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Equivariant D-modules on rigid analytic spaces & the BGG category $\mathcal{O}$

(jt work in progress with K. Ardakov)

$F/\mathbb{Q}_p$ finite

$\mathbb{G}_F$ connected reductive gp$/F$

$G = \mathbb{G}_F(F)$ p-adic Lie group $\rightsquigarrow$ $D(G,K)$ locally analytic distributions

$F \subseteq K$ complete ext. s.t. $\mathbb{G}_F \times_F K$ is split $=: \mathbb{G}$

$\cup\!\!\!\mid$

$\mathbb{B}$ Borel subgroup, $\mathfrak{b} = \mathfrak{t} + \mathfrak{n}$

§1. The parabolic BGG category

$P \subseteq G$ parabolic subgroup $\qquad \mathfrak{p} \subseteq \mathfrak{g}$ Lie algebras over K

$\mathcal{O}$ BGG category $\subseteq \{ \mathcal{U}(\mathfrak{g})\text{-modules} \}$

* f.g. over $\mathcal{U}(\mathfrak{g})$
* $\mathfrak{h}$-semisimple
* locally $\mathfrak{n}$-finite

Ex: • $\underline{\underline{\mathcal{O}^{\mathfrak{g}}}} := \{ \text{fin. dim. } \mathfrak{g}\text{-modules} \} \subseteq \mathcal{O}$

• $\lambda : \mathfrak{h} \longrightarrow K \qquad M(\lambda) := \mathcal{U}(\mathfrak{g}) \underset{\mathcal{U}(\mathfrak{b}),\lambda}{\otimes} K \in \mathcal{O}$

"vermamodule"

$\mathcal{O}^{\mathfrak{p}} \subseteq \mathcal{O}$ full subcategory of all $M \in \mathcal{O}$ s.t. $\mathfrak{p}$ acts locally finite.

Ex: $\mathfrak{p} = \mathfrak{l} + \mathfrak{u}_{\mathfrak{p}}$, $\mathfrak{l} \longrightarrow \mathrm{End}_k(V)$ fin. dim.

$$M(V) := U(\mathfrak{g}) \underset{U(\mathfrak{p})}{\otimes} V \in \mathcal{O}^{\mathfrak{p}}$$

"parabolic Vermamodule"

$\mathfrak{m}_0 := \underbrace{Z(\mathfrak{g})}_{\text{center of } U(\mathfrak{g})} \cap U(\mathfrak{g})\mathfrak{g}$ maximal ideal in Spec $Z(\mathfrak{g})$

$(-)_0 \rightsquigarrow \mathcal{O}^{\mathfrak{p}}_0$

$\rightsquigarrow \mathcal{C}_{D(G,K)_0}$ etc

<u>Thm</u> (Orlik-Strauch 2014) (G_F split, $p > 3$)

$\exists$ exact functor

$$\mathcal{F}^G_P : \mathcal{O}^{\mathfrak{p}}_0 \longrightarrow \mathcal{C}_{D(G,K)_0}$$

which "preserves" irreducibility.

(if $M \in \mathcal{O}^{\mathfrak{p}}_{0}$ simple and $\mathfrak{p}$ is <u>maximal</u> for $M \Rightarrow \mathcal{F}^{G}_{P}(M)$ simple)

Ex: $\mathcal{F}^{G}_{P} M(V) \cong \mathrm{Ind}^{G}_{P}(V')$

"locally analytic parabolic induction"

<u>Q</u>: Can we understand / generalise this functor on the D-module side?

<u>§ 2. Equivariant D-modules on the rigid flag variety</u>

$X = (\mathbb{G}/\mathbb{B})^{an}$, smooth K-analytic rigid space

$G \curvearrowright$

$\mathcal{D}_X$ <u>al</u>ge<u>b</u>raic differential operators

$P \subseteq G$; $\mathcal{C}_{X/P}$ = coadmissible P-equivariant D_X-modules

I. Locally analytic localisation thm

(Huyghe - Patel - Sch. - Strauch, Ardakov)

$$\mathcal{C}_{X/G} \xrightarrow[\cong]{H(X,-)} \mathcal{C}_{D(G,K)_0}$$ is an equivalence of categories.

II Induction equivalence (Ardakov)

$P \subseteq G$ closed subgp

$\exists$ functor $\mathcal{C}_{X/P} \xrightarrow{\mathrm{ind}_P^G} \mathcal{C}_{X/G}$ "left adjoint" to res_P^G. If $Y \subseteq X$ Zariski closed, irreducible s.t. $\begin{cases} G_Y := \mathrm{Stab}_G(Y) \subseteq G \text{ cocompact} \\ gY \cap Y \neq \emptyset \Rightarrow gY = Y \quad \forall g \in G. \quad (*) \end{cases}$

then $\mathcal{C}^{Y}_{X/G_Y} \xrightarrow[\mathrm{ind}^{G}_{G_Y}]{\cong} \mathcal{C}^{GY}_{X/G}$.

