(\mathcal{U}(\mathfrak{t})) = \mathrm{Sym}(\mathfrak{t})$, and it is clear that $\mathrm{gr}_\bullet(\mathbb{C}_\lambda) = \mathbb{C}_\lambda$. From this information, we obtain a morphism of graded \mathbb{C} -algebras

$$\mathrm{Sym}(\mathcal{T}_X) \xrightarrow{\mathrm{Sym}(\nu)} \mathrm{Sym}(\widetilde{\mathcal{T}}) \otimes_{\mathrm{Sym}(\mathfrak{t})} \mathbb{C}_\lambda \rightarrow \mathrm{gr}_\bullet(\mathcal{D}_{X,\lambda})$$

where the first isomorphism comes from the short exact sequence (104). This composition is seen to be an isomorphism over any $U \in \mathcal{S}$ by the first part of the proposition. \square

Remark 5.2.30. (i) By construction, the sheaves \mathcal{D}_X^λ defined in 5.2.3 are endowed with a \mathbb{G} -equivariant structure which respects the multiplicative structure. Moreover, the map $\Phi_\lambda : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{D}_X^\lambda$ is a morphism of \mathbb{G} -modules and \mathfrak{g} acts on \mathcal{D}_X^λ via $T \rightarrow [\Phi_\lambda^\circ(\bullet), T]$.

(i') We have analogue statements for the sheaves $\mathcal{D}_{X,\lambda}$ introduced in this section, by considering the canonical map

$$\Psi_\lambda^\circ : \mathcal{U}^\circ \xrightarrow{\widetilde{\Psi}^\circ} \widetilde{\mathcal{D}} \rightarrow \mathcal{D}_{X,\lambda}$$

where the second map $\widetilde{\mathcal{D}} \rightarrow \mathcal{D}_{X,\lambda}$ is the canonical projection.

We end this subsection by relating the preceding constructions. The reader can find the proof of the preceding proposition in [50, Theorem 2.4].

Proposition 5.2.31. *The map $\Psi_\lambda^\circ : \mathcal{U}^\circ \rightarrow \mathcal{D}_{X,\lambda}$ defines a canonical isomorphism*

$$\mathcal{D}_X^\lambda \rightarrow \mathcal{D}_{X,\lambda+\rho}$$

of sheaves of filtered \mathbb{C} -algebras.

Remark 5.2.32. (i) The preceding proposition together with 5.2.29 ends the proof of 5.2.12.

(ii) If $\lambda \in \mathrm{Hom}_{\mathrm{gps}}(\mathbb{T}, \mathbb{G}_m)$ is an algebraic character, then

$$\mathcal{D}_X^\lambda = \mathcal{D}_X^{\mathcal{L}(\lambda+\rho)}$$

and we have $\mathrm{Hom}_{\mathrm{gps}}(\mathbb{T}, \mathbb{G}_m) \subseteq \mathfrak{t}^\vee \subseteq H^1(X, \mathcal{Z}_X^1)$, where the first inclusion is defined by differentiation.

(iii) $\mathcal{D}_X = \mathcal{D}_X^{-\rho}$.

5.2.5 Global sections

The goal of this subsection is to compute the global sections of the sheaves \mathcal{D}_X^λ . To do that, we start by analysing the following situations. [First of all, from the Poincaré–Birkhoff–Witt theorem, we have constructed a projection](#)

$$\pi : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$$

which is in fact a morphism of \mathbb{C} -algebras. This morphism can be reinterpreted as a map

$$\begin{aligned} \mathfrak{m}_\bullet^{\mathfrak{b}} : \mathfrak{t}^* &\rightarrow \text{MaxSpec}(\mathfrak{z}) \\ \lambda &\mapsto \mathfrak{m}_\lambda^{\mathfrak{b}} := \ker(\chi_\lambda^{\mathfrak{b}} : \mathfrak{z} \xrightarrow{\pi_{\mathfrak{b}}} \mathcal{U}(\mathfrak{h}) \xrightarrow{\lambda} \mathbb{C}) \end{aligned}$$

where $\mathfrak{z} \subseteq \mathcal{U}(\mathfrak{g})$ is the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and $\mathfrak{t}^* = \text{MaxSpec}(\text{Sym}(\mathfrak{t}))$, after having identified $\mathcal{U}(\mathfrak{t}) = \text{Sym}(\mathfrak{t}) = \mathbb{C}[\mathfrak{t}]$. We have the following important fact (Reference from the section BGG):

- For any arbitrary character $\lambda \in \mathfrak{t}^\vee$, the center \mathfrak{z} acts via $\chi_\lambda^{\mathfrak{b}}$ on the Verma module

$$M_\lambda^{\mathfrak{b}} := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} \mathbb{C}_\lambda.$$

We insist in remarking for the reader that $\pi_{\mathfrak{b}} : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{t})$ is a relative version of the **Harish-Chandra** isomorphism, because $\pi_{\mathfrak{b}}$ depends of the Borel subalgebra \mathfrak{b} and the Cartan subalgebra \mathfrak{t} . This construction has the problem of not being able to consider the Verma modules

$$M_\lambda^{\mathfrak{b}_x} := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_x)} \mathbb{C}_\lambda$$

relatives to arbitrary Borel subalgebras $\mathfrak{b}_x \subseteq \mathfrak{g}$ (here $x \in X$ is the respective Borel subgroup). As the reader may notice, this restriction is a disadvantage compared to the definition of the sheaf of λ -twisted differential operators given in 5.2.11, where we have encoded the action of the varying Borel subalgebras \mathfrak{b}_x via a character λ° (given in (99)). To fix this problem, we have shifted this projection by using the Weyl character ρ , obtaining an isomorphism

$$\begin{aligned} \mathfrak{z} &\xrightarrow{\cong} \text{Sym}(\mathfrak{t})^W \\ z &\mapsto \pi_{\mathfrak{b}}(z - \rho(z)) \end{aligned}$$

which is independent of $\mathfrak{t} \subseteq \mathfrak{b}$. In particular, applying this shift to the previous fact, we have the following fundamental remarks:

- Let $\lambda \in \mathfrak{t}$ be an arbitrary character. For any Borel subalgebra $\mathfrak{b}_x \subseteq \mathfrak{g}$ we can consider ρ_x the corresponding longest positive root. Then the center \mathfrak{z} acts on the Verma module (of highest weigh λ and relative to \mathfrak{b}_x)

$$M_\lambda^{\mathfrak{b}_x} := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}_x)} \mathbb{C}_\lambda$$

via $\chi_{\lambda+\rho_x}$ (defined in (98)).

- The action of the center \mathfrak{z} on the Verma module $M_{\lambda-\rho_x}^{\mathfrak{b}_x}$ is given by χ_λ (in (98)) and therefore it is independent of the choice of \mathfrak{b}_x .
- The center \mathfrak{z} acts on the right $\mathcal{U}(\mathfrak{g})$ -module

$${}_\lambda M^{\mathfrak{b}_x} := \mathbb{C}_\lambda \otimes_{\mathcal{U}(\mathfrak{b}_x)} \mathcal{U}(\mathfrak{g})$$

via the character $\chi_{\lambda-\rho_x}$

We can now start the computation of $\Gamma(X, \mathcal{D}_X^\lambda)$. We recall that in (101), we have defined a canonical morphism of filtered \mathbb{C} -algebras

$$\Phi_\lambda : \mathcal{U}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^\lambda).$$

Lemma 5.2.33. *For any character $\lambda \in \mathfrak{t}^\vee$, the map $\Phi_\lambda : \mathcal{U}(\mathfrak{g}) \rightarrow \Gamma(X, \mathcal{D}_X^\lambda)$ factors through $\mathcal{U}(\mathfrak{g})_{\chi_\lambda}$.*

Proof. Let $\mathcal{J}^\lambda(\mathfrak{g}^\circ) \subseteq \mathcal{U}(\mathfrak{g}^\circ)$ be the ideal generated by $z - \chi_\lambda(z)$ for $z \in \mathfrak{z}$. We need to prove that the composition

$$\mathcal{J}^\lambda(\mathfrak{g}^\circ) \hookrightarrow \mathcal{U}(\mathfrak{g}^\circ) \xrightarrow{\Phi_\lambda^\circ} \mathcal{D}_X^\lambda$$

is zero. By \mathbb{G} -equivariance, it is enough to check this on the fibers. For $x \in X$, we have

$$\mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X^\lambda \simeq \mathcal{U}(\mathfrak{g}) / (\mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{I}^\lambda(\mathfrak{g}^\circ)).$$

By definition, the fiber $\mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{I}^\lambda(\mathfrak{g}^\circ)$ is the right ideal of $\mathcal{U}(\mathfrak{g})$ generated by $\eta - (\lambda + \rho_x)(\eta)$ for $\eta \in \mathfrak{b}_x$. In other words, we have

$$\mathbb{C}_x \otimes_{\mathcal{O}_X} \mathcal{D}_X^\lambda \simeq \mathbb{C}_{\lambda + \rho_x} \otimes_{\mathcal{U}(\mathfrak{b}_x)} \mathcal{U}(\mathfrak{g})$$

is a right Verma module. The preceding facts imply that \mathfrak{z} acts via the character $\chi_{\lambda + \rho_x - \rho_x} = \chi_\lambda$. In particular $\mathcal{J}^\lambda(\mathfrak{g}^\circ)_x$ is annihilated on each fiber, as needed. \square

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Proposition 5.2.35. *The ideal in $S(\mathfrak{g})$ defining \mathcal{N} is generated by*

$$S(\mathfrak{g})_+^{\mathbb{G}} = S(\mathfrak{g})^{\mathbb{G}} \cap (\oplus_{p \in \mathbb{Z}_{>0}} S(\mathfrak{g})_p).$$

Now, lifting the elements of $S(\mathfrak{g})_+^{\mathbb{G}} = S(\mathfrak{g})^{\mathbb{G}} \cap (\oplus_{p \in \mathbb{Z}_{>0}} S(\mathfrak{g})_p)$ to $\mathcal{U}(\mathfrak{g})$, we have an epimorphism $\mathbb{C}[\mathcal{N}] \rightarrow \text{gr}(\mathcal{U}(\mathfrak{g})_{\chi_\lambda})$, giving rise to the following commutative diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\quad} & \text{gr}_\bullet \Gamma(X, \mathcal{D}_X^\lambda) \\ & \searrow & \nearrow \\ & \text{gr}_\bullet (\mathcal{U}(\mathfrak{g})_{\chi_\lambda}) & \\ & \uparrow & \\ & \mathbb{C}[\mathcal{N}] & \end{array}$$

We consider the Springer resolution $\gamma : T^*X \rightarrow \mathcal{N}$ and the associated comorphism $\gamma^\# : \mathbb{C}[\mathcal{N}] \rightarrow \Gamma(T^*X, \mathcal{O}_{T^*X})$. Given that γ is a resolution of singularities and \mathcal{N} is normal, then $\gamma^\#$ is actually an isomorphism. Let us consider the commutative diagram

$$\begin{array}{ccc} S(\mathfrak{g}) & \xrightarrow{\quad} & \text{gr}_\bullet (X, \mathcal{D}_X^\lambda) \\ & \searrow & \nearrow \text{gr}(\Phi_\lambda) \\ & \text{gr}_\bullet (\mathcal{U}(\mathfrak{g})_{\chi_\lambda}) & \Gamma(X, \text{gr}_\bullet (\mathcal{D}_X^\lambda)) \\ & \uparrow & \parallel \\ \mathbb{C}[\mathcal{N}] & \xrightarrow{\gamma^\#} & \Gamma \end{array}$$

Since $\text{gr}_\bullet \Gamma(X, \mathcal{D}_X^\lambda) \rightarrow \Gamma(X, \text{gr}_\bullet (\mathcal{D}_X^\lambda))$ is injective and $\gamma^\#$ is surjective, it follows that $\text{gr}(\Phi_\lambda)$ is an isomorphism. From this we can conclude that Φ_λ is an isomorphism.

