

# Lie algebra of a linear algebraic group and general properties

Francesco Zerman

November 7, 2020

## Notation

- $K$  an algebraically closed field of zero characteristic;
- $\mathbb{G}$  a linear algebraic group over  $K$ ;
- $I(\mathbb{G})$  the ideal of definition for  $\mathbb{G}$ ;
- $K[\mathbb{G}]$  the ring of regular functions on  $\mathbb{G}$ .

## 1 Lie algebras

**Definition.** A *Lie algebra* is a vector space  $L$  over  $K$  endowed with a bilinear operation (called *bracket* or *commutator*)

$$[\cdot, \cdot] : L \times L \longrightarrow L$$

that satisfies the following properties:

1.  $[x, x] = 0$  for every  $x \in L$ .
2.  $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$  for every  $x, y, z \in L$ .

The second property of the above definition is called *Jacobi identity*. One important result is that, given any associative algebra  $(A, +, \cdot)$ , we can turn it into a Lie algebra  $(A, +, [\cdot, \cdot])$  by defining the commutator  $[x, y] := xy - yx$  for every  $x, y \in A$ . During this notes we always assume that  $L$  is a finitely generated vector space over  $K$ .

## 2 The tangent space to a linear algebraic group

In this section we define the tangent space to a linear algebraic group at 1 in three ways, and eventually we see that they are equivalent definitions. Tangent spaces to algebraic groups will be our main example of Lie algebras.

## 2.1 First definition

**Definition.** Let  $\mathbb{G}$  be a linear algebraic group over  $K$ ,  $x \in \mathbb{G}$ . Then we define the *tangent space to  $\mathbb{G}$  at  $x$*  as

$$T_x(\mathbb{G}) := \{\delta \in \text{Hom}(K[\mathbb{G}], K) : \delta(fg) = (\delta f)g(x) + f(x)(\delta g) \text{ for every } f, g \in K[\mathbb{G}]\}.$$

## 2.2 Second definition

Let  $D := K[t]/(t^2)$  and  $\varepsilon := t + (t^2) \in D$ . We have that

$$D \cong K + K\varepsilon$$

as vector spaces. Let also  $\mathbb{G} \subseteq \mathbb{A}^n(K)$  be a linear algebraic group.

**Definition.** Let  $x \in \mathbb{G}$ . A vector  $v \in K^n$  is *tangent to  $\mathbb{G}$  at  $x$*  if  $f(x + \varepsilon v) = 0$  in  $D$  for every  $f \in I(\mathbb{G})$ . We call  $\text{Tan}_x(\mathbb{G})$  the set of tangent vectors to  $\mathbb{G}$  at  $x$ .

**Remark 2.1.** A vector  $v \in K^n$  is tangent to  $\mathbb{G}$  at  $x$  if and only if the map

$$\begin{aligned} \phi_v : K[\mathbb{G}] &\longrightarrow D \\ f &\longmapsto f(x + \varepsilon v) \end{aligned}$$

is an algebra map.

We can use this definition in order to compute  $\text{Tan}_1(\mathbb{G})$  for some famous algebraic groups  $\mathbb{G}$ .

**Example 2.2.**  $\mathbb{G} = \text{GL}_n(K)$ . Remember the description of  $\text{GL}_n(K)$  as a linear algebraic group:

$$\mathbb{A}^{n^2+1} \supseteq \text{GL}_n(K) := \{(A, t) \in \mathbb{A}^{n^2+1} : A = (a_{ij})_{i,j=1,\dots,n}, \quad (\det(A))t - 1 = 0\}.$$

We want to compute  $\text{Tan}_{(\mathbb{I}_n, 1)} \text{GL}_n(K)$ . Let  $v := ((a_{ij}), s) \in \mathbb{A}^{n^2+1}$  and consider the coordinate functions

$$\begin{aligned} x_{ij} : \text{GL}_n(K) &\longrightarrow K & t : \text{GL}_n(K) &\longrightarrow K \\ ((a_{ij}), s) &\longmapsto a_{ij} & ((a_{ij}), s) &\longmapsto s. \end{aligned}$$

Then we have that

$$\begin{aligned} x_{ij}((\mathbb{I}, 1) + \varepsilon((a_{ij}), s)) &= \delta_{ij} + \varepsilon a_{ij} \\ t((\mathbb{I}, 1) + \varepsilon((a_{ij}), s)) &= 1 + \varepsilon s. \end{aligned}$$

Remembering that  $\varepsilon^2 = 0$  in  $D$ , we can compute

$$\begin{aligned} \phi_v(\det(A)t - 1) &= \det(\delta_{ij} + \varepsilon a_{ij}) \cdot (1 + \varepsilon s) - 1 = (1 + \varepsilon s) \prod_{i=1}^n (1 + \varepsilon a_{ii}) - 1 = \\ &= \varepsilon(\text{Tr}(A) + s) \end{aligned}$$

Hence, by definition we have that  $(A, s)$  is tangent to  $\mathrm{GL}_n(K)$  at 1 if and only if  $s = -\mathrm{Tr}(A)$ . Hence  $T_1(\mathrm{GL}_n(K)) \cong M_n(K)$ , the set of matrices of order  $n$  with entries in  $K$ .

**Exercise 2.3.** With similar computations, one can show that  $T_1(\mathrm{SL}_n(K))$  is in one to one correspondence with the set of  $n \times n$  traceless matrices (i.e. matrices with zero trace).

### 2.3 Third definition

**Definition.** Let  $\mathbb{G}$  be a linear algebraic group. The set of *derivations* supported on the  $K$ -algebra  $K[\mathbb{G}]$  is

$$\mathrm{Der}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{End}_K(K[\mathbb{G}]) : \tilde{\delta}(fg) = \tilde{\delta}(f)g + f\tilde{\delta}(g) \text{ for every } f, g \in K[\mathbb{G}]\}$$

where  $\mathrm{End}_K(K[\mathbb{G}])$  is the set of  $K$ -linear maps from  $K[\mathbb{G}]$  to itself.

Analyzing the structure of  $\mathrm{Der}(K[\mathbb{G}])$ , one can easily see that whenever  $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$  and  $a, b \in K$  we have that  $a\tilde{\delta} + b\tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$ , but in general  $\tilde{\delta} \cdot \tilde{\eta}$  does not lie in  $\mathrm{Der}(K[\mathbb{G}])$ .