Ex: $X_w = \left(\overline{\mathbb{B}w\mathbb{B}/_{\mathbb{B}}}\right)^{\mathrm{an}}$ $\quad w \in W$ Weyl group

irreducible closed subvariety of X

$W_I = \langle s_\alpha \mid \alpha \in I \rangle \subset W, \quad I \subset \{\alpha_1, \ldots, \alpha_e\}$ simple roots of $\mathbb{G}$ w.r.t. $\mathbb{B}$

$w_{0,I}$ = longest element in W_I

$\underset{\substack{\| \\ Y}}{X_{w_{0,I}}}$ satisfies the condition $(*)$ and is an aff dim $= \ell(w_{0,I})$.

In fact, $X_{w_{0,I}} = \left(\mathbb{P}_I \mathbb{B} / \mathbb{B} \right)$

$$\mathbb{P}_I = W_I \mathbb{B} W_I \subset \mathbb{G} \quad \text{parabolic subgp.}$$

$$\mathrm{Stab}_G (X_{w_{0,I}}) \overset{?}{=} \mathbb{P}_I(F).$$

$$(\circ) \quad gY \cap Y \neq \emptyset \implies g \in \mathbb{P}_I(F)$$

$$f : \mathbb{G}/\mathbb{B} \longrightarrow \mathbb{G}/\mathbb{P}_I \ ; \quad f^{-1}(\mathbb{P}_I) = \mathbb{P}_I \mathbb{B}/\mathbb{B}$$

apply f to $(\circ)$ $\implies g\mathbb{P}_I = \mathbb{P}_I \implies g \in \mathrm{Stab}_G(Y)$.

Local structure of ind_P^G: $\mathcal{U} \subseteq X$ open affinoid

$H \underset{\text{c.o.}}{\subseteq} \mathrm{Stab}_G(\mathcal{U})$, $N \in \mathcal{C}_{X/P}$

$$(\mathrm{ind}_P^G N)(\mathcal{U}) = \bigoplus_{s \in H\backslash G/P} \widehat{D}(\mathcal{U}, H) \,\widehat{\otimes}\, [s]N(s^{-1}\mathcal{U})$$

$$\widehat{D}(\mathcal{U}, \underbrace{{}^sP \cap H}_{\subset H})$$

$$\cong \quad [s]N(s^{-1}\mathcal{U})$$

$$\begin{array}{ccc} N(s^{-1}\mathcal{U}) & & [s]N(s^{-1}\mathcal{U}) \\ \circlearrowleft & & \circlearrowleft \\ \widehat{D}(s^{-1}\mathcal{U}, P \cap H^s) & \xleftarrow{\ \cong\ } & \widehat{D}(\mathcal{U}, {}^sP \cap H) \\ \text{"} s^{-1} x s & \longleftarrow\!\shortmid & x \text{"} \end{array}$$

$H^s = s^{-1} H s$

${}^sP = s P s^{-1}$

§3 Results : X rigid flag variety

$\mathcal{D}_X \subset \widehat{\mathcal{D}}_X$ infinite order differential operators

$$\begin{cases} E(\mathcal{M}) := \widehat{\mathcal{D}}_X \otimes_{\mathcal{D}_X} \mathcal{M} & \mathcal{M} \in \mathcal{D}_X\text{-module} \quad \text{"extension functor"} \\ \operatorname{Loc}(M) = \mathcal{D}_X \otimes_{\mathcal{U}(\mathfrak{g})} M & M \in \mathcal{U}(\mathfrak{g})_0\text{-mod} \quad \text{"classical localization"} \end{cases}$$

$$\operatorname{Irr}(\mathcal{O}_0) = \{L(w)\}_{w \in W}, \quad \rho = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$$

$$M(w) := M(-w(\rho)-\rho) \longrightarrow L(-w(\rho)-\rho) =: L(w)$$

Ex: $w = 1$ $\quad L(1) = M(1) = M(-2\rho)$ "antidominant Vermamodule"

$w = w_0$ $\quad L(w_0) \Leftarrow M(w_0) = M(0)$

is the trivial one dim rep.

$$\underline{\mathrm{Supp}(\mathrm{Loc}(L(w)))} = X_w .$$

Lemma 1: $\quad E \circ \mathrm{Loc}(L(w)) \in \mathcal{C}_{X/P}$ simple

Thm^A (expected): Fix $w \in W$. $P = \mathrm{Stab}_G(X_w)$

Then $\mathrm{ind}_P^G \, E \circ \mathrm{Loc}(L(w)) \in \mathcal{C}^{G \cdot X_w}_{X/G}$

is simple.

(ok. $G = GL_2$, GL_3, $w = w_{0,I}$ longest element in W_I, ...)

Lemma 1 $\Rightarrow$ if $M \in \mathcal{O}_0^P$ then $E \circ \mathrm{loc}(M) \in \mathcal{C}_{X|P}$

<u>Thm B</u>:
$$\begin{tikzcd} \mathcal{O}_0^P \arrow[r, "\mathcal{F}_P^G"] \arrow[d, "E\circ\mathrm{loc}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{C}_{D(G,K)_0} \\ \mathcal{C}_{X|P} \arrow[r, "\mathrm{ind}_P^G"'] & \mathcal{C}_{X|G} \arrow[u, "\cong", "H(X,-)"'] \end{tikzcd}$$

is a commutative diagramm.

<u>Rmk</u>: "Thm A" + Thm B (*) proves the irreducibility of $\mathcal{F}_{G_{X_w}}^G L(w)$ (Orlik-Strauch 2014, G_F split, $p>3$)