C Derived categories and derived functors

Our intention in the next two sections is to give a brief reference to the results about derived categories used along this work. The reader can look up the proof of the assertions in [32] or [43].

C.1 The derived category

Definition C.1.1. Let \mathcal{A} be a category and \mathcal{S} be a family morphism of \mathcal{A} . We say that \mathcal{S} is a **multiplicative system** if it satisfies the following properties.

- (i) For every object $X \in \mathbf{Ob}(\mathcal{A})$ the identity morphism $id_X \in Hom_{\mathcal{A}}(X, X)$ is in \mathcal{S} .
- (ii) For every couple (f, g) of morphisms of \mathcal{S} , if the composition $f \circ g$ is well-defined in \mathcal{A} , then $f \circ g \in \mathcal{S}$.
- (iii) For every $X, Y, Z \in \mathbf{Ob}(\mathcal{A})$, $u \in Hom_{\mathcal{A}}(X, Y)$ (resp. $u \in Hom_{\mathcal{A}}(Y, X)$) and $s \in Hom_{\mathcal{A}}(Z, Y) \cap \mathcal{S}$ (resp. $s \in Hom_{\mathcal{A}}(Y, Z) \cap \mathcal{S}$). There exist $W \in \mathbf{ob}(\mathcal{A})$, $v \in Hom_{\mathcal{A}}(W, Z)$ and $t \in Hom_{\mathcal{A}}(W, X) \cap \mathcal{S}$ such that

$$\begin{array}{ccc} W & \xrightarrow{v} & Z \\ \downarrow t & & \downarrow s \\ X & \xrightarrow{u} & Y \end{array}$$

is a commutative diagram (resp. a diagram with the arrows reversed).

- (iv) For every $X, Y \in \mathbf{Ob}(\mathcal{A})$ and $u, v \in Hom_{\mathcal{A}}(X, Y)$ the following conditions are equivalent.
 - (a) There exist $Y' \in \mathbf{Ob}(\mathcal{A})$ and $s \in Hom_{\mathcal{A}}(Y, Y') \cap \mathcal{S}$ such that $s \circ u = s \circ v$.
 - (b) There exist $X' \in \mathbf{Ob}(\mathcal{A})$ and $t \in Hom_{\mathcal{A}}(X', X) \cap \mathcal{S}$ such that $u \circ t = v \circ t$.

Definition C.1.2. Let \mathcal{A} be a category. We say that a morphism of categories $T : \mathcal{A} \rightarrow \mathcal{A}$ is a **translation** of \mathcal{A} if T is an equivalence of categories.

Definition C.1.3. Let \mathcal{A} be a category with a translation T . A sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$$

where $X, Y, Z \in \mathbf{ob}(\mathcal{A})$, $f \in Hom_{\mathcal{A}}(X, Y)$, $g \in Hom_{\mathcal{A}}(Y, Z)$ and $h \in Hom_{\mathcal{A}}(Z, T(X))$, is called a **triangle** in \mathcal{A} . We usually denote a triangle by

$$X \rightarrow Y \rightarrow Z \xrightarrow{+1}.$$

Definition C.1.4. Let \mathcal{A} be an additive category with a translation T . We say that \mathcal{A} is a **triangulated category** if there exists a family \mathfrak{T} of triangles of \mathcal{A} , called **distinguished triangles**, that satisfies the following properties:

- (i) A triangle isomorphic to a distinguished triangle is a distinguished triangle.

- (ii) For all $f \in \text{Hom}_{\mathcal{A}}(X, Y)$ there exist $g \in \text{Hom}_{\mathcal{A}}(Y, Z)$ and $h \in \text{Hom}_{\mathcal{A}}(Z, T(X))$ such that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a distinguished triangle.
- (iii) For all $X \in \mathbf{Ob}(\mathcal{A})$, the triangle $X \rightarrow X \rightarrow 0 \rightarrow 0 = T(0)$ is a distinguished triangle.
- (iv) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a distinguished triangle if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is a distinguished triangle.
- (v) for every couple of distinguished triangles $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$, and for every couple of morphisms $\alpha \in \text{Hom}_{\mathcal{A}}(X, X')$ and $\beta \in \text{Hom}_{\mathcal{A}}(Y, Y')$ such that $f' \circ \alpha = \beta \circ f$, there exists $\gamma \in \text{Hom}_{\mathcal{A}}(Z, Z')$ such that the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow T(\alpha) \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X') \end{array}$$

is commutative.

- (iii) The family \mathfrak{T} must satisfy a complicated property called the octahedral axiom. The reader is invited to take a look at [32, B.3.3] or [43, 10.1.6].

Furthermore, if \mathfrak{T} is a family of distinguished triangles and \mathcal{S} is a multiplicative system, we say that \mathcal{S} is **compatible** with the triangulated structure of \mathcal{A} if the following conditions are satisfied.

- (i) If $s \in \mathcal{S}$ then $T(s) \in \mathcal{S}$.
- (ii) For every morphism of distinguished triangles, this is, for a commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & T(X) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \xi \\ X' & \xrightarrow{f'} & Y' & \xrightarrow{g'} & Z' & \xrightarrow{h'} & T(X'), \end{array}$$

if $\alpha, \gamma \in \mathcal{S}$ then $\gamma \in \mathcal{S}$.

Theorem C.1.5. Let \mathcal{A} be a category and \mathcal{S} be a multiplicative system in \mathcal{A} . There exist a category $\mathcal{A}_{\mathcal{S}}$ and a functor $Q : \mathcal{A} \rightarrow \mathcal{A}_{\mathcal{S}}$ such that:

- (i) For every $s \in \mathcal{S}$, $Q(s)$ is an isomorphism of $\mathcal{A}_{\mathcal{S}}$.
- (ii) If $F : \mathcal{A} \rightarrow \mathcal{A}'$ is a functor that transforms the elements of \mathcal{S} in isomorphisms of \mathcal{A}' , then there exists a unique functor $F_{\mathcal{S}} : \mathcal{A}_{\mathcal{S}} \rightarrow \mathcal{A}'$ such that $F \simeq F_{\mathcal{S}} \circ Q$.

Remark C.1.6. If \mathcal{A} is an additive category and \mathcal{S} is a multiplicative system of \mathcal{A} , then $\mathcal{A}_{\mathcal{S}}$ is an additive category and Q is an additive functor. Moreover, if \mathcal{A} is a triangulated category with translation T and \mathcal{S} is multiplicative system compatible with the triangulated structure of \mathcal{A} , then $\mathcal{A}_{\mathcal{S}}$ is a triangulated category.

Let's suppose that \mathcal{A} is an abelian category. Let $\mathcal{C}(\mathcal{A})$ denote the category of complexes of objects of \mathcal{A} and $\mathcal{K}(\mathcal{A})$ the **homotopy category** defined by

$$(i) \quad \mathbf{Ob}(\mathcal{K}(\mathcal{A})) = \mathbf{Ob}(\mathcal{C}(\mathcal{A})).$$

$$(ii) \quad \mathit{Hom}_{\mathcal{K}(\mathcal{A})}(X^\bullet, Y^\bullet) := \mathit{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet) / \sum_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet). \text{ Where,}$$

$$\sum_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet) = \{f^\bullet \in \mathit{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet) | f \sim 0\}.$$

Definition C.1.7. A *quasi-isomorphism* in $\mathcal{K}(\mathcal{A})$ is a morphism $[f^\bullet]$ in $\mathcal{K}(\mathcal{A})$ such that $H^i([f^\bullet])$ is an isomorphism for every $i \in \mathbb{Z}$.

Remark C.1.8. The collection of quasi-isomorphisms of $\mathcal{K}(\mathcal{A})$, which we will denote by $\mathbf{Qis}_{\mathcal{A}}$, it is a multiplicative system of $\mathcal{K}(\mathcal{A})$.

Definition C.1.9. Let \mathcal{A} be an abelian category. Under the notations of theorem C.1.5, the category $\mathcal{K}(\mathcal{A})_{\mathbf{Qis}_{\mathcal{A}}}$, denoted by $\mathbf{D}(\mathcal{A})$, it is called the *derived category* of \mathcal{A} .

Now, for $X^\bullet, Y^\bullet \in \mathbf{Ob}(\mathcal{C}(\mathcal{A}))$ and $f^\bullet \in \mathit{Hom}_{\mathcal{C}(\mathcal{A})}(X^\bullet, Y^\bullet)$, we can define the **shifted complex** $X^\bullet[k]$, $k \in \mathbb{Z}$, by

$$(i) \quad X^n[k] := X^{n+k},$$

$$(ii) \quad d_{X^\bullet[k]}^n := (-1)^k d_{X^\bullet}^{n+k},$$

and for f^\bullet we can define the **mapping cone** $M_{f^\bullet} \in \mathbf{Ob}(\mathcal{C}(\mathcal{A}))$ by

$$(i) \quad M_{f^\bullet}^n = X^{n+1} \oplus Y^n,$$

$$(ii) \quad d_{M_{f^\bullet}}(x^{n+1}, y^n) = (-d_{X^\bullet}^{n+1}(x^{n+1}), f^{n+1}(x^{n+1}) + d_{Y^\bullet}^n(y^n)).$$

From this considerations, there exists a short exact sequence

$$0 \rightarrow Y^\bullet \xrightarrow{\alpha(f^\bullet)} M_{f^\bullet} \xrightarrow{\beta(f^\bullet)} X^\bullet[1] \rightarrow 0,$$

where $\alpha(f^\bullet)^n(y^n) = (0, y^n)$ and $\beta(f^\bullet)^n(x^{n+1}, y^n) = x^{n+1}$.

Remark C.1.10. The shift functor $(\bullet)[1] : \mathcal{K}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ defines a translation on $\mathcal{K}(\mathcal{A})$. The family of triangles $X^\bullet \rightarrow Y^\bullet \rightarrow Z^\bullet \rightarrow X^\bullet[1]$ for which there exists a morphism $f^\bullet \in \mathit{Hom}_{\mathcal{K}(\mathcal{A})}(X_0^\bullet, Y_0^\bullet)$ such that

$$\begin{array}{ccccccc} X_0^\bullet & \xrightarrow{f^\bullet} & Y_0^\bullet & \xrightarrow{\alpha(f^\bullet)} & M_{f^\bullet} & \xrightarrow{\beta(f^\bullet)} & X_0^\bullet[1] \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ X^\bullet & \rightarrow & Y^\bullet & \rightarrow & Z^\bullet & \rightarrow & X^\bullet[1] \end{array}$$

is a commutative diagram in $\mathcal{K}(\mathcal{A})$, defines a family of distinguished triangles on $\mathcal{K}(\mathcal{A})$ in the sense of definition C.1.4.

