Semisimple Lie algebras and Jordan decomposition

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Simple Lie algebras

Definition

A Lie algebra L is *simple* if it has no nontrivial ideals and it is not abelian.

Example

Consider the Lie algebra $\mathfrak{sl}_2(K)$ of traceless 2×2 matrices. Define

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\mathfrak{sl}_2(K)$ is generated by e, f, h. By a straight computation, one finds that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

Also, $\mathfrak{sl}_2(K)$ is simple.

Semisimple Lie algebras

Definition

A Lie algebra L is called *semisimple* if $\operatorname{Rad}(L) = 0$.

Example

- If L is a Lie algebra, then $L/\operatorname{Rad} L$ is always semisimple.
- Simple Lie algebras are semisimple.

Lemma

A Lie algebra L is semisimple if and only if L has no nonzero abelian ideals.

Proof

- If L is semisimple, $\operatorname{Rad} L = 0$ and abelian ideals are contained in $\operatorname{Rad} L$.
- Let j be minimal s.t. $(\operatorname{Rad} L)^{(j)} = 0$. Then $(\operatorname{Rad} L)^{(j-1)}$ is an abelian ideal, since $[(\operatorname{Rad} L)^{(j-1)}, (\operatorname{Rad} L)^{(j-1)}] = 0$.

The killing form

Definition

Let L be a Lie algebra. The *killing form* on L is the map

$$\begin{aligned}\kappa_L(\cdot,\cdot) : L \times L &\longrightarrow K\\ (x,y) &\longmapsto \mathrm{Tr}(\mathrm{ad}_L(x)\,\mathrm{ad}_L(y)).\end{aligned}$$

Proposition

(a) κ_L *is symmetric and bilinear.*

(b) $\kappa_L(x,[y,z]) = \kappa_L([x,y],z)$ *for every* $x,y,z \in L$.

(c) *If* I *is an ideal of* L*, then* $\kappa_I = \kappa_{L|_{I\times I}}$.

Proof of (c)

Let $x,y \in I$, $\mathrm{ad}_L(x) = \begin{pmatrix} \mathrm{ad}_I(x) & * \\ 0 & 0 \end{pmatrix}$, $\mathrm{ad}_L(x)\,\mathrm{ad}_L(y) = \begin{pmatrix} \mathrm{ad}_I(x)\,\mathrm{ad}_I(y) & * \\ 0 & 0 \end{pmatrix}$.

The killing form

Lemma

Let $I \subseteq L$ be an ideal. Then $I^{\perp}$ is an ideal in L and $L^{\perp} \subseteq \operatorname{Rad} L$.

Theorem

A Lie algebra L is semisimple if and only if κ_L is nondegenerate.

Proof

- Let $L^{\perp} = 0$ and consider $I \subseteq L$ an abelian ideal.
- For every $x \in I$, $y \in L$ we have $\operatorname{ad}_L(x) \operatorname{ad}_L(y) : L \longrightarrow I$.
- $(\operatorname{ad}_L(x) \operatorname{ad}_L(y))^2 : L \longrightarrow 0$ since I is abelian.
- Then $\operatorname{ad}_L(x) \operatorname{ad}_L(y)$ is nilpotent, hence $\operatorname{Tr}(\operatorname{ad}_L(x) \operatorname{ad}_L(y)) = 0$.
- Then $x \in L^{\perp} = 0$, so L has no nonzero abelian ideals.

Structure of semisimple Lie algebras

Theorem

Let L be semisimple. Then there exist $L_1, \dots, L_r$ simple ideals of L such that $L = L_1 \oplus \cdots \oplus L_r$, $[L_i, L_j] = \delta_{ij} L_i$ and every simple ideal of L is one of these.

Proof (Idea)

- Ideals I of L correspond to subrepresentations for ad_L, and $I \oplus I^\perp = L$.
- Taking the irreducible representations, we find the decomposition $L = L_1 \oplus \cdots \oplus L_r$.
- If I is a simple ideal, then $I = [L, I] = \bigoplus_i [L_i, I] = \bigoplus_{\text{some } i} L_i = L_j$ for some j.

Corollary

Let L be a semisimple Lie algebra. Then

- $[L, L] = L$.
- *Every ideal and homomorphic image of L is semisimple.*
- *Every ideal of L is a sum of certain simple ideals of L.*

Theorem (Weyl)

Every finitely dimensional representation of a semisimple Lie algebra is completely reducible.

Jordan decomposition

For any $x \in \mathfrak{gl}_n(K)$ there exist a unique $x_s \in \mathfrak{gl}_n(K)$ semisimple and a unique $x_n \in \mathfrak{gl}_n(K)$ nilpotent such that $x = x_s + x_n$ and $[x_s, x_n] = 0$.