**Definition.** Let  $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$ . We define the commutator of  $\tilde{\delta}$  and  $\tilde{\eta}$  as

$$[\tilde{\delta}, \tilde{\eta}] := \tilde{\delta}\tilde{\eta} - \tilde{\eta}\tilde{\delta}.$$

**Lemma 2.4.** *If  $\tilde{\delta}, \tilde{\eta} \in \mathrm{Der}(K[\mathbb{G}])$ , then  $[\tilde{\delta}, \tilde{\eta}] \in \mathrm{Der}(K[\mathbb{G}])$ . Moreover, we have that  $[\cdot, \cdot]$  is bilinear and*

$$(a) \quad [\tilde{\delta}, \tilde{\delta}] = 0 \text{ for every } \tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}]).$$

$$(b) \quad [\tilde{\delta}, [\tilde{\eta}, \tilde{\theta}]] + [\tilde{\eta}, [\tilde{\theta}, \tilde{\delta}]] + [\tilde{\theta}, [\tilde{\delta}, \tilde{\eta}]] = 0 \text{ for every } \tilde{\delta}, \tilde{\eta}, \tilde{\theta} \in \mathrm{Der}(K[\mathbb{G}]).$$

In other words,  $(\mathrm{Der}(K[\mathbb{G}]), [\cdot, \cdot])$  is a Lie algebra.

*Proof.* In order to prove that  $[\tilde{\delta}, \tilde{\eta}] \in \mathrm{Der}(K[\mathbb{G}])$ , let  $f, g \in K[\mathbb{G}]$  and compute

$$\begin{aligned} [\tilde{\delta}, \tilde{\eta}](fg) &= \tilde{\delta}\tilde{\eta}(fg) - \tilde{\eta}\tilde{\delta}(fg) = \tilde{\delta}(\tilde{\eta}(f)g + f\tilde{\eta}(g)) - \tilde{\eta}(\tilde{\delta}(f)g + f\tilde{\delta}(g)) = \\ &= \tilde{\delta}\tilde{\eta}f \cdot g + \tilde{\eta}f \cdot \tilde{\delta}g + \tilde{\delta}f \cdot \tilde{\eta}g + f \cdot \tilde{\delta}\tilde{\eta}g - \tilde{\eta}\tilde{\delta}f \cdot g - \tilde{\delta}f \cdot \tilde{\eta}g - \tilde{\eta}f \cdot \tilde{\delta}g - f \cdot \tilde{\eta}\tilde{\delta}g = \\ &= [\tilde{\delta}, \tilde{\eta}](f) \cdot g + f \cdot [\tilde{\delta}, \tilde{\eta}](g). \end{aligned}$$

Point (a) is clear from the definition of  $[\cdot, \cdot]$  and point (b) is just another matter of computations.  $\square$

**Definition.** Let  $x \in \mathbb{G}$ . Consider  $\lambda_x : K[\mathbb{G}] \rightarrow K[\mathbb{G}]$  to be the algebra map defined as  $(\lambda_x(f))(h) := f(x^{-1}h)$  for every  $f \in K[\mathbb{G}]$  and  $h \in \mathbb{G}$ . Then the set of *left invariant  $\mathbb{G}$ -derivations* is

$$\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}]) := \{\tilde{\delta} \in \mathrm{Der}(K[\mathbb{G}]) : \tilde{\delta}\lambda_x = \lambda_x\tilde{\delta} \text{ for every } x \in \mathbb{G}\}.$$

It is easy to see that  $\mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}])$  is closed with respect to the commutator  $[\cdot, \cdot]$ , hence it is a Lie algebra. For this reason it is usually called the *Lie algebra of  $\mathbb{G}$*  and we write  $\mathrm{Lie}(\mathbb{G}) := \mathrm{Der}_{\mathbb{G}}(K[\mathbb{G}])$ .

## 2.4 Identification theorem

**Theorem 2.5.** *Let  $\mathbb{G}$  be a linear algebraic group. We have a 1 : 1 correspondence between the elements of  $T_1(\mathbb{G})$ ,  $\text{Tan}_1(\mathbb{G})$  and  $\text{Lie}(\mathbb{G})$ . In particular, the tangent space at the identity is naturally a Lie algebra.*

*Proof.* contenuto... □

**Example 2.6.** We call  $\mathfrak{gl}_n(K) := \text{Lie}(\text{GL}_n(K))$  and  $\mathfrak{sl}_n(K) := \text{Lie}(\text{SL}_n(K))$ . These are the tangent spaces we computed in example 2.2 and exercise 2.3, hence  $\mathfrak{gl}_n(K)$  is the set of  $n \times n$  matrices and  $\mathfrak{sl}_n(K)$  the set of  $n \times n$  traceless matrices. It is just a matter of computations to show that the Lie algebra structure on  $\mathfrak{gl}_n(K)$  and  $\mathfrak{sl}_n(K)$  that comes from the correspondence in theorem 2.5 is exactly the Lie algebra structure that comes from the usual commutator operator.

## 3 First properties of Lie algebras

**Definition.** Let  $L$  and  $M$  be two Lie algebras. A vector spaces morphism  $\phi : L \rightarrow M$  is called a *Lie algebra map* if  $[\phi(x), \phi(y)] = \phi([x, y])$  for every  $x, y \in L$ .

We have seen that to any linear algebraic group  $\mathbb{G}$  we can associate the Lie algebra  $\text{Lie}(\mathbb{G})$ , and this is the main motivation that will lead us to the systematic study of Lie algebras. Moreover, if we have  $\phi : \mathbb{G} \rightarrow \mathbb{H}$  a morphism of linear algebraic groups, we can always define a morphism of Lie algebras  $d\phi : \text{Lie}(\mathbb{G}) \rightarrow \text{Lie}(\mathbb{H})$  in the following way. Let  $f \in \text{Tan}_1(\mathbb{G}) \cong \text{Lie}(\mathbb{G})$  and consider the pull-back  $\phi^* : K[\mathbb{H}] \rightarrow K[\mathbb{G}]$ . Then we define  $d\phi(f) := f \circ \phi^*$ . It can be proved that  $d\phi$  is well defined and that it is a Lie algebra map. This means also that if we have a linear representation

$$\rho : \mathbb{G} \rightarrow \text{GL}_n(K)$$

than we can always canonically construct the Lie algebra map

$$d\rho : \text{Lie}(\mathbb{G}) \rightarrow \mathfrak{gl}_n(K).$$

This fact leads us to the following definition.

