p-adic representations and arithmetic $\mathcal{D}$-modules

Roots systems, Weyl groups and Bruhat order

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UNIVERSITÀ DEGLI STUDI DI PADOVA

Root systems and Weyl groups

Classification of complex semisimple Lie algebras

Bruhat order

Representations and geometry of flag varieties

Root systems: definitions and first properties

V a real vector space.

Definition. $s \in \mathrm{End}(V)$ is a **symmetry with vector** $\alpha \in V \setminus \{0\}$ if:

- $s(\alpha) = -\alpha$
- $\mathrm{Codim}(H := \{\beta \in V \mid s(\beta) = \beta\}) = 1$

Facts.

- $$V = H \oplus \mathbb{R}\alpha.$$
- There exists a unique $\alpha^\vee \in V^\vee$ such that

  $< \alpha^\vee, H >= 0$ and $< \alpha^\vee, \alpha >= 2.$
- If $< \alpha^\vee, \alpha >= 2$, then $s_\alpha(v) := v - < \alpha^\vee, v > \alpha$ is an α-symmetry.

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Proof.

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Examples

Root system of rank 1: $R = \{\pm\alpha\}$

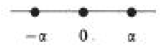

Figure: Type A_1

Root systems in $\mathbb{R}^2$.

- $R = \{\pm\alpha, \pm\beta\}$, with $\alpha = (1, 0)$ and $\beta = (0, 1)$.

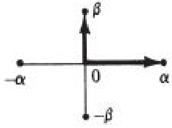

Figure: Type $A_1 \times A_1$

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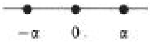

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Root systems in $\mathbb{R}^2$.

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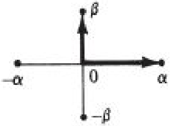

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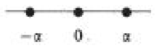

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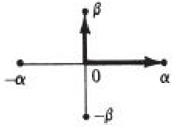

Figure: Type $A_1 \times A_1$

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- $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$, with $\alpha = (1,0)$ and $\beta = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$.

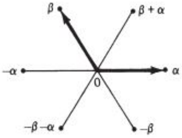

Figure: Type A_2

- $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$, $\alpha = (1,0)$ and $\beta = (-1,1)$

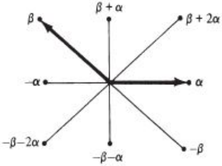

Figure: Type $B_2 = C_2$

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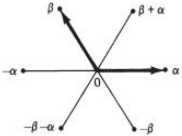

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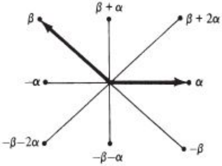


Figure: Type $B_2 = C_2$

Examples

- $R = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta) \pm (3\alpha+2\beta)\}$, with $\alpha = (1,0)$ and $\beta = (-\frac{3}{2}, \frac{\sqrt{3}}{2})$.

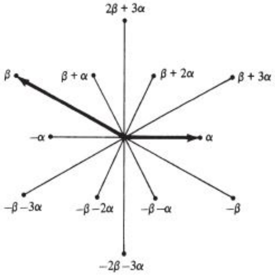
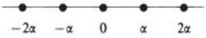


Figure: Type G_2

The only non reduced system of rank 1 is



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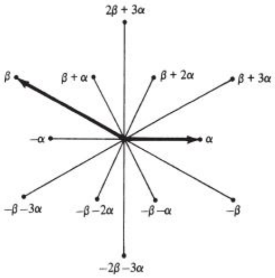

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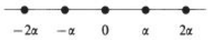

Figure: Non reduced rank 1

Weyl group

R a root system in V. We will consider $\mathrm{Aut}(R) \subset \mathrm{GL}(V)$.

Definition. $W := < s_\alpha \mid \alpha \in R > \subset \mathrm{Aut}(R)$ is the **Weyl group**.

- $Aut(R) \hookrightarrow \mathfrak{S}(R)$ and $us_\alpha u^{-1} = s_{u(\alpha)}$ ($u \in \mathrm{Aut}(R)$ and $\alpha \in R$). So

Fact. $W \trianglelefteq Aut(R)$ and $0 < |W| \leq |\mathrm{Aut}(R)| < \infty$.