Proposition

Let L be a Lie algebra. Then Der L *contains the semisimple and nilpotent parts of its elements.*

- $\mathrm{ad}_L(L) \subseteq \mathrm{Der}\, L \subseteq \mathfrak{gl}(L)$.
- Let $L \subseteq \mathfrak{gl}_n(K)$. Then $\mathrm{ad}_L(x) = \mathrm{ad}_L(x_s) + \mathrm{ad}_L(x_n)$ is the Jordan decomposition for $\mathrm{ad}_L(x)$.
- $\mathrm{ad}_L(x_s)$ and $\mathrm{ad}_L(x_n)$ are elements of $\mathrm{Der}\, L$.

Jordan decomposition

Lemma

Let L be a Lie algebra. Then $\mathrm{ad}_L(L)$ is an ideal of $\mathrm{Der}\, L$ and $[\delta, \mathrm{ad}_L(x)] = \mathrm{ad}_L(\delta(x))$ for every $\delta \in \mathrm{Der}\, L$ and $x \in L$.

Proposition

If L is a semisimple Lie algebra, then $\mathrm{ad}_L(L) = \mathrm{Der}(L)$.

Proof

- $\zeta(L) = 0$, hence $L \cong \mathrm{ad}_L(L)$ is a semisimple ideal of $\mathrm{Der}\, L$.
- Call $M := \mathrm{ad}_L(L)$ and $D := \mathrm{Der}\, L$. Then $\kappa_M = \kappa_{D|_{M\times M}}$.
- Since κ_M is nondegenerate, $M \cap M^\perp = 0$.
- Let $\delta \in M^\perp$. Then $0 = [\delta, \mathrm{ad}_L(x)] = \mathrm{ad}_L(\delta x)$ for every $x \in L$.
- Since ad_L is an isomorphism, $\delta = 0$, hence $M^\perp = 0$, hence $M = D$.

Jordan decomposition

Consequences

Let L be a semisimple Lie algebra, $x \in L$.

- $\mathrm{ad}_L(x) = (\mathrm{ad}_L(x))_s + (\mathrm{ad}_L(x))_n$ is the Jordan decomposition for $\mathrm{ad}_L(x)$ in $\mathrm{ad}_L(L) = \mathrm{Der}\, L$.
- Since ad_L is an isomorphism, there exist x_s ad-semisimple and x_n ad-nilpotent such that $\mathrm{ad}_L(x_s) = (\mathrm{ad}_L(x))_s$ and $\mathrm{ad}_L(x_n) = (\mathrm{ad}_L(x))_n$.
- $[x_s, x_n] = 0$.

Definition

The decomposition $x = x_s + x_n$ is called the *abstract Jordan decomposition* of the element $x \in L$.

Jordan decomposition

Question

If $L \subseteq \mathfrak{gl}(V)$ is a semisimple Lie algebra, what are the relations between usual and abstract Jordan decomposition?

Theorem

Let $L \subseteq \mathfrak{gl}(V)$ be a semisimple Lie algebra, $x \in L$. Then the usual Jordan decomposition for x coincides with the abstract Jordan decomposition.

Proof (Idea)

- Let $x = s + n$ be the usual Jordan decomposition.
- $s, n \in L$.
- s and n are respectively ad-semisimple and ad-nilpotent.
- $x = s + n$ is the abstract Jordan decomposition.

Corollary

Let L be a semisimple Lie algebra, $\rho : L \longrightarrow \mathfrak{gl}(V)$ a finitely dimensional representation. If $x = s + n$ is the abstract Jordan decomposition for $x \in L$, then $\rho(x) = \rho(s) + \rho(n)$ is the Jordan decomposition for $\rho(x)$ in $\mathfrak{gl}(V)$.

Proof

- $\rho(L)$ is semisimple.
- Since L is spanned by the eigenvectors of $\mathrm{ad}_L(s)$, $\rho(L)$ is spanned by the eigenvectors of $\mathrm{ad}_{\rho(L)}\, \rho(s)$. Then $\rho(s)$ is ad-semisimple.
- Since $\mathrm{ad}_L(n)$ is nilpotent, $\mathrm{ad}_{\rho(L)}\, \rho(n)$ is nilpotent.
- $[\rho(s), \rho(n)] = \rho([s, n]) = 0$.
- $\rho(x) = \rho(s) + \rho(n)$ is the abstract (and hence usual) Jordan decomposition of ρx.

Toral subalgebras

Lemma

If L is a semisimple Lie algebra, then L contains a semisimple element.

Proof

- For every $x \in L$, $x = s + n$ is the Jordan decomposition.
- If every s is zero, then L is nilpotent.

Definition

Let L be a semisimple Lie algebra. A subalgebra $T \subseteq L$ consisting of semisimple elements is called a *toral subalgebra*.