**Definition.** Let  $L$  be a Lie algebra. A *representation* of  $L$  is a Lie algebra map  $L \rightarrow \mathfrak{gl}_n(K)$  for some  $n \in \mathbb{N}$ .

**Definition.** Let  $L$  be a Lie algebra. The *adjoint representation* is the representation

$$\begin{aligned} \text{ad}_L : L &\rightarrow \mathfrak{gl}(L) \\ x &\mapsto [x, \cdot], \end{aligned}$$

where  $\mathfrak{gl}(L)$  is the group of  $K$ -endomorphisms of  $L$  as a vector space, endowed with the structure of a Lie algebra by using the commutator.

The adjoint representation allows us find a bridge between our abstract algebra  $L$  and an algebra of linear transformations over a (finitely dimensional) vector space, that can be represented by an algebra of matrices.

**Definition.** Let  $L$  be a Lie algebra,  $E$  a subspace of  $L$  such that  $[E, E] \subseteq E$ . Then  $E$  is called a *subalgebra* of  $L$ . If, moreover,  $[E, L] \subseteq E$ , then  $E$  is called an *ideal* of  $L$ .

**Exercise 3.1.** Prove that kernels of Lie algebra morphisms are ideals, and that quotients of Lie algebras by Lie ideals are still canonically Lie algebras. Also, the usual isomorphism theorems hold.

**Definition.** The *center* of  $L$  is  $\zeta(L) := \ker(\text{ad}_L)$ . If  $\zeta(L) = L$ , we say that  $L$  is *abelian*.

**Exercise 3.2.** Every 1-dimensional Lie algebra is abelian.

**Lemma 3.3.**  $\text{Im}(\text{ad}_L) \subseteq \text{Der}(L)$  and we have an injection

$$L/\zeta(L) \longrightarrow \text{Der}(L) \subseteq \mathfrak{gl}(L).$$

*Proof.* Let  $x \in L$ : we see that  $\text{ad}_L(x) = [x, \cdot]$  is a derivation. Take  $y, z \in L$ . Then

$$\begin{aligned} \text{ad}_L(x)([y, z]) &= [x, [y, z]] = -[y, [x, z]] - [z, [x, y]] = [y, [x, z]] + [[x, y], z] = \\ &= [y, \text{ad}_L(x)(z)] + [\text{ad}_L(x)(y), z]. \end{aligned}$$

Since  $\zeta(L) = \ker(\text{ad}_L)$ , we have that

$$L/\zeta(L) \cong \text{ad}_L(L) \subseteq \text{Der}(L) \subseteq \mathfrak{gl}(L).$$

□

**Definition.** A Lie algebra  $L$  is *simple* if it has no nontrivial ideals and it is not abelian.

**Example 3.4.** Consider the Lie algebra  $\mathfrak{sl}_2(K)$  of traceless  $2 \times 2$  matrices. Define

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then  $\mathfrak{sl}_2(K)$  is generated by  $e, f, h$  as a vector space over  $K$ . By a straight computation, one finds that

$$[e, f] = h, \quad [h, e] = 2e, \quad [h, f] = -2f. \quad (1)$$

We prove now that  $\mathfrak{sl}_2(K)$  is simple. Suppose that  $I \subseteq \mathfrak{sl}_2(K)$  is an ideal and let  $x \in I$ . Then there exist  $a, b, c \in K$  such that  $x = ae + bf + ch$ . Using bilinearity and the relations in equation (1), we obtain that  $[x, e] = 2ec - bh$ .

Hence  $[[x, e], e] = -2be$ . If  $b \neq 0$ , since  $I$  is an ideal, we have that  $-2be \in I$ , hence  $e \in I$ . But, using relations in equation (1), we can easily deduce that  $h, f \in I$  and then  $I = \mathfrak{sl}_2(K)$ . If  $b = 0$  and  $c \neq 0$ , then we obtain the same result by considering  $[x, e] = 2ec \in I$ . If  $b = c = 0$  and  $a \neq 0$ , we directly have that  $ae \in I$  and we can conclude as before. This means that either  $I = \mathfrak{sl}_2(K)$  or  $I = 0$ .

## 4 Nilpotent Lie algebras

**Definition.** Let  $L$  be a Lie algebra and consider the sequence

$$L^1 := [L, L], \quad L^j := [L, L^{j-1}] \quad \text{for } j > 1.$$

$L$  is called *nilpotent* if  $L^j = 0$  for some  $j > 0$ .

**Example 4.1.** Here there are two important examples of nilpotent Lie algebras.

1. Abelian Lie algebras are nilpotent.
2. The subalgebra  $\mathfrak{n}_n(K)$  of  $\mathfrak{sl}_n(K)$  consisting of strictly upper triangular matrices, i.e. matrices of the form

$$\begin{pmatrix} 0 & * & \dots & * \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & \dots & 0 \end{pmatrix},$$

is nilpotent.

**Proposition 4.2.** Let  $L$  be a Lie algebra.

- (a) If  $L$  is nilpotent, then its subalgebras and homomorphic images are so.
- (b) If  $L/\zeta(L)$  is nilpotent, then  $L$  is nilpotent.
- (c) If  $L$  is nilpotent, then  $\zeta(L) \neq 0$ .

*Proof.* (a) If  $E$  is a subalgebra of  $L$ , we have that  $E^j \subseteq L^j$ , hence if  $L$  is nilpotent then  $E$  is so. If  $f : L \rightarrow N$  is a morphism of Lie algebras, then  $f(L^j) = f(L)^j$ . Hence if  $L$  is nilpotent, then  $f(L)$  is so.

(b) If  $L/\zeta(L)$  is nilpotent, then there exists a  $j \in \mathbb{N}$  with the property that  $(L/\zeta(L))^j = L^j/(\zeta(L) \cap L^j) = 0$ . This means that  $L^j \subseteq \zeta(L)$ , therefore  $L^{j+1} \subseteq [L, \zeta(L)] = 0$ .