Theorem C.1.11. Let \mathcal{A} be an abelian category. The category $\mathbf{D}(\mathcal{A})$ is an additive triangulated category. The distinguished triangles of $\mathbf{D}(\mathcal{A})$ are precisely the image under $Q_{\mathcal{A}}$ of the distinguished triangles in $\mathcal{K}(\mathcal{A})$. The functor $Q_{\mathcal{A}} : \mathcal{K}(\mathcal{A}) \rightarrow \mathbf{D}(\mathcal{A})$ is additive and triangulated, this means that $Q_{\mathcal{A}}$ sends distinguished triangles to distinguished triangles.

Definition C.1.12. For a complex $X^\bullet \in \mathbf{Ob}(\mathcal{C}(\mathcal{A}))$, with \mathcal{A} an abelian category, we define **truncated complexes** by

$$\tau^{\leq k} X^\bullet := [\dots \rightarrow X^{k-1} \rightarrow \ker d_k \rightarrow 0 \rightarrow 0 \rightarrow \dots],$$

$$\tau^{\geq k} X^\bullet := [\dots \rightarrow 0 \rightarrow 0 \rightarrow \operatorname{Im} d_{k+1} \rightarrow X^{k+1} \rightarrow \dots].$$

The reader can easily verify that we have a short exact sequence

$$0 \rightarrow \tau^{\leq k} X^\bullet \rightarrow X^\bullet \rightarrow \tau^{\geq k} X^\bullet \rightarrow 0. \quad (106)$$

Proposition C.1.13. Let \mathcal{A} be an abelian category. Any short exact sequence

$$0 \rightarrow X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \rightarrow 0$$

in $\mathcal{C}(\mathcal{A})$ can be embedded into a distinguished triangle

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \rightarrow X^\bullet[1]$$

in $\mathbf{D}(\mathcal{A})$.

Proof. Let's denote by d the differential morphism of the mapping cone $M_{id_{X^\bullet}}$. The morphism h^\bullet defined by $h^{n+1}(x^{n+1}, x^n) = (x^n, 0)$, satisfies

$$h^{n+2}d(x^{n+1}, x^n) = (x^{n+1} + d^n(x^n), 0), \quad (107)$$

and

$$dh^{n+1}(x^{n+1}, x^n) = d(x^n, 0) = (-d^n(x^n), x^n). \quad (108)$$

Taking (107) and (108) it is clear that the complex $M_{id_{X^\bullet}}$ is quasi-isomorphic to 0. With this and the exact sequence

$$0 \rightarrow M_{id_{X^\bullet}} \xrightarrow{A} M_{f^\bullet} \xrightarrow{B} Z^\bullet \rightarrow 0$$

where

$$A = \begin{pmatrix} id_{X^\bullet} & 0 \\ 0 & f^\bullet \end{pmatrix} \quad \text{and} \quad B = (0, g^\bullet),$$

there exists an isomorphism $\phi : M_{f^\bullet} \simeq Z^\bullet$ in $\mathbf{D}(\mathcal{A})$. Hence there exists a commutative diagram

$$\begin{array}{ccccccc} X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{g^\bullet} & M_{f^\bullet} & \xrightarrow{\beta(f^\bullet)} & X^\bullet[1] \\ \downarrow id_{X^\bullet} & & \downarrow id_{Y^\bullet} & & \downarrow \phi & & \downarrow id_{X^\bullet} \\ X^\bullet & \xrightarrow{f^\bullet} & Y^\bullet & \xrightarrow{g^\bullet} & Z^\bullet & \xrightarrow{\beta(f^\bullet) \circ \phi^{-1}} & X^\bullet[1] \end{array}$$

in $\mathbf{D}(\mathcal{A})$ which shows that $X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{g^\bullet} Z^\bullet \rightarrow X^\bullet[1]$ is a distinguished triangle. \square

C.2 Derived functors

Definition C.2.1. Let \mathcal{A} and \mathcal{B} be abelian categories and $*$ $\in \{+, -, b\}$ (see the preliminaries to the section 2.2). Let $F : \mathcal{K}^*(\mathcal{A}) \rightarrow \mathcal{K}^*(\mathcal{B})$ be a triangulated functor. The **right derived functor** of F is a couple (R^*F, ξ_F) where

$$R^*F : \mathbf{D}^*(\mathcal{A}) \rightarrow \mathbf{D}^*(\mathcal{B})$$

is a ∂ -functor (i.e. compatible with translations and preserving distinguished triangles) and

$$\xi_F : Q_{\mathcal{B}} \circ F \rightarrow R^*F \circ Q_{\mathcal{A}}$$

is a morphism of functors ($Q_{\mathcal{A}}$ and $Q_{\mathcal{B}}$ the **localisation functors**) that satisfies the following **universal property**: for every triangulated functor $G : \mathbf{D}^*(\mathcal{A}) \rightarrow \mathbf{D}^*(\mathcal{B})$ and every morphism of functors $\Phi : Q_{\mathcal{B}} \circ F \rightarrow G \circ Q_{\mathcal{A}}$ there exists a unique morphism of functors $\eta : R^*F \rightarrow G$ such that $\Phi = (\eta \circ Q_{\mathcal{A}}) \circ \xi_F$.

Definition C.2.2. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories. We say that a full additive subcategory \mathcal{J} of \mathcal{A} is **F -injective** if the following conditions are satisfied:

- (i) For any $X \in \mathbf{Ob}(\mathcal{A})$ there exists an object $I \in \mathbf{Ob}(\mathcal{J})$ and an exact sequence $0 \rightarrow X \rightarrow I$.
- (ii) If $0 \rightarrow X' \rightarrow X'' \rightarrow X''' \rightarrow 0$ is an exact sequence in \mathcal{A} and $X', X''' \in \mathbf{Ob}(\mathcal{J})$, then $X'' \in \mathbf{Ob}(\mathcal{J})$.
- (iii) For any exact sequence $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ such that $X', X, X'' \in \mathbf{Ob}(\mathcal{J})$, the sequence $0 \rightarrow F(X') \rightarrow F(X) \rightarrow F(X'') \rightarrow 0$ in \mathcal{B} is also exact.

Similarly, we define **F -projective** categories reversing all arrows in the conditions above.

Remark C.2.3. If $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor as in the preceding definition, then $X^\bullet \rightarrow F(X^\bullet)$ defines a triangulated functor $F : \mathcal{K}^*(\mathcal{A}) \rightarrow \mathcal{K}^*(\mathcal{B})$.

Theorem C.2.4. Let $F : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between abelian categories which is left (resp. right) exact functor and assume that there exists and F -injective (resp. F -projective) subcategory of \mathcal{A} . Then the right (resp. left) derived functors

$$R^+F : \mathbf{D}^+(\mathcal{A}) \rightarrow \mathbf{D}^+(\mathcal{B})$$

of F exists (resp. the left derived functors $L^-F : \mathbf{D}^-(\mathcal{A}) \rightarrow \mathbf{D}^-(\mathcal{B})$ of F exists).

Remark C.2.5. Under the hypothesis of the preceding theorem, the right derived functor is given by the following. By [43, 13.2.1] there exists a quasi-isomorphism $I^\bullet \simeq X^\bullet$ with $I^\bullet \in \mathcal{K}^+(\mathcal{J})$, called an injective resolution of X^\bullet , and the right derived functor is given by $RF(X^\bullet) = F(I^\bullet)$.

6 Non-archimedean functional analysis

In this section we will introduce the necessary notions on non-archimedean functional analysis that we will need in the coming sections.

6.1 The ring of p -adic integers

In this subsection we will introduce the most basic theory about non-archimedean fields. We will follow the arguments and definitions given in [53].

Let us start by noting that a repeated use of Euclidean division gives us for every natural number $f \in \mathbb{N}$ a **p -adic expansion**

$$f = a_0 + a_1p + \cdots + a_2p^n.$$

More explicitly

$$\begin{aligned} f &= a_0 + pf_1 \\ f_1 &= a_1 + pf_2 \\ &\vdots \\ f_{n-1} &= a_{n-1} + pf_n \\ f_n &= a_n. \end{aligned}$$

It is therefore clear that $a_i \in \{0, 1, \dots, p-1\}$ and they also denote the representative of f_i in $\mathbb{Z}/p\mathbb{Z}$. It is easy to see that if we want to consider negative integers then we are forced to allow infinite series

$$\sum_{k=0}^{\infty} a_k p^k := \{s_n := \sum_{k=0}^{\infty} a_k p^k\}_{n \in \mathbb{N}}.$$

We have the following (formal) definition.

Definition 6.1.1. *Let p be a fixed prime number. A **p -adic integer** is a formal infinite series $a_0 + a_1p + \cdots$, where $a_i \in \{0, 1, \dots, p-1\}$. The set of all p -adic integers will be denoted by \mathbb{Z}_p .*

Let us denote now by $\mathbb{Z}_{(p)}$ the localization of \mathbb{Z} at the prime ideal (p) . Let us explain how to define the p -adic expansion for a rational number $f \in \mathbb{Z}_{(p)}$. We recall for the reader that $\mathbb{Z}_{(p)}/p^n\mathbb{Z}_{(p)} = \mathbb{Z}/p^n\mathbb{Z}$. Moreover a routine inductive argument proves that the residue class $a \bmod p^n \in \mathbb{Z}/p^n\mathbb{Z}$ can be uniquely written in the form

$$a \equiv a_0 + a_1p + a_2p^2 + \cdots + a_{n-1}p^{n-1} \bmod p^n$$

where $0 \leq a_i < p$. With this in mind, we see that $f \in \mathbb{Z}_{(p)}$ defines a sequence of residue classes

$$\bar{s}_n = f \bmod p^n \in \mathbb{Z}/p^n\mathbb{Z} \equiv a_0 + a_1p + a_2p^2 + \cdots + a_{n-1}p^{n-1} \bmod p^n,$$

and the sequence of numbers $s_n := a_0 + a_1p + a_2p^2 + \cdots + a_{n-1}p^{n-1}$ defines a p -adic integer

$$\sum_{k=0}^{\infty} a_k p^k \in \mathbb{Z}_p. \quad (109)$$

This is the **p -adic expansion** of f .