Invariant bilinear form

Let B be any positive definite bilinear form on V.

$$(u, v) := \frac{1}{|W|} \sum_{w \in W} B(w(u), w(v))$$

is a positive definite symmetric bilinear form, invariant under W.

- Identifying $V \overset{(\cdot,\cdot)}{\simeq} V^\vee$ it is possible to see

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$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} \Rightarrow s_\alpha(v) = v - 2\frac{(\alpha, v)}{(\alpha, \alpha)} v$$

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Duals and relative positions

Facts.

- $R^\vee := \{\alpha^\vee \mid \alpha \in R\} \subset V^\vee$ is a root system.
- $(R^\vee)^\vee = R$ and $R^\vee$ is the **dual root system**.
- $W(R) \simeq W(R^\vee)$.

Relative positions.

Let $\alpha, \beta \in R$ and ϕ the angle between α and β.

$$2|\beta||\alpha|^{-1}cos(\phi) =< \alpha^\vee, \beta > \Rightarrow 4cos^2(\phi) =< \alpha^\vee, \beta >< \beta^\vee, \alpha > \in \mathbb{Z}$$

(i) $< \alpha^\vee, \beta > = < \beta^\vee, \alpha > = 0$ and $\phi = \pi/2$.

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Example.

(i) is of type $A_1 \times A_1$, (ii) is of type A_2, (iii) is of type B_2 and (iv) of type G_2.

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Simple roots and basis

Definition.

$S \subset R$ is a **system of simple roots** or a **basis** if:

- S is a basis for V.
- $\forall \beta \in R$, $\exists \beta = \sum_{\alpha \in S} \gamma_\alpha \alpha$ such that $\gamma_\alpha \geq 0$ or $\gamma_\alpha \leq 0$, $\forall \alpha$.

Example

For the root systems in $\mathbb{R}^2$ we have $S = \{\alpha, \beta\}$.

Theorem. There exists a basis.

Important remark from the proof and relations with the Weyl group.

- For all $\alpha, \beta \in S$, we have $< \alpha^\vee, \beta > \leq 0$.
- If S' is another basis, there exists $w \in W$, such that $w(S') = S$.
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One can be more precise and to prove that the Weyl group is generated by s_α, $\alpha \in S$ satisfying the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \quad m(\alpha, \beta) \in \{2, 3, 4, 6\}$$

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Example

For the root systems in $\mathbb{R}^2$ we have $S = \{\alpha, \beta\}$.

Theorem. There exists a basis.

Important remark from the proof and relations with the Weyl group.

- For all $\alpha, \beta \in S$, we have $< \alpha^\vee, \beta > \leq 0$.
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One can be more precise and to prove that the Weyl group is generated by s_α, $\alpha \in S$ satisfying the relations

$$(s_\alpha s_\beta)^{m(\alpha,\beta)} = 1, \quad m(\alpha, \beta) \in \{2, 3, 4, 6\}$$

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Example.

Let us take $V = (\mathbb{R}^2, (\bullet, \bullet))$ with the usual inner product.

Using the relation $< \alpha^{\vee}, \bullet >= 2(\alpha, \bullet)/||\alpha||^2$ we can compute the Cartan matrix of the root systems of $\mathbb{R}^2$ studied so far.

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- The Cartan matrix gives us also information about ϕ.
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Coxeter graph

R root system and S a basis.

Definition. A **Coxeter graph** is a finite graph such that the vertices are linked by 0, 1, 2 or 3 edges.

Definition. The **Coxeter graph** of R with respect to S is the graph whose vertices are the elements of S and such that two vertices α and β are linked by $< \alpha^\vee, \beta >< \beta^\vee, \alpha >$.

Proposition. The Coxeter graph is independent of the basis.

Proof.

Let S and S' be basis for R.

There exists $w \in W$ such that $w(S') = S$.

It is straightforward to prove $< w(f), w(v) >=< f, v >$.

