p-adic representations and arithmetic $\mathscr{D}$-modules

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The aim of the seminar is to give an introduction to the theory of representations of p-adic locally analytic groups, and the theory of $\widehat{\mathscr{D}}$-modules introduced in [2, 4]. We also aim to study localisation of admissible (locally analytic) representations in the sens of Huyghe-Patel-Schmidt-Strauch [17, 25, 26]. In [27] the author of this notes has generalise the results of [18, 19] for general characters, and in [28, 29] the results of [17] are generalised for algebraic characters.

The foundations in these subjects were done by P. Schneider, J. Teitelbaum, K. Ardakov, S. Wadsley, C. Huyghe, D. Patel, T. Schmidt and M. Strauch, whose articles will be our main references. Later papers ([23]) will allow us to deepen our work. For instance, in the reference just cited the authors haves studied the irreducibility of a certain class of admissible locally analytic representations. We consider that a natural approach will be to treat in the examples the locally analytic principal series representations. Papers like [31, 33, 35] give an outlook of this part.

1 Grassmannian and flag varieties

Fundamental object in our work will be the **flag varieties**. We will begin our seminar reviewing the definition and the most important geometric properties. We will study:

- Grassmannians, cf. [14].
- Algebraic groups, cf [15, 21, 38]
- Flag varieties, cf. [10].

2 Lie algebras

In the second part, we will start with the classical aspects of the representations theory of complex Lie algebras. The key notion is that of root systems arising after fixing a maximal torus $\mathbb{T}$ and a Borel subgroup (of GL_n for example) which contains $\mathbb{T}$. This will allow us to study (in a geometric way) representations attached to certain central characters. We will review:

- Lie algebras, cf. [21, 37]
- Root systems, cf. [37]
- Representation theory, cf. [12]

3 $\mathscr{D}$-modules

The fundamental aims of geometric representation theory are to discover deep geometric structures hidden in the familiar objects of representation theory and analysis, and to apply the resulting properties to the resolution of classical problems. An excellent example is the Beilinson-Bernstein localization theorem [6] which allows us to classify all representations of Lie groups via the geometric study of differential equations on flag varieties. In order to understand the statement of this theorem, we need to complement the preceding algebraic topics with the following geometric notions:

- $\mathscr{D}$-modules on complex algebraic varieties, cf. [11, 13].
- Holonomic $\mathscr{D}$-modules, cf. [5, 13]
- Minimal extensions and classification of irreducible representations of the Weyl algebra $A_1(\mathbb{C})$, cf. [8, 13].
- Beilinson-Bernstein localization theorem [13, 20]

4 p-adic groups

We will start the study of representations of p-adic groups. We will define the notion of locally analytic representations, and distribution algebra. The purpose is to *algebraize* the category of locally analytic representations. We will discuss:

- Basic notions of non-archimedean functional analysis, cf. [36]
- Locally analytic manifolds and groups over completely valued fields, cf. [9]
- Locally analytic representations, cf. [31, 33]
- Distribution algebras and co-admissible modules, cf. [23, 34]

5 $\widehat{\mathscr{D}}$-modules on smooth rigid analytic spaces

Let G be a p-adic Lie group, and let us suppose that the ground field K is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers. In the third part of our seminar we have studied admissible locally analytic K-representations of G. As we have "remarked", this category is (anti)-equivalent to the category of co-admissible $D(G,K)$-modules. where $D(G,K)$ is the distribution algebra which can be considered as the K-Fréchet completion of the abstract group ring $K[G]$. This completion contains the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g} := \mathrm{Lie}(G)$ and the closure $\overline{U(\mathfrak{g})}$ in $D(G,K)$ equals the Arens-Michael envelope $\widehat{U(\mathfrak{g})}$ ([30]).

Let us suppose that $\mathfrak{g}$ is a split semi-simple Lie algebra over K. Motivated to stablish an analogue of the Beilinson-Bernstein localization theorem for co-admissible $\widehat{U(\mathfrak{g})}$-modules, in [2, 4] the authors have introduced a sheaf $\widehat{\mathscr{D}}$ of infinite-order differential operators on smooth rigid analytic spaces. This part is aimed in understand their constructions. We will treat here:

- $\widehat{\mathscr{D}}$-modules on rigid analytic spaces, cf. [2, 4].
- Equivariant $\widehat{\mathscr{D}}$-modules and Kashiwara's equivalence, cf. [3, 4].
- Weak holomicity and weak holonomicity for equivariant $\widehat{\mathscr{D}}$-modules, cf. [1, 22].

6 Arithmetic $\mathscr{D}$-modules

Let us suppose that $\mathfrak{X}$ is a smooth formal scheme over the ring of integers $\mathfrak{o}$ of a finite extension L of $\mathbb{Q}_p$. In this part, we pretend to study the classical sheaf of infinity differential operators $\mathscr{D}^\dagger$ introduced by P. Berthelot in [7]. Even if we dispose of a twisted version of these sheaves ([16, 27]) we will study the untwisted version and we will review the analogous construction on admissible formal models of rigid analytic flag varieties [17].

- The sheaf $\mathscr{D}^\dagger$
- Arithmetic differential operators on admissible formal models of rigid analytic flag varieties.
- Localisation of admissible locally analytic representations.

7 Perspectives

In [22] T.M. Phuong VU proves that localization of co-admissible modules, attached to the trivial character, and arising via the Orlik-strauch functor ([24]) are examples of equivariant weakly holonomic modules. We hope to extend these results to arbitrary dominant and regular characters.

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