Let us now consider formally extend the set of p -adic integers into that of formal series

$$\sum_{k=-m}^{\infty} a_k p^k,$$

where $m \in \mathbb{Z}$ and $0 \leq a_k < p$. We will call such a series a **p -adic number** and we write \mathbb{Q}_p for the set of all these p -adic numbers. Moreover, we can use (109) to define the p -adic expansion of any rational number $f \in \mathbb{Q}$. To do that, let us write $f = \frac{g}{h} p^{-m}$, where $g, h, m \in \mathbb{Z}$ and $(gh, p) = 1$. By (109), we can attach to $\frac{g}{h}$ the p -adic integer

$$a_0 + a_1 p + a_2 p^2 + \cdots$$

and we can associate to f the p -adic number

$$a_0 p^{-m} + a_1 p^{-m+1} + \cdots + a_m + a_{m+1} p + \cdots \in \mathbb{Q}_p \quad (110)$$

which is the p -adic expansion of $f \in \mathbb{Q}$. We thus get a canonical commutative diagram

$$\begin{array}{ccc} \mathbb{Q} & \hookrightarrow & \mathbb{Q}_p \\ \uparrow & & \uparrow \\ \mathbb{Z} & \hookrightarrow & \mathbb{Z}_p. \end{array} \quad (111)$$

Injectivity of the map on the bottom follows from the fact that if $a, b \in \mathbb{Z}$ has the same p -adic expansion, then $a - b$ is divisible by p^n for every n , and therefore $a = b$.

In order to endow \mathbb{Z}_p with an algebraic structure, let us discuss another way to introduce this set. By definition, every p -adic integer $f \in \mathbb{Z}_p$ can be considered as a sequence of residue classes

$$\bar{s}_n = s_n \bmod p^n \in \mathbb{Z}/p^n \mathbb{Z}$$

which are related via the canonical projections

$$\cdots \rightarrow \mathbb{Z}/p^2 \mathbb{Z} \rightarrow \mathbb{Z}/p \mathbb{Z}.$$

In other words, $\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^{n+1} \mathbb{Z}$ and \mathbb{Z}_p is a ring if we consider

$$\varprojlim_{n \in \mathbb{N}} \mathbb{Z}/p^{n+1} \mathbb{Z} \subseteq \prod_{n \in \mathbb{N}} \mathbb{Z}/p^{n+1} \mathbb{Z}.$$

We can now say that \mathbb{Z}_p is the **ring of p -adic integers**. Moreover, by definition, every p -adic number $f \in \mathbb{Q}_p$ can be written $f = p^{-m} g$ with $g \in \mathbb{Z}_p$ and \mathbb{Q}_p becomes the field of fractions of \mathbb{Z}_p . In fact, under the preceding identification, every integer $a \in \mathbb{Z}$ can be regarded as a p -adic integer via the sequence

$$(a \bmod p, a \bmod p^2, \cdots) \in \mathbb{Z}_p.$$

The reinterpretation of the diagram (111) is clear.

6.2 The p -adic valuation

The idea in this section is to introduce the p -adic valuation (or equivalent the p -adic norm). The reader should keep in mind that using this valuation, \mathbb{Q}_p will be constructed from the field \mathbb{Q} in the same fashion as the field of real numbers \mathbb{R} . The key point is to replace the usual Euclidean valuation by the p -adic valuation in which the sequence (110) converges. Let $a = \frac{b}{c} \in \mathbb{Q}$. We can suppose that

$$a = p^m \frac{b'}{c'}, \quad (b'c', p) = 1$$

and we put

$$|a|_p = \frac{1}{p^m}. \quad (112)$$

Roughly speaking the summands of a p -adic series like in (109) form a sequence converging to 0 with respect to $|\cdot|_p$.

Definition 6.2.1. *The exponent m in (112) is called the p -adic valuation of the integer a . It is usually denoted by $v_p(a)$, and we extend it to $\mathbb{Z} \cup \{\infty\}$ by defining $v_p(0) := \infty$.*

The reader may easily check the following properties

- (1) $v_p(a) = \infty$ if and only if $a = 0$,
- (2) $v_p(ab) = v_p(a) + v_p(b)$,
- (3) $v_p(a + b) \geq \min\{v_p(a), v_p(b)\}$.

It is also clear that by (112) we have the following properties

- (4) $|a|_p = 0$ if and only if $a = 0$,
- (5) $|ab|_p = |a|_p |b|_p$,
- (6) $|a + b|_p \leq \max\{|a|_p, |b|_p\}$.

With the last inequality known as the **non-archimedean inequality**.

Now, we will say that a sequence of rational numbers $\{x_n\}_{n \in \mathbb{N}}$ is a **Cauchy sequence** with respect to $|\cdot|_p$, if for every $\epsilon > 0$, there exists a positive integer $N \in \mathbb{Z}_{>0}$, such that for every $n, m \geq N$ we have

$$|x_n - x_m|_p < \epsilon.$$

An important example is given by a series $\sum_{k=0}^{\infty} a_k p^k$ representing a p -adic integer. In other words $0 \leq a_k < p$ for every $k \in \mathbb{N}$. In this case, for every $n > m$, the non-archimedean gives us

$$|x_n - x_m|_p = \left| \sum_{k=m}^n a_k p^k \right|_p \leq \max_{m \leq k \leq n} \{|a_k p^k|_p\} \leq \frac{1}{p^m}.$$

Other examples are given by sequences of rational numbers $\{x_n\}_{n \in \mathbb{N}}$ which are **nullsequences** with respect to $|\cdot|_p$. These are just the sequences such that $|x_n|_p \rightarrow 0$ in the usual sense.

Let us denote by \mathbf{C} the ring of the Cauchy sequences, and by \mathfrak{m}_0 its maximal ideal consisting of the nullsequences. It is clear that we can embed \mathbb{Q} into the residue field $\mathbf{C}/\mathfrak{m}_0$ by associating to every rational number a the residue class of the constant sequence (a, a, \dots) . Furthermore, the p -**adic norm** $|\cdot|_p$ on \mathbb{Q} extends to $\{x_n\}_{n \in \mathbb{N}} \in \mathbf{C}/\mathfrak{m}_0$ by defining

$$|\{x_n\}_{n \in \mathbb{N}}|_p := \lim_{n \rightarrow \infty} |x_n|_p. \quad (113)$$

This extension is clearly well-defined and independent of the class of $\{x_n\}_{n \in \mathbb{N}}$.

As for the field of real numbers one has the following result.

Proposition 6.2.2. *The field $\mathbf{C}/\mathfrak{m}_0$ is complete with respect to the absolute value $|\cdot|_p$ defined in (113).*

Proposition 6.2.3. [53, Chapter II, proposition 2.3] *The set*

$$\mathfrak{o}_p := \{x \in \mathbf{C}/\mathfrak{m}_0 \mid |x|_p \leq 1\}$$

is a subring of $\mathbf{C}/\mathfrak{m}_0$. It is the closure of the ring \mathbb{Z} with respect to inner topology of \mathbb{Q}_p induced by (113).

Proof. The algebraic structure of \mathfrak{o}_p comes from the properties (5)-(6) given in the page 155. Let us prove the topological property. We want to see that

$$\mathfrak{o}_p = \{x \in \mathbf{C}/\mathfrak{m}_0 \mid x = \lim_{n \rightarrow \infty} x_n, x_n \in \mathbb{Z}, \forall n \in \mathbb{N}\} (= \overline{\mathbb{Z}}).$$

First of all, if $\{x_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathbb{Z} and $x = \lim_{n \rightarrow \infty} x_n$, then $|x_n|_p \leq 1$ implies that $|x|_p \leq 1$, and therefore $x \in \mathbb{Z}_p$.

On the other hand, let $x \in \mathfrak{o}_p$ and let us suppose that $x = \lim_{n \rightarrow \infty} x_n$, for a Cauchy sequence in \mathbb{Q} . Let us also take $N \in \mathbb{N}$ such that $|x_n|_p \leq 1$ for every $n \geq N$, i.e., $x_n = \frac{h_n}{g_n}$, with $h_n, g_n \in \mathbb{Z}$ and $(g_n, p) = 1$. Choosing a solution $y_n \in \mathbb{Z}$ of the congruence $b_n y_n \equiv a_n \pmod{p^n}$ we see that $|x_n - y_n|_p \leq \frac{1}{p^n}$ and hence $x = \lim_{n \rightarrow \infty} y_n$, and x belongs to the closure of \mathbb{Z} . \square

in order to relate the previous definitions we will need the following proposition [53, Chapter II, proposition 2.3].

Proposition 6.2.4. *The non-zero ideals of the ring \mathfrak{o}_p are the principal ideals*

$$p^n \mathbb{Z}_p = \{x \in \mathbf{C}/\mathfrak{m}_0 \mid |x|_p \leq \frac{1}{p^n}\}. \quad (114)$$

Moreover,

$$\mathfrak{o}_p / p^n \mathfrak{o}_p \cong \mathbb{Z} / p^n \mathbb{Z}.$$

Proof. We start by remarking for the reader that every element $x \in (\mathbf{C}/\mathfrak{m}_0)^*$ admits a unique representation $x = p^m u$ with $m \in \mathbb{Z}$ and $u \in \mathfrak{o}_p^*$. Let $\mathfrak{a} \neq (0)$ be an ideal of \mathfrak{o}_p and $x = p^m u$ and element of \mathfrak{a} with smallest possible m . It is an easy exercise to see that $\mathfrak{a} = p^m \mathfrak{o}_p$. \square

7 The Orlik-Strauch functor \mathcal{F}_P^G

Throughout this part of these notes L will be a finite extension of \mathbb{Q}_p and K a finite extension of L which will be our coefficient field. We will also assume that \mathbb{G} is a split connected reductive algebraic group over L , and we will take $\mathbb{T} \subseteq \mathbb{B} \subseteq \mathbb{P} \subseteq \mathbb{G}$ a maximal torus contained in a Borel subgroup, which in turns is contained in a parabolic subgroup of \mathbb{G} . We will use capital letters to denote the respective groups of L -points, for instance $G := \mathbb{G}(L)$, $P := \mathbb{P}(L)$, and so on. Moreover, we will denote by gothic letters the Lie algebra of the group concerned, this is $\mathfrak{g} := \text{Lie}(\mathbb{G})$, $\mathfrak{b} := \text{Lie}(\mathbb{B})$ and $\mathfrak{p} := \text{Lie}(\mathbb{P})$. Furthermore, once we fix the Levi decomposition [39, Part II, 1.8]

$$\mathbb{P} := \mathbb{L}_{\mathbb{P}}\mathbb{U}_{\mathbb{P}},$$

where $\mathbb{L}_{\mathbb{P}}$ is the **Levi factor** and $\mathbb{U}_{\mathbb{P}}$ is the **unipotent radical**, we set $\mathfrak{l}_{\mathbb{P}} := \text{Lie}(\mathbb{L}_{\mathbb{P}})$ and $\mathfrak{u}_{\mathbb{P}} := \text{Lie}(\mathbb{U}_{\mathbb{P}})$. Finally, the base change of an L -vector space (or an L -scheme) to K will always be denoted by the subscript K , in other words, $\mathfrak{g}_K := \mathfrak{g} \otimes_L K$ and $\mathbb{G}_K := \mathbb{G} \times_{\text{Spec}(L)} \text{Spec}(K)$.