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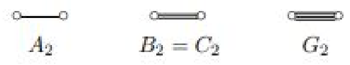

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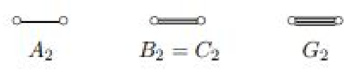

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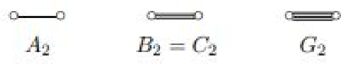

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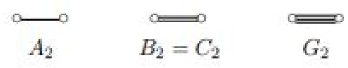

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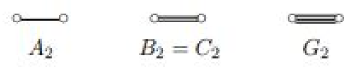

Coxeter graph

R root system and S a basis.

Definition. A **Coxeter graph** is a finite graph such that the vertices are linked by 0, 1, 2 or 3 edges.

Definition. The **Coxeter graph** of R with respect to S is the graph whose vertices are the elements of S and such that two vertices α and β are linked by $< \alpha^\vee, \beta >< \beta^\vee, \alpha >$.

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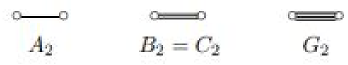

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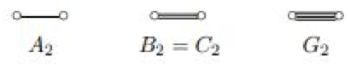

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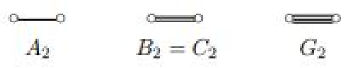

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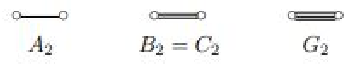

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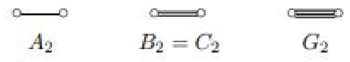

Classification of Coxeter graphs

(Non official) Definition. A root system is **irreducible** if its Coxeter graph is non-empty and connected.

$C = (U, E)$ a Coxeter graph is **non degenerated** if

$$m := m(u, u') \in \{0, 1, 2, 3\} \quad , q_{u,u'} := -cos\left(\frac{\pi}{m+2}\right), \quad q_{u,u} = 0$$

is positive definite over $V_C := \mathrm{Span}(U)$.

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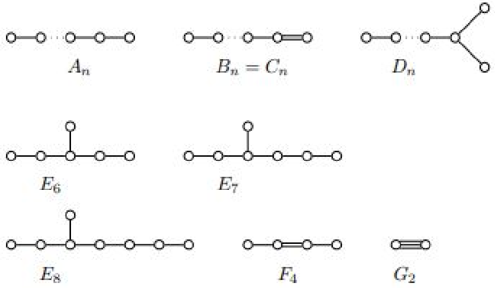
The connected non degenerated Coxeter graphs are the following:



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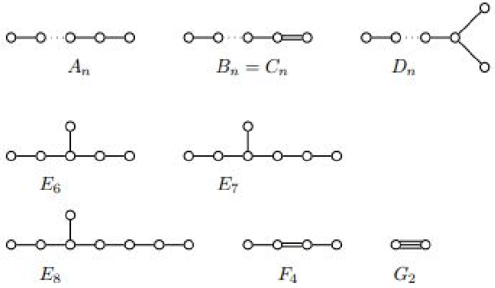

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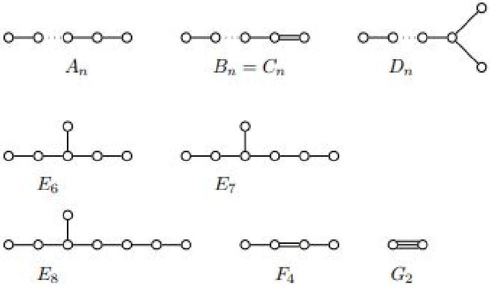

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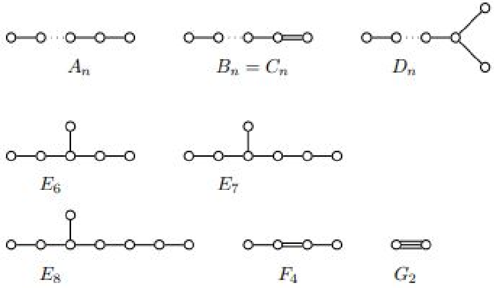
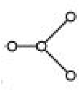


Proof (sketch)

To control the local structure of the graph at one vertex.