(c) Let  $L$  be nilpotent and consider  $j$  minimal such that  $L^j = 0$ . Then  $L^{j-1} \neq 0$  and the condition  $0 = L^j = [L, L^{j-1}]$  implies that  $L^{j-1} \subseteq \zeta(L)$ .  $\square$

**Definition.** Let  $L$  be a Lie algebra and  $x \in L$ . Then  $x$  is *ad-nilpotent in  $L$*  if  $\text{ad}_L(x)$  is a nilpotent element of  $\mathfrak{gl}(L)$ , i.e. if there exists a  $j \in \mathbb{N}$  such that  $(\text{ad}_L(x))^j \equiv 0$ .

**Lemma 4.3.** *Let  $x \in \mathfrak{gl}(V)$  for some vector space  $V$ . If  $x$  is a nilpotent endomorphism, then  $x$  is ad-nilpotent in  $\mathfrak{gl}(V)$ .*

*Proof.* Let  $x \in \mathfrak{gl}(V)$  be a nilpotent endomorphism. Define

$$\begin{aligned} \lambda_x : \text{End}_K(V) &\longrightarrow \text{End}_K(V) & \rho_x : \text{End}_K(V) &\longrightarrow \text{End}_K(V) \\ y &\longmapsto x \circ y & y &\longmapsto y \circ x \end{aligned}$$

Clearly we have that  $\lambda_x, \rho_x \in \text{End}_K(\text{End}_K(V))$ , they commute and they are nilpotent (since  $x$  is so). Then also  $\lambda_x - \rho_x$  is nilpotent (just use the binomial formula to prove it). Hence we have that  $\text{ad}_{\mathfrak{gl}(V)}(x) = \lambda_x - \rho_x$  is nilpotent.  $\square$

**Theorem 4.4.** *Let  $L \subseteq \mathfrak{gl}(V)$  be a Lie algebra, with  $V$  a finitely dimensional vector space. If  $V \neq 0$  and  $L$  consists of nilpotent endomorphisms, then there exists  $v \in V$ ,  $v \neq 0$ , such that  $Lv = 0$ .*

*Proof.* Use induction on  $\dim L$ . If  $\dim L = 1$ , then  $L = Kx$  for some  $x \in L$ , and  $\ker(x) = \ker(ax) \neq 0$  for every  $a \in K \setminus \{0\}$ , since  $x$  is nilpotent. Taking  $v \in \ker(x)$ , we are done.

Assume now that the statement holds for every  $L'$  subalgebra of  $\mathfrak{gl}(V)$  with  $\dim L' < \dim L$ . Let  $M \neq L$  be a subalgebra of  $L$ . According to the previous lemma,  $M$  acts via  $\text{ad}_L$  as a lie algebra of nilpotent endomorphisms on the vector space  $L$ , and hence it acts also on the vector space  $L/M$  since  $[M, M] \subseteq M$ . Since  $\dim M < \dim L$ , the inductive hypothesis guarantees the existence of a vector  $x + M \in L/M$ ,  $x \notin M$ , killed by the image of  $M$  in  $\mathfrak{gl}(L/M)$ . This just means that  $[y, x] \in M$  for every  $y \in M$ , whereas  $x \notin M$ . In other words,  $M$  is properly included in

$$N_L(M) := \{y \in L : [y, z] \in M \text{ for every } z \in M\},$$

the normalizer of  $M$  in  $L$ .

Consider now  $M$  to be a maximal proper subalgebra of  $L$ . The preceding argument forces  $N_L(M) = L$ , i.e.  $M$  is an ideal of  $L$ . If  $\dim L/M$  were greater than one, then the inverse image in  $L$  of a one dimensional subalgebra of  $L/M$  (which always exists) would be a proper subalgebra properly containing  $M$ , which is absurd. Therefore,  $M$  has codimension one and so we can write  $L = M + Kz$  for any  $z \in L \setminus M$ .

By induction,  $W := \{v \in V : Mv = 0\}$  is nonzero. Since  $M$  is an ideal,  $W$  is stable under the action of  $L$ : for every  $x \in L$ ,  $y \in M$ ,  $w \in W$  we have

$$yx(w) = xy(w) - [x, y]w = 0.$$

Choose  $z \in L \setminus M$ . Then the nilpotent endomorphism  $z$ , acting on the subspace  $W$ , has an eigenvector of eigenvalue 0, i.e there exists a nonzero  $v \in W$  for which  $zv = 0$ . Therefore  $Lv = 0$  as desired.  $\square$

**Theorem 4.5.** (Engel) *Let  $L$  be a finite dimensional Lie algebra. Then  $L$  is nilpotent if and only if every element of  $L$  is  $\text{ad}_L$ -nilpotent.*

*Proof.* If  $L$  is nilpotent, then there exists a  $j \in \mathbb{N}$  such that  $L^j = 0$ , and so for every  $x \in L$  we have that  $(\text{ad}_L(x))^j = 0$ , hence  $x$  is ad-nilpotent.

Conversely, let  $L$  be composed of ad-nilpotent elements, and use induction on  $\dim L$ . If  $\dim L = 1$ , then  $L$  is abelian and we are done. Let now  $\dim L > 1$  and work with  $\text{ad}_L(L) \subseteq \mathfrak{gl}(L)$ . We can use the above theorem to find a nonzero  $x \in L$  such that  $[x, L] = 0$ , i.e.  $x \in \zeta(L)$ . Hence  $L/\zeta(L)$  has smaller dimension than  $L$ , and its elements are again ad-nilpotent. Using the inductive hypothesis, we obtain that  $L/\zeta(L)$  is nilpotent. Thanks to point (c) of proposition 4.2, we conclude that  $L$  is nilpotent.  $\square$

**Definition.** Let  $V$  be a finitely dimensional vector space of dimension  $n \in \mathbb{N}$ . A *flag* on  $V$  is a collection of subspaces  $0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$  such that  $\dim V_i = i$ .