Proposition

Let L be a semisimple Lie algebra. Any toral subalgebra T of L is abelian.

Definition

Let L be a Lie algebra and E be a subalgebra. Then the *centralizer of E in L* is

$$c_L(E) := \{x \in L : [x, e] = 0 \quad \text{for every } e \in E\}.$$

Lemma

Let L be a semisimple Lie algebra. Any toral subalgebra T of L such that $c_L(T) = T$ it is a maximal toral subalgebra.

Proof

If $T \subseteq T' \subseteq L$ with T' toral, then $[T, T'] = 0$ and so $T' \subseteq c_L(T) = T$.

Example

The algebra of traceless diagonal matrices is a maximal toral subalgebra inside $\mathfrak{sl}_n(K)$.

Maximal toral subalgebras

Let L be a semisimple Lie algebra and $H \subseteq L$ a maximal toral subalgebra.

- We study the action of $\mathrm{ad}_L(H)$ on L. All elements of $\mathrm{ad}_L(H)$ are commuting and diagonalizible, hence simultaneously diagonalizable.

Definition

For every $\alpha \in H^*$, where H^* is the dual of H, we call

$$L_\alpha := \{x \in L : [h,x] = \alpha(h)x \quad \text{for every } h \in H\}.$$

We call Φ the set of $\alpha \in H^* \setminus \{0\}$ such that L_α is nonzero.

- We have the decomposition $L = L_0 \oplus \left(\bigoplus_{\alpha \in \Phi} L_\alpha \right)$.
- $L_0 = H$.
- $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and the elements of L_α are nilpotent if $\alpha \neq 0$.

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- $\mathfrak{sl}_2(K)$ is generated by the elements e, f, h and

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f.$$

- Kh is a maximal toral subalgebra, e and f are eigenvectors for the adjoint of h.
- $\mathfrak{sl}_2(K) = \mathfrak{sl}_2(K)_0 \oplus \mathfrak{sl}_2(K)_2 \oplus \mathfrak{sl}_2(K)_{-2} = Kh \oplus Ke \oplus Kf.$
- Ke, Kf are maximal nilpotent subalgebras of $\mathfrak{sl}_2(K)$.
- $Ke \oplus Kh$, $Kf \oplus Kh$ are maximal solvable Lie algebras.

Representations of $\mathfrak{sl}_2(K)$

We consider now a representation $\rho : \mathfrak{sl}_2(K) \longrightarrow \mathfrak{gl}(V)$.

- Using Weyl's theorem we know that ρ is completely reducible.
- Consider the action of h on V: then $V = \bigoplus_\lambda V_\lambda$ where λ varies among the eigenvalues for the action of h.

Definition

The spaces V_λ are called *weight spaces* and the vectors $v \in V_\lambda$ are called *weight vectors*.

Lemma

We have that $eV_\lambda \subseteq V_{\lambda+2}$ *and* $fV_\lambda \subseteq V_{\lambda-2}$.

Proof

Let $v \in V_\lambda$. Then $hev = ehv + [h, e]v = \lambda ev + 2ev = (\lambda + 2)ev$.

Representations of $\mathfrak{sl}_2(K)$

Definition

A weight vector $0 \neq v \in V_\lambda$ is called a *maximal vector* if $ev = 0$.

Lemma

Let $v_0 \in V_\lambda$ be a maximal vector. Define $v_{-1} := 0$ and $v_i := \frac{1}{i!} f^i v_0$ for $i \geq 0$. Then for every $i \geq 0$ we have that

(a) $hv_i = (\lambda - 2i) v_i$.

(b) $ev_i = (\lambda - i + 1) v_{i-1}$.

(c) $fv_i = (i + 1) v_{i+1}$.

- The v_i's are zero or linearly independent.
- Since V is finitely dimensional, there exists an m such that $v_m = 0$ and $v_{m-1} \neq 0$. The space generated by $v_0, \dots, v_{m-1}$ is $\mathfrak{sl}_2(K)$-stable.

Representations of $\mathfrak{sl}_2(K)$

Let now ρ be an irreducible representation of $\mathfrak{sl}_2(K)$.

- V has dimension m and is spanned by $v_0, \dots, v_{m-1}$.
- Since $0 = v_m = ev_m = (\lambda - m + 1)v_{m-1}$, we find that $m = \lambda + 1$.
- The weights are in $\mathbb{Z}$.

CONCLUSION

- Given an irreducible representation V for $\mathfrak{sl}_2$ of dimension m, we find a maximal vector v_0 (unique up to a scalar) of weight $m - 1$.
- The vector v_i has weight $m - 1 - 2i$, the vector v_{m-1} has weight $-(m - 1)$.
- $\{\text{Irreducible representations of } \mathfrak{sl}_2(K)\} \longrightarrow \mathbb{N}$ is a $1:1$ correspondence.