The first part of this section is dedicated to recall the definition of the functor (cf. [54, Section 4])

$$\mathcal{F}_P^G := \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \times \text{Rep}_K^{\infty, \text{adm}}(L_P) \rightarrow \text{Rep}_K^{\text{la}}(G) \quad (115)$$

which depends of the parabolic subgroup $P \subseteq G$. We recall for the reader that $\text{Rep}_K^{\infty, \text{adm}}(L_P)$ denotes the category of **smooth admissible representations** of the Levi subgroup $L_P \subseteq P$ on K -vector spaces. We also recall that $\text{Rep}_K^{\text{la}}(G)$ is the category of **locally analytic representations** of G on K -vector spaces.

7.1 The algebraic BGG category \mathcal{O}

By $\mathcal{O}^{\mathfrak{b}_K}$ we will consider the following adaptation of the BGG category \mathcal{O} when the coefficient field is not algebraic [54, 2.5]:

- (1) M is a finitely generated $U(\mathfrak{g}_K)$ -module.
- (2) M decomposes as a direct sum of one-dimensional \mathfrak{t}_K -representations.
- (3) The action of \mathfrak{b}_K on M is locally finite in the usual sense (cf. [36, (1.1)]).⁴³

We will also consider $\mathcal{O}^{\mathfrak{p}_K}$ the subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ consisting of those modules M in $\mathcal{O}^{\mathfrak{b}_K}$ on which \mathfrak{p}_K acts locally finitely. In [54] the authors were mainly interested in the following subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ (resp. $\mathcal{O}^{\mathfrak{p}_K}$). First of all, let us note that property (2) tells us that any object $M \in \mathcal{O}^{\mathfrak{b}_K}$ (resp. in $\mathcal{O}^{\mathfrak{p}_K}$) can be written as a direct sum on one-dimensional \mathfrak{t}_K -representations

$$M = \bigoplus_{\lambda \in \mathfrak{t}_K^*} M_{\lambda} \quad (116)$$

where $M_{\lambda} := \{m \in M \mid \forall \mathfrak{t} \in \mathfrak{t}_K, \mathfrak{r} \cdot m = \lambda(\mathfrak{t})m\}$ is the **eigenspace** associated to $\lambda \in \mathfrak{t}_K^* := \text{Hom}_K(\mathfrak{t}_K, K)$. Let $X^*(\mathbb{T}_K) := \text{Hom}_{\text{alg. gps}}(\mathbb{T}_K, \mathbb{G}_m)$ be the **group**

⁴³To see also the notes of Giovanna Carnovale on "The Bernstein-Gelfand-Gelfand category \mathcal{O} ".

of (algebraic) characters of the torus \mathbb{T}_K , which can be considered as a subgroup (lattice) of \mathfrak{t}_K^* via derivation. We have the following fundamental definition ([54, Definition 2.6]).

Definition 7.1.1. *We let $\mathcal{O}_{\text{alg}}^{\mathfrak{b}_K}$ be the full subcategory of $\mathcal{O}^{\mathfrak{b}_K}$ whose objects are $U(\mathfrak{g}_K)$ -modules such that the decomposition (116) is algebraic. In other words, all λ appearing in (116), for which $M_\lambda \neq 0$, are contained in $X^*(\mathbb{T}_K) \subseteq \mathfrak{t}_K^*$.*

Let \mathcal{O}^P be the category whose objects are pairs $\underline{M} := (M, \tau)$, where $M \in \mathcal{O}^P$ and $\tau : P \rightarrow \text{End}_K(M)^*$ is locally analytic locally finite P -representation, e.g, $M = \cup_{i \in \mathbb{N}} M_i$ is an increasing union of finite-dimensional locally analytic P -stable subspaces, such that the derived action of \mathfrak{p}_k lifts the initial \mathfrak{p}_k -action, and such that the actions of P and \mathfrak{g}_k are compatible. The category \mathcal{O}^P is abelian ([55, Lemma 2.5]) and any object is of finite length ([55, Lemma 2.7]). It is clear that we have a forgetful functor

$$\begin{aligned} \omega : \mathcal{O}^P &\rightarrow \mathcal{O}^{\mathfrak{p}_K} \\ \underline{M} &\mapsto M. \end{aligned} \tag{117}$$

On the other hand, we will denote by $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}_K}$ formed by objects M such that in the weight decomposition $M = \bigoplus_{\lambda \in \mathfrak{t}_K^*} M_\lambda$ all occurring λ lie in the lattice of algebraic characters $X^*(\mathbb{T}_K) \subseteq \mathfrak{t}_K^*$. There exists a fully faithful embedding

$$\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \rightarrow \mathcal{O}^P \tag{118}$$

whose composition with the forgetful functor equals the inclusion $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \subseteq \mathcal{O}^{\mathfrak{p}_K}$.⁴⁴ To see this, let us remark first that the algebraic \mathbb{T}_K -action on an object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ lifts uniquely to an algebraic $\mathbb{L}_{\mathbb{P}, K}$ -action on each finite dimensional simple $\mathfrak{p}_{\mathbb{P}, K}$ -constituent of M (see the proof of [54, Lemma 2.8]). Moreover, via the exponential map, the action of the Lie algebra $\mathfrak{u}_{\mathbb{P}, K}$ integrates uniquely to give an algebraic action of $\mathbb{U}_{\mathbb{P}, K}$ on M (see the proof of [54, Lemma 3.2]). Both actions combined give an algebraic \mathbb{P}_K -action on M , and therefore $M \in \mathcal{O}^P$.

Example 7.1.2. (i) ([54, Example 2.7]) Let $\lambda \in \mathfrak{t}_K^*$ and K_λ be the one-dimensional \mathfrak{t}_K -representation whose \mathfrak{t}_K -action is given by λ . This action extends uniquely to a \mathfrak{b}_K -module structure. Let

$$M(\lambda) := U(\mathfrak{g}_K) \otimes_{U(\mathfrak{b}_K)} K_\lambda$$

be the corresponding Verma module, which is an object in $\mathcal{O}^{\mathfrak{b}_K}$. Denoting by $L(\lambda)$ its simple quotient, we have that $M(\lambda)$ and $L(\lambda)$ are objects in $\mathcal{O}^{\mathfrak{b}_K}$ if and only if $\lambda \in X^*(\mathbb{T}_K)$.

(ii) ([54, Example 2.10]) Let \mathcal{S} be the set of simple roots of \mathbb{G}_K with respect to $\mathbb{T}_K \subseteq \mathbb{B}_K$. Let $\lambda \in X^*(\mathbb{T}_K)$ be an algebraic character, and let us consider $I := \{\alpha \in \mathcal{S} \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\}$. Let \mathbb{P}_I be the standard parabolic subgroup of \mathbb{G}_K attached to I . It is known that λ is dominant with respect to the Levi factor $\mathbb{L}_{\mathbb{P}_I}$. Denote by $V_I(\lambda)$ the corresponding irreducible finite-dimensional algebraic $\mathbb{L}_{\mathbb{P}_I}$ -representation ([39, Part II, 2.14] and [46, Page 4]), which we consider as a \mathbb{P}_I -module by letting act $\mathbb{U}_{\mathbb{P}}$ trivially on it.

⁴⁴The following reasoning has been taken from [2, 4.1].

The **generalized parabolic Verma module** (in the sense of [46, Page 4]) attached to the weight λ is defined by

$$M_I(\lambda) := U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_I)} V_I(\lambda).$$

Then $M_I(\lambda)$ is an object of $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Moreover, there exists a surjective map

$$M(\lambda) \rightarrow M_I(\lambda)$$

where the kernel is given by the image of $\bigoplus_{\alpha \in I} M(s_\alpha \cdot \lambda) \rightarrow M(\lambda)$ ([39, Theorem 9.4 (b)]). Given that λ is algebraic, it follows from [39, Theorem 9.4 (a)] and the first part of the example that $L(\lambda)$ is an object in $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$.

7.2 From \mathcal{O}_{alg} to locally analytic representations

The goal of this subsection is to show how to attach, in a natural way, to any object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ a locally analytic representation. This process will define a functor

$$\mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} \rightarrow \text{Rep}_K^{\text{la}}(G),$$

which we will extend naturally to define the bi-functor (115). To start with, we remark for the reader that the defining properties (1) and (3) for $\mathcal{O}^{\mathfrak{p}_K}$ allow us to take a finite-dimensional \mathfrak{p}_K -representation $W \subseteq M$ which generates M as a $U(\mathfrak{g}_K)$ -module. In other words, for any $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$, we have a short exact sequence of $U(\mathfrak{g}_K)$ -modules

$$0 \rightarrow \mathfrak{d} \rightarrow U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M \rightarrow 0, \quad (119)$$

where \mathfrak{d} is the kernel of the canonical map $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M$. Furthermore, the same reasoning given just before the example 7.1.2 shows that W can be endowed with a structure of a locally analytic P -representation, and therefore its K -dual space W' is also a locally analytic P -representation. By [54, Lemma 2.4] the canonical map of $D(G, K)$ -modules

$$\begin{aligned} D(G, K) \otimes_{D(P, K)} W &\rightarrow (\text{Ind}_P^G(W'))' \\ \delta \otimes w &\mapsto [f \mapsto \delta(f)(w)] \end{aligned}$$

is an isomorphism (I think that the definition of the function on the right hand side uses the fact that W' is of compact type and therefore we have the identification $C^{\text{an}}(G, W') = C^{\text{an}}(G, K) \hat{\otimes}_{K, \pi} W'$) and we dispose of a canonical paring

$$\langle \cdot, \cdot \rangle : (D(G, K) \otimes_{D(P, K)} W) \otimes_K \text{Ind}_P^G(W') \rightarrow K \quad (120)$$

which identifies the left hand side with the topological dual of the right hand side and vice versa. We can give a more explicit description of the preceding paring (cf. [54, (3.2.2)]). First of all, let us consider the $C^{\text{an}}(G, K)$ -valued paring

$$\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, K)} : \begin{aligned} (D(G, K) \otimes_{D(P, K)} W) \otimes_K \text{Ind}_P^G(W') &\rightarrow C^{\text{an}}(G, K) \\ (\delta \otimes w) \otimes f &\mapsto [g \mapsto (\delta_r(f(\cdot)(w)))(g)] \end{aligned}$$

where $(\delta_r(f(\cdot)(w)))(g) := \delta(x \mapsto f(gx)(w))$. On the other hand, using the arguments exhibited in [56, 3.4.1], it is possible to prove that the canonical map

$$U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow D(G, K) \otimes_{D(P, K)} W$$

is injective, and we can suppose that $U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W$ is a subspace of $(\text{Ind}_P^G(W'))'$. We then denote by $\text{Ind}_P^G(W')^\mathfrak{d}$ the closed subspace of $\text{Ind}_P^G(W')$ annihilated by \mathfrak{d} via the paring $\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, K)}$:

$$\text{Ind}_P^G(W')^\mathfrak{d} := \{f \in \text{Ind}_P^G(W') \mid \forall \delta \in \mathfrak{d}, \langle \delta, f \rangle_{C^{\text{an}}(G, K)} = 0\}.$$

This is, by construction, a G -invariant subspace of $\text{Ind}_P^G(W')$ (cf. [54, Comment before the proposition 3.3]). Moreover, by [54, Lemma 2.4] the representation $\text{Ind}_P^G(W')$ is strongly admissible (in the sense of [69]), and therefore $\text{Ind}_P^G(W')^\mathfrak{d}$ is also a strongly admissible locally analytic G -representation, being a closed invariant subspace of a strongly admissible representation [69, Lemma 3.5]. Finally, it is not very hard to prove that the annihilator, under the canonical paring defined in (120), of $\text{Ind}_P^G(W')^\mathfrak{d}$ in $D(G, K) \otimes_{D(P, K)} W$ is equal to $D(G, K)\mathfrak{d}$ ([54, Proposition 3.3 (ii)]). Therefore we have a canonical isomorphism of coadmissible $D(G, K)$ -modules⁴⁵

$$(\text{Ind}_P^G(W')^\mathfrak{d})' \simeq (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d} \quad (121)$$

In practice, we are interested in the following description of $(\text{Ind}_P^G(W')^\mathfrak{d})'$, cf. (122) below.