**Corollary 4.6.** *Let  $V$  be a vector space of dimension  $n$ . If  $L \subseteq \mathfrak{gl}(V)$  is a Lie algebra consisting of nilpotent elements, then there exists a flag  $0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$  such that  $xV_i \subseteq V_{i-1}$  for every  $x \in L$ ,  $i = 1, \dots, n$ .*

*Proof.* Using theorem 4.4, we find a vector  $v_1 \in V$  such that  $Lv_1 = 0$ . Hence we define  $V_1 := Kv_1$ . Work on  $V/V_1$ :  $L$  acts on  $V/V_1$  by nilpotent endomorphisms, hence by theorem 4.4 there exists a  $v_2 \in V \setminus V_1$  such that  $L(v_2) \subseteq V_1$ . Define  $V_2 := Kv_1 + Kv_2$ . Proceeding inductively, we conclude.  $\square$

**Corollary 4.7.** *Let  $L \subseteq \mathfrak{gl}(V)$  be a Lie algebra consisting of nilpotent elements,  $V$  a vector space of dimension  $n$ . Then there exists a basis of  $V$  such that, with respect to this basis,  $L$  is a subalgebra of the algebra  $\mathfrak{n}$  of strictly upper triangular matrices.*

## 5 Solvable Lie algebras

**Definition.** Let  $L$  be a Lie algebra and consider the sequence

$$L^{(1)} := [L, L], \quad L^{(j)} := [L^{(j-1)}, L^{(j-1)}] \quad \text{for } j > 1.$$

$L$  is called *solvable* if  $L^j = 0$  for some  $j > 0$ .

**Example 5.1.** Here there are three important examples of nilpotent Lie algebras.

1. Abelian Lie algebras are solvable.
2. Nilpotent Lie algebras are solvable, as  $L^j \subseteq L^{(j)}$ .
3. The subalgebra  $\mathfrak{b}_n(K)$  of  $\mathfrak{sl}_n(K)$  consisting of all upper triangular matrices is solvable.

**Proposition 5.2.** *Let  $L$  be a Lie algebra.*

(a) *If  $L$  is solvable, then its subalgebras and homomorphic images are so.*

(b) *If an ideal  $I$  of  $L$  is solvable and  $L/I$  is solvable, then  $L$  is solvable.*

(c) *If  $I, J$  are solvable ideals of  $L$ , then  $I + J$  is solvable.*

*Proof.* (a) If  $E$  is a subalgebra of  $L$ , we have that  $E^{(j)} \subseteq L^{(j)}$ , hence if  $L$  is solvable then  $E$  is so. If  $f : L \rightarrow N$  is a morphism of Lie algebras, then  $f(L^{(j)}) = f(L)^{(j)}$ . Hence if  $L$  is solvable, then  $f(L)$  is so.

(b) Consider the projection  $\pi : L \rightarrow L/I$ . Then there exists a  $j$  such that  $0 = (\pi(L))^{(j)} = \pi(L^{(j)})$ , i.e.  $L^j \subseteq I$ . There exists also a  $k$  such that  $I^{(k)} = 0$ , hence  $L^{(j+k)} \subseteq I^{(k)} = 0$ .

(c) We have that  $(I+J)/I \cong J/(I \cap J)$ . Since  $I$  is solvable and  $J/(I \cap J)$  is solvable (by point (a)), then by point (b) we have that  $I + J$  is solvable.  $\square$

Point (c) of the above proposition implies that inside a Lie algebra  $L$  there is always a maximal solvable ideal, namely the sum of all solvable ideals.

**Definition.** Let  $L$  be a Lie algebra. The maximal solvable ideal of  $L$  is called  $\text{Rad}(L)$ , the *radical ideal* of  $L$ .

Following the ideas of the nilpotent case, we give a description of every solvable Lie algebra  $L \subseteq \mathfrak{gl}(V)$ , for some finitely dimensional vector space  $V$ .

**Theorem 5.3.** *Let  $L \subseteq \mathfrak{gl}(V)$  a solvable Lie algebra, with  $V$  a finitely dimensional vector space. If  $V \neq 0$  then there exists  $v \in V$ ,  $v \neq 0$ , such that  $Lv = Kv$ .*

*Proof.* Use induction on  $\dim L$ , the case of  $\dim L = 0$  being trivial. The proof follows now some steps similar the proof of theorem 4.4, but here the initial hypotheses that  $\text{char } K = 0$  and  $K = \bar{K}$  play a fundamental role.

**Step 1: Find an ideal  $M$  of  $L$  of codimension 1.** Since  $L$  is solvable of positive dimension,  $L$  properly contains  $[L, L]$ . Since  $L/[L, L]$  is abelian, any subspace is automatically an ideal. Call  $M$  the inverse image in  $L$  of a subspace of codimension 1 in  $L/[L, L]$ . Then  $M$  is an ideal of  $L$  of codimension 1.

**Step 2: Common eigenvectors exist for  $M$ .** Use induction to find a common eigenvector  $v \in V$  for  $M$ . This means that for every  $x \in M$  we have that  $xv = \lambda(x)v$ , with  $\lambda : M \rightarrow K$  some linear function. Fix this  $\lambda$  and define the nonzero subspace

$$W := \{w \in V : xw = \lambda(x)w \text{ for every } x \in M\}.$$

**Step 3:  $L$  stabilizes  $W$ .** Let  $w \in W$ ,  $x \in L$ . To test whether or not  $xw$  lies in  $W$ , we must take an arbitrary  $y \in M$  and examine

$$yxw = xyw - [x, y]w = \lambda(y)xw - \lambda([x, y])w.$$

In particular, we have to prove that  $\lambda([x, y]) = 0$ . For this, let  $n > 0$  be the smallest integer for which  $w, xw, \dots, x^{n-1}w$  are linearly independent. Let  $W_i$  be the subspace of  $V$  generated by  $w, xw, \dots, x^{i-1}w$ , so that  $\dim W_i = i$ ,  $W_n = W_{n+j}$  for every  $j \geq 0$  and clearly  $x$  maps  $W_n$  to  $W_n$ . It is easy to prove by induction on  $i$  that each  $y \in M$  leaves each  $W_i$  invariant. This implies that, with respect to the basis  $w, xw, \dots, x^{n-1}w$ , every  $y \in M$  is represented by an upper triangular matrix. Moreover, an induction similar to the previous one shows that the diagonal entries of those matrices are equal to  $\lambda(y)$ . Hence we have that  $\text{Tr}_{W_n}(y) = n\lambda(y)$ . In particular, this is true for elements of  $M$  of the special form  $[x, y]$ , with  $x$  as above,  $y \in K$ . But  $x$  and  $y$  both stabilize  $W_n$ , so  $[x, y]$  acts on  $W_n$  as the commutator of two endomorphisms of  $W_n$  and hence its trace is equal to 0. We conclude that  $n\lambda([x, y]) = 0$ . Since  $\text{char } K = 0$ , this implies that  $\lambda([x, y]) = 0$  as required.