Key point. $\sum_{u' \neq u} q_{u,u'}^2 < 1$

(*i*) If $u \in U$ is connected to 3 different vertices u_1, u_2 and u_3, then $m(u, u_i) = 1$, and $m(u, u') = 0$ $\forall u' \neq u_i$.



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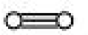

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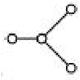

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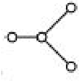

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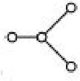

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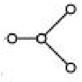

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Reason:

Let $(u_i)_{1\leq i\leq n}$ be the vertices connected to u.
Assume there n_k vertices u_i with $m(u,u_i)=k$ for $k\in\{1,2,3\}$.

$$q_{u,u_i}^2 \geq \frac{k}{4} \quad \Rightarrow \quad 1 > \sum_{i=1}^{n} q_{u,u_i}^2 \geq \frac{n_1+2n_2+3n_3}{4}$$

$$n_1+2n_2+3n_3 \leq 3$$

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To control the number of vertices in the graph.

- C has one **ramification point** u with exactly 3 simple edges and all the edges of the graph are simple.
- The graph has no ramification point and at most one double edge.

Reason.

We already know the result for 1,2 and 3.

By induction, let us assume $|U| = n+1$. If none of the vertices related to u is related to another vertex, then $n+1 = 4$ and we are done.

If at least one of the vertex, u_1, is related to another via an edge e, we can collapse this edge and the results follows by induction.

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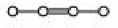
So far we have classified G_2 and

We are left with two types of graphs:

- Chains with no ramifications and one double edge. The graph C will be the following F_4:



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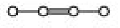

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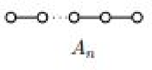

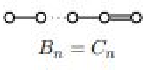

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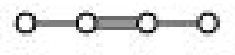
- Chains with no ramifications and one double edge. The graph C will be the following F_4:



- A graph with only simple edges and a unique ramification point u.

  C is the union of u and three chains $(u_k)_{1 \leq k \leq l}$, $(v_i)_{1 \leq i \leq n}$ and $(w_j)_{1 \leq j \leq m}$. It is possible to prove that

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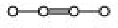

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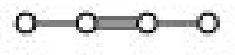

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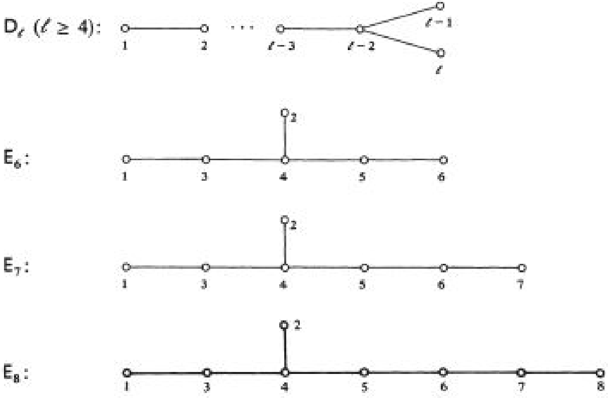
The only possible values of (l, m, n) are $E_6 = (1, 2, 2)$, $E_7 = (1, 2, 3)$, $E_8 = (1, 2, 4)$ and $D_n = (1, 1, n)$.



Dynkin diagrams

R reduced and irreducible, and $C(R)$ the Cartan matrix

Problem: Even if the Coxeter graph does not depend of the choice of a basis, this does not determine the Cartan matrix. In fact a Cartan matrix and its transpose have the same Coxeter graph.

Key point. The preceding problem comes from the fact that the Coxeter diagram only determines the angle but not which of the roots is the longest.

Definition.