In what follows, we will denote by \mathfrak{o} the ring of integers of the finite extension $L|\mathbb{Q}_p$. We will also take smooth integral models $\mathbb{T}_0 \subseteq \mathbb{B}_0 \subseteq \mathbb{P}_0 \subseteq \mathbb{G}_0$, which are by definition groups \mathfrak{o} -schemes, of $\mathbb{T} \subseteq \mathbb{B} \subseteq \mathbb{P} \subseteq \mathbb{G}$, respectively, and we will denote by $G_0 := \mathbb{G}_0(\mathfrak{o})$, which is a maximal open compact subgroup G .

Let $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. By definition, M is a union of finite dimensional \mathfrak{p}_K -modules. Let us denote by M_0 one of these finite-dimensional submodules. As we have remarked, M_0 lifts to a locally analytic P -representation which extends to a unique $D(P, K)$ -module structure, in the sense that the Dirac distributions acts as group element on P . cf. [69, Proposition 3.2] and the paragraph before the lemma 3.1 in [69].

The algebra $D(\mathfrak{g}_K, P)$: In what follows, we will denote by $D(\mathfrak{g}_K, P)$ the subring⁴⁶ of $D(G, K)$ generated by $U(\mathfrak{g}_K)$ and $D(P, K)$. Actually, this ring is equal to $U(\mathfrak{g}_K)D(P, K)$, which means that every element in $D(\mathfrak{g}_K, P)$ can be written as a finite sum $\sum_j \mathfrak{z}_j \delta_j$ with $\mathfrak{z}_j \in U(\mathfrak{g}_K)$ and $\delta_j \in D(P, K)$ ([54, Proposition 3.5]). From this description and the discussion in the previous paragraph, we can conclude that any object $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ carries a unique $D(\mathfrak{g}_K, P)$ -structure, such that the $U(\mathfrak{p}_K)$ -action, as a subring of $U(\mathfrak{g}_K)$, coincides with the $U(\mathfrak{p}_K)$ -action as a subring of $D(P, K)$. Furthermore, the Dirac distributions $\delta_p \in D(P, K)$ act like the group elements $p \in P$ and, by uniqueness of the $D(\mathfrak{g}_K, P)$ -module structure, any morphism $M_1 \rightarrow M_2$ in $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ is in particular a homomorphism of $D(\mathfrak{g}_K, P)$ -modules.⁴⁷

⁴⁵The coadmissible structure on the right hand side becomes from the fact that $\mathfrak{d} \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Hence $D(G, K)\mathfrak{d}$ is finitely generated as $D(G, K)$ -module and therefore closed. The assertion now follows from [70, Lemma 3.6].

⁴⁶We recall for the reader that we have an inclusion $U(\mathfrak{g}_K) \rightarrow D(G, K)$ via the exponential map, cf. discussion after the proposition 2.3 in [69]. The same reasoning applies for the compact subgroup $G_0 \subseteq G$.

⁴⁷This reasoning has been taken from the proof of the corollary 3.6 in [54].

Remark 7.2.1. If $D(\mathfrak{g}_K, P_0)$ denotes the subring of the **Fréchet-Stein algebra** $D(G_0, K)$ generated by $U(\mathfrak{g}_K)$ and $D(P_0, K)$, then we have the same description $D(\mathfrak{g}_K, P_0) = U(\mathfrak{g}_K)D(P_0, K)$ than for $D(\mathfrak{g}_K, P)$.

Let us take, as before, $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Since, by definition, M is a finitely generated $U(\mathfrak{g}_K)$ -module, we see that M is a finitely generated $D(\mathfrak{g}_K, P)$ -module, and therefore $D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M$ is a finitely generated $D(G, K)$ -module. It is clear that the same reasoning applies for the algebra $D(\mathfrak{g}_K, P_0)$, i.e. $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$ is also a finitely generated $D(G_0, K)$ -module.

Remark 7.2.2. Using the Iwasawa decomposition⁴⁸ $G = G_0P$, it is possible to prove that the canonical map

$$D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M \rightarrow D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M$$

is an isomorphism of $D(G_0, K)$ -modules ([72, Lemma 6.1 (i)]).

We have introduced the preceding information because we pretend to prove that we have a canonical isomorphism of $D(G_0, K)$ -modules

$$D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M \simeq \left(\text{Ind}_P^G(W')^{\mathfrak{d}} \right)', \quad (122)$$

which in particular implies that $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$ is a coadmissible $D(G_0, K)$ -module (**in fact strongly coadmissible**). By using the previous remark and the isomorphism (121), we only need to exhibit a canonical isomorphism of $D(G, K)$ -modules

$$D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M \simeq (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d},$$

which we construct as follows (this is exactly as in [54, Proposition 3.7]). We start by considering the canonical map

$$\iota : M = (U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W) / \mathfrak{d} \rightarrow (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d}$$

and we point out that if this map is in fact $D(\mathfrak{g}_K, P)$ -linear, then we can define a homomorphism of $D(G, K)$ -modules

$$\Phi : D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M \rightarrow (D(G, K) \otimes_{D(P, K)} W) / D(G, K)\mathfrak{d}$$

by extending the relation⁴⁹ $\Phi(\delta \otimes w) := \delta \iota(w)$, for $w \in W$. It turns out that this map is in fact an isomorphism of $D(G, K)$ -modules whose inverse Ψ is given by $\Psi((\delta \otimes w) + D(G, K)\mathfrak{d}) := \delta \otimes (w + \mathfrak{d})$. The $D(\mathfrak{g}_K, P)$ -linearity of ι follows from the fact that the $U(\mathfrak{p}_K)$ -action on W is compatible as a subalgebra of both $U(\mathfrak{g}_K)$ and $D(P, K)$. This clearly implies that the natural map

$$U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow D(G, K) \otimes_{D(P, K)} W$$

is $D(\mathfrak{g}_K, P)$ -linear and so is ι .

⁴⁸This has been explained by Stefano Morra in his talk. The interested reader can also find a discussion about this decomposition in [20, 3.5].

⁴⁹ $\Phi(\delta \otimes m) = \Phi(\delta \otimes \sum \mathfrak{z}_i w_i) = \Phi(\sum \delta \mathfrak{z}_i \otimes w_i) = \sum \delta \mathfrak{z}_i \iota(w_i) = \sum \delta \iota(\mathfrak{z}_i w_i) = \delta \iota(m)$, the last equality by $D(\mathfrak{g}_K, P)$ -linearity.

All in all, we can pass to define the functor \mathcal{F}_P^G ([54, 4.1]). For $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$, we have constructed a coadmissible⁵⁰ $D(G, K)$ -module in (122). This motivates the definition of the functor

$$\begin{array}{ccc} \mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} & \rightarrow & \text{Rep}_K^{\text{la}}(G) \\ M & \rightsquigarrow & \mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)' \end{array} \quad (123)$$

Proposition 7.2.3. *The functor \mathcal{F}_P^G is exact.*

Remark 7.2.4. We will follow the arguments given by Matthias Strauch in his talk, which in turn are inspired by the appendix B in [1]. These arguments introduce interesting relations between the algebra of locally analytic distributions and Kohlhaase's ring of distributions supported in a closed subset. The interested reader can also find an alternative proof of the proposition 7.2.3 in [54, Proposition 4.2].

To explain the ideas given by Matthias Strauch, we will need the following notions introduced in [1, B.5, B.6, B.7 and B.8]. Let us denote by $D(G, K)_P$ the **Kohlhaase's ring of distributions supported in P** [45, 1.2.1 - 1.2.6]. This can be considered as the topological closure of $D(\mathfrak{g}_K, P)$ inside $D(G, K)$ [45, Lemma 1.2.10].⁵¹ In particular, $D(G_0, K)_{P_0}$ is Fréchet. The same constructions apply for the maximal compact open subgroup $G_0 \subset G$, and we want to prove that, in this case, $D(G_0, K)_{P_0}$ is a Fréchet-Stein algebra, where $P_0 := P \cap G_0$. To do that, let us consider the Fréchet-Stein structure

$$D(G_0, K) = \varprojlim_{r \in (1/p, 1) \cap \mathbb{P}^{\mathbb{Q}}} D_r(G_0, K)$$

of the distribution algebra $D(G_0, K)$ constructed in [70, Theorem 5.1], and let us denote by $D_r(G_0, K)_{P_0}$ the closure of $D(G_0, K)_{P_0}$ inside $D_r(G_0, K)$. If $\mathfrak{z}_1, \dots, \mathfrak{z}_d$ is an L -basis of \mathfrak{g} and

$$U_r(\mathfrak{g}_K) := \left\{ \sum_{\beta \in \mathbb{N}^d} s_{\beta} \mathfrak{z}^{\beta} \mid s_{\beta} \in K \text{ and } \|s_{\beta} \mathfrak{z}^{\beta}\|_r \xrightarrow{|\beta| \rightarrow \infty} 0 \right\}$$

where $\|\cdot\|_r$ is the so-called r -norm [70, Section 4], it is proved in [45, Theorem 1.4.2 and corollary 1.4.3] (see also [67, Theorem 2.3]) that there exists a cofinal system (which we fix from now on) defining the Fréchet-Stein structure of $D(G_0, K)$, such that $U_r(\mathfrak{g}_K)$ is noetherian and $D_r(G_0, K)_{P_0}$ is a finitely free $U_r(\mathfrak{g}_K)$ -module. In other words $D_r(G_0, K)_{P_0}$ is noetherian. Furthermore, it is not so hard to prove, by using the results cited before ([45, 1.4.2 and 1.4.3]), that $D_r(G_0, K)$ is finite free as a right module over $D_r(G_0, K)_{P_0}$ -module, cf. [1, Lemma B.5.2]. From these facts Agrawal-Strauch conclude that $D(G_0, K)_{P_0}$ is a **Fréchet-Stein algebra** by comparing the transition morphisms via the commutative diagram

$$\begin{array}{ccc} D_r(G_0, K)_{P_0} & \longrightarrow & D_{r'}(G_0, K)_{P_0} \\ \downarrow & & \downarrow \\ D_r(G_0, K) & \longrightarrow & D_{r'}(G_0, K). \end{array} \quad (124)$$

⁵⁰Let us recall that in the non compact case, admissibility is tested over a open compact subgroup. In this case $G_0 \subset G$.