**Step 4: Conclusion.** Write  $L = M + Kz$  for any  $z \in L \setminus M$ . Since  $K = \bar{K}$  and  $W$  is stable under the action of  $L$ , we find an eigenvector  $v \in W$  for the action of  $z$ . Then  $v$  is obviously an eigenvector for the whole  $L$ .  $\square$

**Theorem 5.4 (Lie).** *Let  $V$  be a vector space of dimension  $n$ . If  $L \subseteq \mathfrak{gl}(V)$  is a solvable Lie algebra, then there exists a flag  $0 \subseteq V_1 \subseteq \dots \subseteq V_n = V$  such that  $xV_i \subseteq V_i$  for every  $x \in L$ ,  $i = 1, \dots, n$ .*

*Proof.* Using the theorem, we find a  $v_1 \in V$  such that  $Lv_1 = Kv_1$ , and we set  $V_1 = Kv_1$ . Then we can work on  $V/V_1$  and find  $v_2 \in V \setminus V_1$  such that  $L(v_2 + V_1) = K(v_2 + V_1)$ . This means that, if we call  $V_2 := Kv_1 + Kv_2$ , then  $Lv_2 \subseteq V_2$ . Proceeding inductively, we conclude.  $\square$

**Corollary 5.5.** *Let  $L \subseteq \mathfrak{gl}(V)$  be a solvable Lie algebra,  $V$  a vector space of dimension  $n$ . Then there exists a basis of  $V$  such that, with respect to this basis,  $L$  is a subalgebra of the algebra  $\mathfrak{b}$  of upper triangular matrices.*

## 6 Semisimple Lie algebras

We begin this section enouncing an important solvability criterion.

**Theorem 6.1 (Cartan's solvability criterion).** *Let  $L \subseteq \mathfrak{gl}(V)$  be a Lie algebra,  $V$  a finitely dimensional vector space. If  $\text{Tr}(xy) = 0$  for every  $x \in [L, L]$  and  $y \in L$ , then  $L$  is solvable.*

*Proof.* See [1, 4.3].  $\square$

**Definition.** A Lie algebra  $L$  is called *semisimple* if  $\text{Rad}(L) = 0$ .

**Remark 6.2.** If  $L$  is a Lie algebra, then  $L/\text{Rad } L$  is always semisimple.

**Lemma 6.3.** *A Lie algebra  $L$  is semisimple if and only if  $L$  has no nonzero abelian ideals.*

*Proof.* If  $L$  is semisimple, then it has no solvable ideals, hence it has no abelian ideals.

Conversely, assume that  $L$  has no abelian ideals and work with the solvable Lie algebra  $\text{Rad } L$ . Let  $j$  be minimal natural number with the property that  $(\text{Rad } L)^{(j)} = 0$ . Then  $(\text{Rad } L)^{(j-1)}$  is a nonzero ideal of  $L$  and  $[(\text{Rad } L)^{(j-1)}, (\text{Rad } L)^{(j-1)}] = 0$ , hence  $(\text{Rad } L)^{(j-1)}$  is abelian. This implies that  $\text{Rad } L = 0$ .  $\square$

**Definition.** Let  $L$  be a Lie algebra. The *killing form* on  $L$  is the map

$$\begin{aligned} \kappa_L(\cdot, \cdot) : L \times L &\longrightarrow K \\ (x, y) &\longmapsto \text{Tr}(\text{ad}_L(x) \text{ad}_L(y)). \end{aligned}$$

**Proposition 6.4.** *Let  $L$  be a Lie algebra.*

- (a)  $\kappa_L$  is symmetric and bilinear.
- (b)  $\kappa_L(x, [y, z]) = \kappa_L([x, y], z)$  for every  $x, y, z \in L$ .
- (c) If  $I$  is an ideal of  $L$ , then  $\kappa_I = \kappa_L|_{I \times I}$ .

*Proof.* Point (a) is trivial. For (b), just use some properties of the adjoint and of the trace, computing

$$\begin{aligned} \kappa_L(x, [y, z]) &= \text{Tr}(\text{ad}_L(x) \text{ad}_L([y, z])) = \text{Tr}(\text{ad}_L(x) [\text{ad}_L(y), \text{ad}_L(z)]) = \\ &= \text{Tr}(\text{ad}_L(x) \text{ad}_L(y) \text{ad}_L(z) - \text{ad}_L(x) \text{ad}_L(z) \text{ad}_L(y)) = \\ &= \text{Tr}(\text{ad}_L(x) \text{ad}_L(y) \text{ad}_L(z) - \text{ad}_L(y) \text{ad}_L(x) \text{ad}_L(z)) = \\ &= \text{Tr}([\text{ad}(x), \text{ad}(y)] \text{ad}(z)) = \kappa_L([x, y], z). \end{aligned}$$

(c) Take  $x_1, \dots, x_r$  a basis for  $I$ , and complete it with  $x_{r+1}, \dots, x_n$  to a basis of  $L$ . Then, for every  $x \in I$  we have that the matrix of  $\text{ad}_L(x)$  with respect to the basis  $x_1, \dots, x_n$  has the form

$$\text{ad}_L(x) = \begin{pmatrix} \text{ad}_I(x) & * \\ 0 & 0 \end{pmatrix}$$

since  $I$  is an ideal. Then, if  $x, y \in I$ , we have that

$$\text{ad}_L(x) \text{ad}_L(y) = \begin{pmatrix} \text{ad}_I(x) \text{ad}_I(y) & * \\ 0 & 0 \end{pmatrix}$$

and so  $\kappa_L(x, y) = \text{Tr}(\text{ad}_L(x) \text{ad}_L(y)) = \text{Tr}(\text{ad}_I(x) \text{ad}_I(y)) = \kappa_I(x, y)$ .  $\square$