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Dynkin $\Rightarrow$ Cartan

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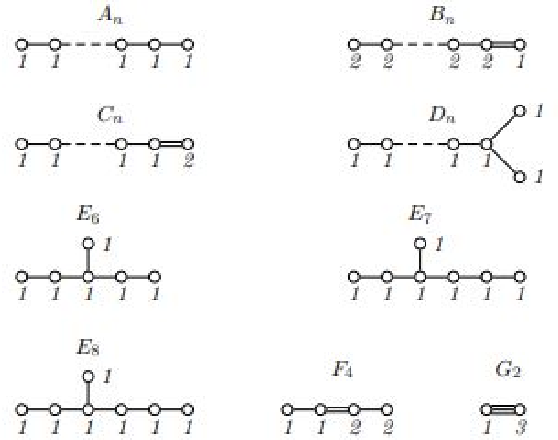
Proposition.

specifying the Dynkin diagram is equivalent to specifying a Cartan matrix. They determine the root system up to isomorphism.

Theorem.

The connected Dynkin diagram are the following



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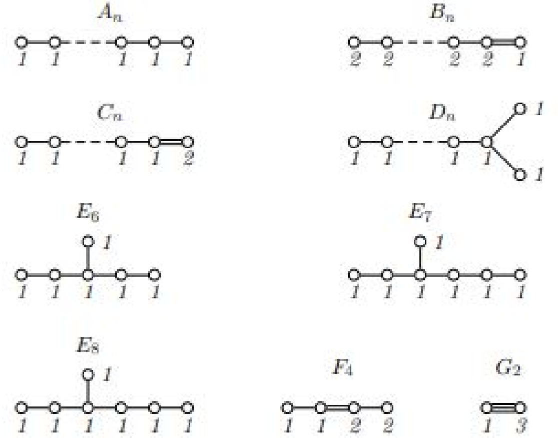

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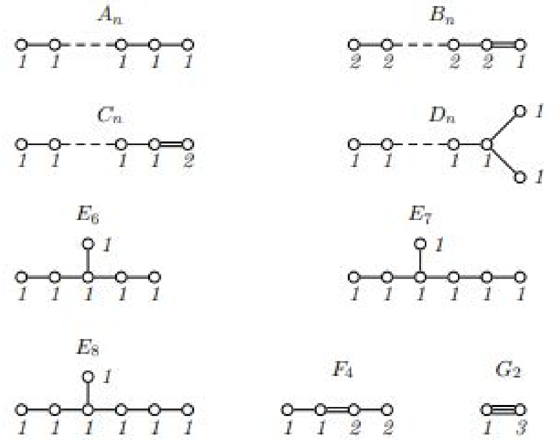

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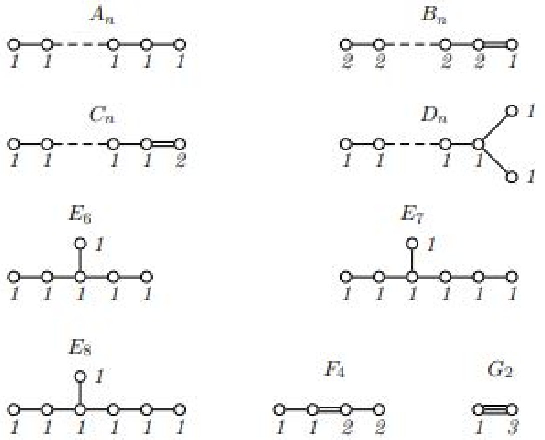

Root systems and Weyl groups

Classification of complex semisimple Lie algebras

Bruhat order

Representations and geometry of flag varieties

From Lie algebras to root systems

$\mathfrak{g}$ complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra

From F. Zerman's presentation we have the **root decomposition**

$$\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in R \subset \mathfrak{h}^\vee \setminus \{0\}} \mathfrak{g}_\alpha \quad \mathfrak{g}_\alpha := \bigcap_{y \in \mathfrak{h}} \mathrm{Ker}(\mathrm{ad}(y) - \alpha I) \neq \{0\}$$

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Proof (sketch)

1st key point. If $h_\alpha \in \mathfrak{h}$ is the dual vector of $\alpha \in R$ under the isomorphism $\mathfrak{h} \overset{\kappa}{\simeq} \mathfrak{h}^\vee$. Then

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Theorem. (Serre presentation)

- $\mathfrak{g} = \mathrm{Span}(H_\alpha, X_\alpha, Y_\alpha)_{\alpha\in S}$, $\mathfrak{n} = \mathrm{Span}(X_\alpha)$ and $\mathfrak{n}_- = \mathrm{Span}(Y_\alpha)$.
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In practice, we use the preceding theorem by "forcing" the Weyl and Serre relations.

theorem.