⁵¹We recall for the reader that the set of Dirac distributions is dense in $D(P, K)$ [69, Lemma 3.1].

Indeed, the horizontal map on the bottom being flat and both vertical maps faithfully flat (by the arguments in the previous paragraph), imply that the morphism on the top is also flat. To see [1, Proposition B.5.3] for more details.

Remark 7.2.5. As we have remarked before, in the noncompact case coadmissibility over the distribution algebra can be tested over any open compact subgroup $H \subset G$. By [45, (1.7)] and [1, Corollary B.7.2] the same holds for the algebra $D(G, K)_P$. The following definition is given in [1, Definition B.7.3].

Definition 7.2.6. A $D(G, K)_P$ -module M is **coadmissible** if M is coadmissible as a module over the Fréchet-Stein algebra $D(G_0, K)_{P_0}$.

Remark 7.2.7. (i) Let $M \in \mathcal{O}_{\text{alg}}^{\text{p}K}$. In particular M is a finitely generated $U(\mathfrak{g}_K)$ -module. We already know that M can be endowed with a structure of $D(\mathfrak{g}_K, P)$ -module, and with this action it is possible to prove that

$$D(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M$$

is a coadmissible $D(G, K)_P$ -module, cf. [74, Lemma 4.3].

(ii) ([74, Lemma 4.6]) For a $D(\mathfrak{g}_K, P)$ -module M , the natural map

$$U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} M \rightarrow D_r(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M$$

is an isomorphism of left $U_r(\mathfrak{g}_K)$ -modules.

(iii) ([1, Corollary B.8.2]) If M is a $D(\mathfrak{g}_K, P)$ -module, then we have a canonical isomorphism

$$D(G, K)_P \otimes_{D(\mathfrak{g}_K, P)} M = D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} M$$

of $D(G_0, K)_{P_0}$ -modules.

⁵² Now, let

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \quad (125)$$

be an exact sequence in $\mathcal{O}_{\text{alg}}^{\text{p}K}$. By (i) and (iii) in the previous remark and [1, Proposition 4.1.5], we know that

$$D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (125) \quad (126)$$

is a sequence of coadmissible modules over the Fréchet-Stein algebra $D(G_0, K)_{P_0}$, so it is exact if and only if

$$\begin{aligned} D_r(G_0, K)_{P_0} \otimes_{D(G_0, K)_{P_0}} (D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (125)) \\ = U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} (125) \end{aligned}$$

is exact, where we have used (ii) in the previous remark, and r in the cofinal set that we have fixed. But $U_r(\mathfrak{g}_K)$ is flat over $U(\mathfrak{g}_K)$, cf. [74, Theorem 3.13] and [70, Remark 3.2], thus $U_r(\mathfrak{g}_K) \otimes_{U(\mathfrak{g}_K)} (125)$ is exact for all r . This clearly says that the functor

$$D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet) : \mathcal{O}_{\text{alg}}^{\text{p}K} \rightarrow \mathcal{C}_{D(G, K)_P}$$

is exact. Here $\mathcal{C}_{D(G, K)_P}$ denotes the category of coadmissible $D(G, K)_P$ -modules.

⁵²This reasoning is an extract of the proof of the proposition B.8.5 in [1].

Remark 7.2.8. This fact illustrates one of the motivations of the authors in [1] to introduce Kohlhaase' ring of distributions $D(G, K)_P$. With this ring we somehow have a *good algebraic control*⁵³ over the modules in the category $\mathcal{O}_{\text{alg}}^{\mathfrak{p}K}$.

Of course, if we now want to deal with the proof of the proposition 7.2.3, the naive reasoning that we can carry out is to consider a short exact sequence like in (125) and then tensoring $D(G_0, K) \otimes_{D(G_0, K)_{P_0}} (\bullet)$ the (short exact) sequence (126). Unfortunately, we do not currently know if $D(G_0, K)$ is right flat over $D(G_0, K)_{P_0}$, cf. [1, Remark B.6.3]. To deal with this gap, Agrawal-Strauch have attacked the problem from a *topological point of view* by introducing Ardakov-Wadsley notion of **c-fatness**, [4]. It turns out that this is the right version of flatness for Fréchet-Stein algebras. Let us recall the definition.

Definition 7.2.9. Let $\varphi : A \rightarrow B$ be a continuous morphism of Fréchet-Stein algebras. There exists a right-exact functor $B \otimes_A^c (\bullet)$ from coadmissible left A -modules to coadmissible left B -modules [4, Section 7]. We say that B is **right c-flat** if this functor is exact, and **right faithfully c-flat** if this functor is faithfully exact.

Remark 7.2.10. Under the notation of the previous definition, if M is a finitely presented A -module, then $B \otimes_A^c M = B \otimes_A M$.

Now, by construction (see for example the diagram (124)) we know that the inclusion $D(G_0, K)_{P_0} \rightarrow D(G_0, K)$ is a continuous morphism of Fréchet-Stein algebras whose transition maps $D_r(G_0, K)_{P_0} \rightarrow D_r(G_0, K)$ are right faithfully flat (paragraph right before the diagram (124)). From this fact, it is not hard to see that $D(G_0, K)_{P_0} \rightarrow D(G_0, K)$ is right faithfully c-flat, cf. [1, Lemma B.6.1] for more details.

All in all, we can finally give the proof of the proposition 7.2.3.

Proof of the proposition 7.2.3. Let us recall that for $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}K}$ we have

$$\mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)'$$

Furthermore, given that $(\bullet)'$ induces an equivalence of categories

$$\mathcal{C}_G \xrightarrow{(\bullet)'} \text{Rep}_K^{\text{adm}}(G)$$

between the category of admissible locally analytic G -representations and the category \mathcal{C}_G of coadmissible $D(G, K)$ -modules, cf. [70, Theorem 6.3], we only need to prove, using (iii) in the remark 7.2.7, that $D(G_0, K) \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet)$ defines an exact functor between $\mathcal{O}_{\text{alg}}^{\mathfrak{p}K}$ and \mathcal{C}_G . But this functor equals the composition of the functors

$$\mathcal{O}_{\text{alg}}^{\mathfrak{p}K} \xrightarrow{D(G_0, K)_{P_0} \otimes_{D(\mathfrak{g}_K, P_0)} (\bullet)} \mathcal{C}_{D(G, K)_P} \xrightarrow{D(G_0, K) \otimes_{D(G_0, K)_{P_0}}^c (\bullet)} \mathcal{C}_G.$$

The exactness now follows from c-flatness of $D(G_0, K)$ over $D(G_0, K)_{P_0}$, the reasoning given after the remark 7.2.7 and the remark 7.2.10. \square

⁵³Shishir Agrawal has indicated to me that we don't have a *good algebraic control* over the ring $D(\mathfrak{g}_K, P)$, so they required a good algebraic property from the modules over this ring, namely, finitely presented $D(\mathfrak{g}_K, P)$ -modules. A good source of examples of such a modules are those $D(\mathfrak{g}_K, P)$ -modules arising from the category $\mathcal{O}_{\text{alg}}^{\mathfrak{p}K}$, cf. [1, Proposition 4.1.5].

Remark 7.2.11. The reasoning given in the proof of the proposition 7.2.3 gives a second motivation to introduce the ring $D(G, K)_P$. From a topological point of view, we have a good knowledge of the morphism $D(G, K)_P \rightarrow D(G, K)$, in the sense it is faithfully c-flat.

7.3 Extending the functor \mathcal{F}_P^G

The goal of this subsection will be to extend the functor defined in (123) in order to obtain the announced bi-functor in (115). We will follow word by word the arguments and definitions given in [54, 4.4]. To start with, let us first recall the definition of a **smooth representation** and an **admissible smooth representation**, cf. [72, Section 2]. We will give the definition for the group G , but we warn the reader that in this subsection we will be interested in L_P -representations.

First of all, we say that a K -vector space V is a **smooth representation** of G , if V is endowed with a K -linear G -action such that the stabilizer of each vector in V is open in G . Furthermore, we say that V is an **admissible smooth representation** if, for any compact open subgroup $H \subset G$, the vector space V^H of H -invariant vectors in V is finite dimensional.

Remark 7.3.1. Let V be an admissible smooth G -representations. Given that the unit element in G has a countable fundamental system of open compact neighbourhoods $\{H_n\}_{n \in \mathbb{N}}$, we have

$$V = \varinjlim_{n \in \mathbb{N}} V^{H_n}.$$

This implies that V is of compact type, being the countable locally convex inductive limit of the finite dimensional spaces V^{H_n} .

From now on, we will assume that V is a smooth admissible representation of the Levi subgroup $L_P \subseteq P$, and we regard it as a representation of P . We will always consider on V the finest locally convex topology exhibited in the previous remark.

Let $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ and write it as a quotient of a generalized Verma module

$$0 \rightarrow U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \rightarrow M \rightarrow 0.$$

We have⁵⁴

$$W' \otimes_K V = \varinjlim_{n \in \mathbb{N}} W' \otimes_K V^{N_n},$$

where $\{N_n\}_{n \in \mathbb{N}}$ runs through a countable fundamental system of open compact subgroups of P . Equipped with the diagonal action $W' \otimes_K V$ is a locally analytic representation. By using the pairing $\langle \cdot, \cdot \rangle_{C^{\text{an}}(G, V)}$ defined in the subsection 7.2, we can consider

$$\begin{aligned} \mathcal{F}_P^G(M, V) &:= \text{Ind}_P^G(W \otimes V')^{\circ} \\ &:= \{f \in \text{Ind}_P^G(W' \otimes_K V) \mid \forall \mathfrak{z} \in \mathfrak{d}, \langle \mathfrak{z}, f \rangle_{C^{\text{an}}(G, V)} = 0\}. \end{aligned}$$

⁵⁴Given that W is finite-dimensional, the injective tensor products $W' \otimes_{K, \pi} V = W' \otimes_{K, \iota} V$ coincide, and we can just write $W' \otimes_K V$.

Remark 7.3.2. Strictly speaking we should have written $\mathcal{F}_P^G(M, W, V)$ instead of $\mathcal{F}_P^G(M, V)$, because it could depend of the chosen P -representation W . In [54, Proposition 4.5] the authors prove that if $W_1 \subseteq W_2$ are two finite-dimensional $U(\mathfrak{g}_K)$ -representations which generates M as a $U(\mathfrak{g}_K)$ -module, then there exists a canonical isomorphism

$$\mathcal{F}_P^G(M, W_2, V) \simeq \mathcal{F}_P^G(M, W_1, V).$$

We can henceforth identify all representations $\mathcal{F}_P^G(M, W, V)$ and to write just $\mathcal{F}_P^G(M, V)$.