**Lemma 6.5.** *Let  $I \subseteq L$  be an ideal. Then the orthogonal of  $I$  with respect to  $\kappa_L$ , namely  $I^\perp$ , is an ideal in  $L$ .*

*Proof.* Let  $x \in I^\perp$ ,  $y \in I$ ,  $z \in L$ . We want to prove that  $[z, x] \in I^\perp$ . Compute

$$\kappa_L([z, x], y) = -\kappa_L([x, z], y) = -\kappa_L(x, [z, y]) = 0$$

since  $[z, y] \in I$ . Hence  $[z, x] \in I^\perp$ .  $\square$

**Definition.** Let  $L$  be a Lie algebra. Then we define  $\text{rad } \kappa_L := L^\perp$ .

**Lemma 6.6.** *Let  $L$  be a Lie algebra. Then  $\text{rad } \kappa_L \subseteq \text{Rad } L$ .*

*Proof.* By definition of  $\text{rad } \kappa_L$ , for every  $x \in L$ ,  $y \in \text{rad } \kappa_L$  we obtain that  $\text{Tr}(\text{ad}_L(y) \text{ad}_L(x)) = 0$ . Using Theorem 6.1 we obtain that  $\text{rad } \kappa_L$  is a solvable ideal, hence it is contained in  $\text{Rad } L$ .  $\square$

**Theorem 6.7.** *A Lie algebra  $L$  is semisimple if and only if  $\kappa_L$  is nondegenerate.*

*Proof.*  $L$  semisimple means that  $\text{Rad } L = 0$ , and so  $\text{rad } \kappa_L = 0$  by the previous lemma.

Conversely, assume that  $\text{rad } \kappa_L = 0$ . Let  $I$  be an abelian ideal. For every  $x \in I$ ,  $y \in L$  we have that  $\text{ad}_L(x) \text{ad}_L(y) : L \rightarrow I$  since  $I$  is an ideal, and so  $(\text{ad}_L(x) \text{ad}_L(y))^2 : L \rightarrow 0$  since  $I$  is abelian. This means that  $\text{ad}_L(x) \text{ad}_L(y)$  is nilpotent, hence  $\text{Tr}(\text{ad}_L(x) \text{ad}_L(y)) = 0$ . This implies that  $x \in \text{rad } \kappa_L = 0$ , and so  $L$  has no nonzero abelian ideals. Using lemma 6.3 we conclude.  $\square$

**Theorem 6.8.** *Let  $L$  be semisimple. Then there exist  $L_1, \dots, L_r$  simple ideals of  $L$  such that  $L = L_1 \oplus \dots \oplus L_r$ ,  $[L_i, L_j] = \delta_{ij} L_i$  and every simple ideal of  $L$  is one of these.*

*Proof.* Let  $I \subseteq L$ . Then  $I$  is an ideal of  $L$  if and only if it is a stable subspace via the action of  $\text{ad}_L(L)$ , i.e.  $I$  is a subrepresentation for  $\text{ad}_L$ . Let  $I$  be an ideal of  $L$ . Then  $I \cap I^\perp$  is an ideal of  $L$ , and using Cartan's criterion in  $I$  we have that  $I \cap I^\perp$  is solvable, hence 0 since  $L$  is semisimple. Therefore we must have that  $L = I \oplus I^\perp$ . Since  $I^\perp$  is an ideal, the adjoint representation is completely reducible, hence by induction on  $\dim L$  we obtain a decomposition  $L = L_1 \oplus \dots \oplus L_r$  where the  $L_i$ 's are ideals, irreducible as representations. Also,  $[L_i, L_j] \subseteq L_i \cap L_j$ , so if  $i \neq j$  we have that  $[L_i, L_j] = 0$ . This implies that the ideals of  $L_i$  are also ideals of  $L$ , but since the  $L_i$ 's are irreducible spaces for the adjoint representation, they must be simple ideals. Hence, since  $[L_i, L_i]$  cannot be zero since  $L$  does not have abelian ideals, we must have  $[L_i, L_i] = L_i$ .

Let again  $I$  be an ideal of  $L$ . Then  $[L, I] = \bigoplus_i [L_i, I]$ . Since the  $L_i$ 's are simple, we have that either  $[L_i, I] = 0$  or  $[L_i, I] = L_i$ , and so  $[L, I]$  is the sum of some of the  $L_i$ 's. If  $I$  is simple,  $[L, I]$  is either 0 or  $I$ . If it was 0, we would have  $I \subseteq \zeta(L)$ , that is impossible since  $L$  is semisimple. Hence we have that  $I = [L, I]$  is the sum of some  $L_i$ 's, hence  $I = L_i$  for some  $i$ , since  $I$  is simple.  $\square$

**Corollary 6.9.** *Let  $L$  be a semisimple Lie algebra. Then*

- $[L, L] = L$ .
- Every ideal and homomorphic image of  $L$  is semisimple.
- Every ideal of  $L$  is a sum of certain simple ideals of  $L$ .

**Theorem 6.10** (Weyl). *Every finitely dimensional representation of a semisimple Lie algebra is completely reducible.*

*Proof.* See [1, 6.3]. □

## 7 Jordan decomposition

Recall from linear algebra that any  $x \in \mathfrak{gl}_n(K)$  can be decomposed as  $x = x_s + x_n$  with  $x_s$  a diagonalizable endomorphism and  $x_n$  a nilpotent endomorphism, with  $[x_s, x_n] = 0$ . We are now going to perform a similar decomposition for the elements of Lie algebras.

**Lemma 7.1.** *Let  $L$  be a Lie algebra. Then  $\text{ad}_L(L) \subseteq \text{Der}(L) \subseteq \mathfrak{gl}(L)$ .*

*Proof.* We show that  $\text{ad}_L(x)$  is a derivation. Let  $x, y, z \in L$ . Then

$$\begin{aligned} \text{ad}_L(x)([y, z]) &= [x, [y, z]] = -[y, [z, x]] - [z, [x, y]] = [y, [x, z]] + [[x, y], z] = \\ &= [y, \text{ad}_L(x)(z)] + [\text{ad}_L(x)(y), z]. \end{aligned}$$

□

Let  $V$  be a finitely dimensional vector space over  $K$ , let  $x \in \text{End}_K(V)$ . Using the Jordan decomposition, we can write uniquely  $x = x_s + x_n$  with  $x_s$  diagonalizable (*semisimple*) and  $x_n$  nilpotent endomorphisms such that  $[x_n, x_s] = 0$ .