- Let R be a reduced root system with basis S and let $\widetilde{\mathfrak{g}}$ be the Lie algebra defined by generators and relations in the previous theorem, then $\widetilde{\mathfrak{g}}$ is a semisimple Lie algebra.
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Classification of complex semisimple Lie algebras

By construction, the Lie algebra $\widetilde{\mathfrak{g}}$ in the previous theorem satisfies

$$\widetilde{\mathfrak{g}} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \widetilde{\mathfrak{g}}_\alpha .$$

In particular, $\dim(\widetilde{\mathfrak{g}}) = |S| + |R|$.

Corollary.
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Fact. The root system does not depend of the choice of $\mathfrak{h}$.

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- 2 Lie algebras are iso. iff they have the same root system.
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Example $\mathfrak{sl}_2$

Let us recall that F. Zerman has introduce the semisimple (in fact simple) Lie algebra

$$\begin{aligned} \mathfrak{sl}_2 &= \mathbb{C}\cdot e \oplus \mathbb{C}\cdot h \oplus \mathbb{C}\cdot f \\ [h,e] &= 2e, \quad [h,f] = -2f \end{aligned} \tag{1}$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Considering the Cartan subalgebra $\mathfrak{h} := \mathbb{C}\cdot h$, we define $\lambda_i \in \mathfrak{h}^\vee$, $i = 1, 2$

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We have

$$R = \{\lambda_1 - \lambda_2, \lambda_2 - \lambda_1\}$$

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Root systems and Weyl groups

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Bruhat order

Representations and geometry of flag varieties

Coxeter group

Let S be a set.

$$m : S \times S \to \mathbb{Z}_{>0}$$

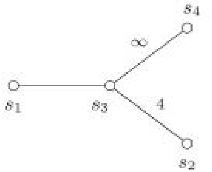
is a **Coxeter matrix** if:

- $m(s, s') = m(s', s)$, $\forall s, s' \in S$.
- $m(s, s') = 1 \Leftrightarrow s = s'$.

Coxeter graph: $C = (S, E)$, where $\{s, s'\} \in E$ if $m(s, s') \geq 3$.

Example

$$\begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 3 & 4 & 1 & \infty \\ 2 & 2 & \infty & 1 \end{pmatrix}$$



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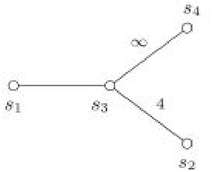

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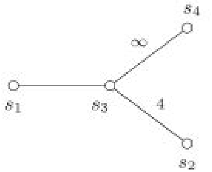

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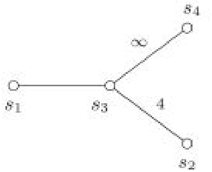

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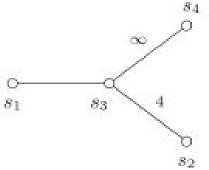

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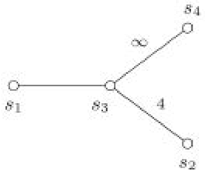

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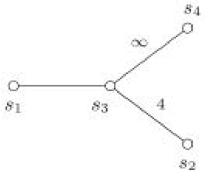

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The **Coxeter group** W with generators in S is the free group generated by S modulo the relation:

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Exchange property and deletion property

Definition.

The **length** $l(w)$ of $w \in W$ is defined by

$$l(w) := \min\{k \in \mathbb{Z}_{\geq 0} \mid w = s_1 \cdots s_k,\ s_i \in S\}.$$

- We say that the system (W, S) has the **Exchange** or **Deletion** property if the following hold:

Exchange property.