Proposition 7.3.3. ([54, Proposition 4.7]) $\mathcal{F}_P^G(\cdot, \cdot)$ is a bi-functor

$$\begin{array}{ccc} \mathcal{O}_{alg}^{\mathfrak{p}_K} \times \text{Rep}_K^{\infty, adm}(L_P) & \rightarrow & \text{Rep}_K^{la}(G) \\ (M, V) & \mapsto & \mathcal{F}_P^G(M, V), \end{array}$$

which is contravariant in M and covariant in V .

Proof. Let $\alpha : M_1 \rightarrow M_2$ be a morphism in $\mathcal{O}_{alg}^{\mathfrak{p}_K}$ and $\beta : V_1 \rightarrow V_2$ be a morphism in $\text{Rep}_K^{\infty, adm}(L_P)$. Let us choose $W_1 \subseteq M_1$ a finite-dimensional $U(\mathfrak{p}_K)$ -submodule which generates M_1 as a $U(\mathfrak{g}_K)$ -module. Then let us choose $W_2 \subset M_2$ a finite-dimensional $U(\mathfrak{p}_K)$ -submodule which generates M_2 as a $U(\mathfrak{g}_K)$ -module and which contains $\alpha(W_1)$. We have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{d}_1 & \longrightarrow & U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W_1 & \longrightarrow & M_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{d}_2 & \longrightarrow & U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W_1 & \longrightarrow & M_2 & \longrightarrow & 0 \end{array}$$

This shows that the map

$$\text{Ind}_P^G(W_2' \otimes_K V_1) \rightarrow \text{Ind}_P^G(W_1' \otimes_K V_2)$$

induced by $\alpha' \otimes \beta$ maps $\mathcal{F}_P^G(M_2, V_1)$ into $\mathcal{F}_P^G(M_1, V_2)$. \square

Proposition 7.3.4. (i) For all $M \in \mathcal{O}_{alg}^{\mathfrak{p}_K}$, and for all smooth admissible L_P -representation V , the G -representation $\mathcal{F}_P^G(M, V)$ is admissible.

(ii) If V is of finite length, then $\mathcal{F}_P^G(M, V)$ is strongly admissible.

Proof. The first item follows from [54, Lemma 2.4 (i)] and [70, Proposition 6.4 (iii)]. The second item follows from [54, Lemma 2.4 (ii)] and [69, Lemma 3.5]. \square

The reader can find the proof of the following proposition in [54, Proposition 4.9].

Proposition 7.3.5. (i) The bi-functor $\mathcal{F}_P^G(\cdot, \cdot)$ is exact in both arguments.

(ii) If $P \subseteq Q$ are parabolic subgroups, $\mathfrak{q}_K := \text{Lie}(Q)$, and $M \in \mathcal{O}_{alg}^{\mathfrak{q}_K}$, then

$$\mathcal{F}_P^G(M, V) = \mathcal{F}_Q^G(M, \text{Ind}_P^Q(V)).$$

Remark 7.3.6. (i) It is clear that if \mathbb{K} denotes the trivial L_P -representation, then

$$\mathcal{F}_P^G(M, \mathbb{K}) = \mathcal{F}_P^G(M)$$

for all $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$.

- (ii) Let us recall the category \mathcal{O}^P , [55], whose objects consist of those pairs $\underline{M} := (M, \tau)$, where M is an object of $\mathcal{O}^{\mathfrak{p}_K}$ and $\tau : P \rightarrow \text{End}_K(M)^*$ is a locally finite-dimensional locally analytic representation on M which lifts the Lie algebra representation of \mathfrak{p}_K on M . As we have remarked, any object $M \in \mathcal{O}^P$ can be endowed with a structure of $D(\mathfrak{g}_K, P)$ -module which allows us to consider

$$\mathcal{F}_P^G(M) := (D(G, K) \otimes_{D(\mathfrak{g}_K, P)} M)'$$

This is an extension of the functor (123) to the category \mathcal{O}^P , in the sense of the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} & \xrightarrow{\quad} & \mathcal{O}^P \\ & \searrow \mathcal{F}_P^G & \swarrow \mathcal{F}_P^G \\ & \text{Rep}_K^{\text{la}}(G) & \end{array}$$

where we have used the fully faithful embedding (118).

8 Local properties of the functor \mathcal{F}_P^G

In this section we will review the main ideas introduced by S. Orlik in his talk. We will consider the aspects of faithfulness, projective and injective objects and we will compute some Ext-groups. The main reference for this section is [57].

8.1 Jacquet functors

Let us recall the Levi decomposition $P = L_P U_P$. For any locally analytic P -representation, we let $V(U_P)$ be the subspace generated by the expressions $uv - v$, with $u \in U_P$, $v \in V$ and let $\overline{V(U_P)}$ be its topological closure which is a P -stable subspace of V . We will denote by

$$\overline{H}_0(U_P, V) := V_{U_P} := V/\overline{V(U_P)}$$

the corresponding naive Jacquet module,⁵⁶ which is the largest Hausdorff quotient of V on which U_P acts trivially. Furthermore, since $\overline{V(U_P)}$ is a closed subspace of V , the quotient $\overline{H}_0(U_P, V)$ is of compact type. Moreover, the orbit maps $P \rightarrow \overline{H}_0(U_P, V)$ are clearly locally analytic since they are induced via the locally analytic orbit maps $P \rightarrow V$, cf. [57, Lemma 4.1]. In other words, $\overline{H}_0(U_P, V)$ has a canonical structure of a locally analytic P -representation.

⁵⁵We remark for the reader that the motivation in considering the full subcategory $\mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ was to have a control over the root decomposition of certain objects in $\mathcal{O}^{\mathfrak{p}_K}$, that allowed us to lift the algebraic \mathbb{T}_K -action to a P -action.

⁵⁶The interested reader can take a look to the notes of Gabriel Dospinescu "A review of Emerton's Jacquet functors".

Now, by the very definition, we know that V' is a K -Fréchet space endowed with a continuous P -action. We let

$$H_0(U_P, V) := \{v' \in V' \mid U_P \cdot v' = v'\} \subseteq V'.$$

Being a closed subspace $H_0(U_P, V)$ inherits the structure of a K -Fréchet space equipped with a P -action, as well. More exactly, we have the following topological isomorphism ([57, Lemma 4.2])

$$H_0(U_P, V) \simeq (\overline{H}_0(U_P, V))'$$

of P -representations.

Finally, if M is a \mathfrak{g}_K -representation, then we can consider the subspace $H^0(\mathfrak{u}_P, M)$ of vectors killed by \mathfrak{u}_P , and the quotient $H_0(\mathfrak{u}_P, M) = M/\mathfrak{u}_P M$. Both are $U(\mathfrak{p}_K)$ -modules.

We will need the following technical result whose proof can be founded in [57, Proposition 4.20].

Proposition 8.1.1. *Let $\underline{M} = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W \in \mathcal{O}^P$ be a generalized Verma module. Then*

$$\overline{H}_0(U_P, \mathcal{F}_P^G(\underline{M})) = H^0(\mathfrak{u}_\mathfrak{p}, \omega(\underline{M}))',$$

where $\omega(\cdot)$ denotes the forgetful functor (117).

Remark 8.1.2. ([57, Remark 4.22]) The same statement holds true for objects $\underline{M} \in \mathcal{O}^P$ of the shape $\underline{M} = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} W$ where W is a finite-dimensional locally analytic P -representation. In particular, it holds if W is a finite dimensional algebraic representation of the levi factor $\mathbb{L}_\mathfrak{p}$.

8.2 Functorial properties

In the first part of this subsection, we want to discuss whether the functors \mathcal{F}_P^G are faithful resp. fully faithful. More exactly, we want to prove that the functor

$$\begin{aligned} \mathcal{F}_P^G : \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K} &\rightarrow \text{Rep}_K^{\text{la}}(G) \\ M &\mapsto \mathcal{F}_P^G(M, \mathcal{K}) \end{aligned}$$

is fully faithful.

Theorem 8.2.1. ([57, Theorem 5.1]) *Let $\underline{M}_1, \underline{M}_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. Then the map*

$$\begin{aligned} \text{Hom}_{\mathcal{O}^P}(\underline{M}_1, \underline{M}_2) &\rightarrow \text{Hom}_G(\mathcal{F}_P^G(\underline{M}_2), \mathcal{F}_P^G(\underline{M}_1)) \\ f &\mapsto \mathcal{F}_P^G(f) \end{aligned}$$

is bijective.

As S. Orlik did in his presentation, we will prove the preceding theorem in the special case when $\underline{M}_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$ and \underline{M}_1 is the generalized Verma module $\underline{M}_1 = U(\mathfrak{g}_K) \otimes_{U(\mathfrak{p}_K)} Z$, where Z is a finite-dimensional locally analytic L_P -representation. We remark for the reader that in this particular case we do not have a set of *differential equations* in the sense that $\mathfrak{d} = 0$ (in (119)) and therefore

$$\mathcal{F}_P^G(\underline{M}_1) = \text{Ind}_P^G(Z'), \tag{127}$$

and U_P acts trivially on Z .

Proof of the theorem 8.2.1. The authors have divided this proof into several steps. We will give the proof in the particular case when M_1 is the generalized Verma module of the preceding paragraph and $M_2 \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}_K}$. The interested reader can find a complete proof in [57, Theorem 5.1].

By proposition 8.1.1 we have $\overline{H}_0(U_P, \mathcal{F}_P^G(M_2))' = H^0(\mathfrak{u}_{\mathfrak{p}}, M_2)$. From this fact we have the following identities

$$\begin{aligned} \text{Hom}_G(\mathcal{F}_P^G(M_2), \mathcal{F}_P^G(M_1)) &= \text{Hom}_P(\mathcal{F}_P^G(M_2), Z') \\ &= \text{Hom}_{L_P}(\overline{H}_0(U_P, \mathcal{F}_P^G(M_2)), Z') \\ &= \text{Hom}_{D(L_P, K)}(Z, H^0(\mathfrak{u}_{\mathfrak{p}}, M_2)) \\ &= \text{Hom}_{D(P, K)}(Z, M_2) \\ &= \text{Hom}_{D(\mathfrak{g}_K, P)}(M_1, M_2). \end{aligned}$$

The first relation follows from (127) and Frobenius reciprocity, the second one uses the fact that Z is a representation of the Levi factor L_P and by definition U_P acts trivially on $\overline{H}_0(U_P, \mathcal{F}_P^G(M_2))$. The third relation is the well-known Schneider-Teitelbaum equivalence of categories [70, Theorem 6.3]. The fourth one has already explained and the last one is just by definition of M_1 . \square

Remark 8.2.2. The following example given by S. orlik in his talk shows that the functor obtained by fixing M and letting vary V is not fully faithful.

Example 8.2.3. Let us take $G = \text{GL}_2(\mathbb{Q}_p)$, $B = P$ the Borel subgroup of upper triangular matrices, $M = \mathbb{K}$ and $V = \chi$ a character of the diagonal torus, such that the induced representation splits

$$V := \text{ind}_P^G(\chi) = V_1 \oplus V_2.$$

By the Schur's lemma is known that

$$\dim(\text{Hom}_B(V, V)) = 1,$$

but

$$\dim(\text{Hom}_G(\mathcal{F}_B^G(M, V), \mathcal{F}_B^G(M, V))) = 2.$$

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