Let $w = s_1 \cdots s_k$ be a reduced word ($k = l(w)$), and $s \in S$. Then $l(sw) < l(w) \Rightarrow sw = s_1 \cdots \hat{s}_i \cdots s_k$.

The deletion property.

If $w = s_1 \cdots s_k$,then $l(w) < k \Rightarrow w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_k$, $\exists i < j$.

Theorem (Characterisation theorem).

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- Let R be a root system and S a basis. We have remarked that the Weyl group W is the group generated by s_α, $\alpha \in S$, satisfying the relations

$$(s_\alpha, s_\beta)^{m(\alpha,\beta)} = 1, \quad m(\alpha, \beta) \in \{2, 3, 4, 6\}$$

  (S, W) is a Coxeter system

- For the Coxeter system $(\mathfrak{S}_n, S)$ we have

$$T = \{(a, b) \mid 1 \leq a < b \leq n\}.$$

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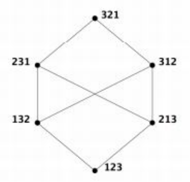

Figure: Bruhat order for $\mathfrak{S}_3$

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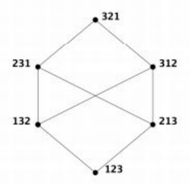

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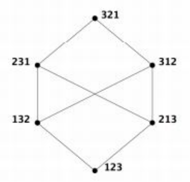

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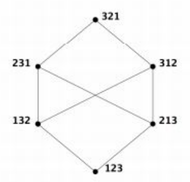

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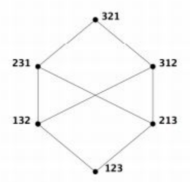

Figure: Bruhat order for $\mathfrak{S}_3$

Examples

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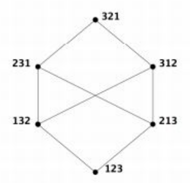

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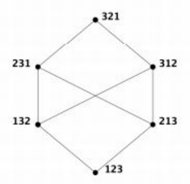

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Remark about Weyl groups

G a semisimple algebraic group and $H \subset G$ a maximal tori.

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- $W := N_G(H)/H$ is a finite group acting naturally on H.
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$$\mathrm{N}_{\mathrm{SL}_n}(D) = \{(a_i \delta_{i\sigma(j)}) \mid \sigma \in \mathfrak{S}_n, (\prod a_i)\mathrm{sgn}(\sigma) = 1\}$$

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Root systems and Weyl groups

Classification of complex semisimple Lie algebras

Bruhat order

Representations and geometry of flag varieties

Representations (but just a taste)

Fact.
Let $SL_{2,\mathbb{C}} \to GL(V)$ be an irreducible rep. There exists a unique weight λ of V which is maximal with respect to the Bruhat order.

Example. Let us consider the algebraic group $G = SL_{2,\mathbb{C}}$.

The subgroup

$$H := \left\{ d(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \,\middle|\, a \in \mathbb{C}^\times \right\}$$

is a maximal torus.

Roots

If $\rho(d(a)) := a$ and $\alpha := 2\rho$, then

$$R = \{\pm\alpha\} \subset X^*(H) = \mathbb{Z}\rho \text{ and } W = \{\pm 1\}$$

The action of $SL_{2,\mathbb{C}}$ on $\mathbb{C}^2$ extends to an action on $\mathrm{Sym}(\mathbb{C}^2)$.
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Geometry of the flag variety

Let $G := \mathrm{GL}_n$ and B the Borel subgroup of upper triangular matrices. By G. Carnovale's talk, we know that

$$X := G/B$$

is a smooth projective variety.

We will see that G decomposes in terms of the **Bruhat decoposition**

$$G = BWB$$

where $W(\simeq \mathfrak{S}_n)$ is the Weyl subgroup of G.

$$X = \bigcup_{w \in W} C_w, \quad C_w := BwB/B.$$

We can connect Bruhat order to the study of flag varieties as follows

- $\overline{C_u} \subseteq \overline{C_w}$ iff $u \leq w$ in the Bruhat order.

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