**Exercise 7.2.** Let  $L$  be a Lie subalgebra of  $\mathfrak{gl}(V)$ . If  $x \in L$  is a nilpotent (resp. semisimple) element of  $\text{End}(V)$ , then  $\text{ad}_L(x)$  is a nilpotent (resp. semisimple) element of  $\text{End}(\text{End}(V))$ .

This exercise implies that if  $x = x_s + x_n$  is the Jordan decomposition of  $x \in \text{End}_K(V)$ , then  $\text{ad}_{\mathfrak{gl}(V)}(x) = \text{ad}_{\mathfrak{gl}(V)}(x_s) + \text{ad}_{\mathfrak{gl}(V)}(x_n)$  is the Jordan decomposition of  $\text{ad}_{\mathfrak{gl}(V)}(x) \in \text{End}(\text{End}(V))$ .

Let now  $L$  be a Lie algebra. We know that  $\text{ad}_L \subseteq \text{Der}(L) \subseteq \mathfrak{gl}(V)$  and for every  $x \in L$  we have a Jordan decomposition for  $\text{ad}_L(x)$  in  $\mathfrak{gl}(V)$ . We prove now that this decomposition in fact lies in  $\text{Der}(L)$ .

**Proposition 7.3.** *Let  $A$  be a finitely dimensional  $K$ -algebra. Then  $\text{Der}(A)$  contains the semisimple and nilpotent parts of its elements.*

*Proof.* Let  $\delta \in \text{Der } A$  and let  $\sigma, \nu \in \text{End}_K(A)$  be its semisimple and nilpotent parts, respectively. It will be enough to show that  $\sigma \in \text{Der } A$ , since then  $\nu = \delta - \sigma \in \text{Der } A$ . Take  $a \in K$  and consider

$$A_a := \{x \in A : (\delta - a \text{id})^k x = 0 \text{ for some } k \text{ (depending on } x)\}$$

Then  $A$  is the direct sum of those  $A_a$  for which  $a$  is an eigenvalue of  $\delta$ , since the  $A_a$  are just generalized eigenspaces for  $\delta$ . Then,  $\sigma$  acts on  $A_a$  as scalar multiplication by  $a$ . We can verify that, for arbitrary  $a, b \in K$ , we have that  $A_a \cdot A_b \subseteq A_{a+b}$ , by means of the general formula

$$(\delta - (a+b)\text{id})^n(xy) = \sum_{i=0}^n \binom{n}{i} ((\delta - a\text{id})^{n-i}x) \cdot ((\delta - b\text{id})^i y)$$

that can be proved by induction on  $n$ , for every  $x, y \in A$ . If  $x \in A_a$  and  $y \in A_b$ , then  $\sigma(xy) = (a+b)xy$ , because  $xy \in A_{a+b}$ . On the other hand, we have  $\sigma(x)y + x\sigma(y) = (a+b)xy$ . Since  $A = \bigoplus_a A_a$ , we have that  $\sigma$  is a derivation.  $\square$

**Lemma 7.4.** *Let  $L$  be a Lie algebra. Then  $\text{ad}_L(L)$  is an ideal of  $\text{Der } L$  and  $[\delta, \text{ad}_L(x)] = \text{ad}_L(\delta(x))$  for every  $\delta \in \text{Der } L$  and  $x \in L$ .*

*Proof.* Take  $x, y \in L$  and  $\delta \in \text{Der } L$ . Then

$$[\delta, \text{ad}_L(x)](y) = \delta[x, y] - \text{ad}_L(x)(\delta y) = [\delta x, y] + [x, \delta y] - [x, \delta y] = \text{ad}_L(\delta x)(y),$$

hence  $[\delta, \text{ad}_L(x)] \in \text{ad}_L(L)$  and so  $\text{ad}_L(L)$  is an ideal of  $\text{Der } L$ .  $\square$

**Proposition 7.5.** *If  $L$  is a semisimple Lie algebra, then  $\text{ad}_L(L) = \text{Der}(L)$ .*

*Proof.* Since  $L$  is semisimple, then by lemma 6.3  $\zeta(L) = 0$  and so  $L \cong \text{ad}_L(L)$ . In particular,  $\text{ad}_L(L)$  is semisimple and has its own nondegenerate killing form by theorem 6.7. Call  $D := \text{Der } L$  and  $M := \text{ad}_L(L)$ . Since by the previous lemma  $M$  is an ideal of  $D$ , using point (c) of proposition 6.4 we obtain that

$$\kappa_D|_{M \times M} = \kappa_M.$$

In particular, if  $M^\perp$  is the subspace of  $D$  orthogonal to  $M$  with respect to  $\kappa_D$ , the nondegeneracy of  $\kappa_M$  forces  $M^\perp \cap M = 0$ , and hence  $[M, M^\perp] = 0$ . If  $\delta \in M^\perp$ , then  $0 = [\delta, \text{ad}_L(x)] = \text{ad}_L(\delta x)$  for every  $x \in L$ , using the formula of the previous lemma. But then  $\delta x = 0$ , since  $\text{ad}_L$  is an isomorphism, hence  $\delta = 0$ . In conclusion, we get that  $M^\perp = 0$ , hence  $\text{Der } L = M = \text{ad}_L(L)$ .  $\square$

If  $L$  is semisimple, then  $L \cong \text{ad}_L(L) = \text{Der}(L)$ . Let now  $x \in L$ . Thanks to proposition 7.3 we have the Jordan decomposition  $\text{ad}_L(x) = (\text{ad}_L(x))_s + (\text{ad}_L(x))_n$  in  $\text{Der}(L)$ . But since  $L \cong \text{ad}_L(L) = \text{Der}(L)$ , there exist two unique  $x_s, x_n \in L$  such that  $(\text{ad}_L(x))_s = \text{ad}_L(x_s)$ ,  $(\text{ad}_L(x))_n = \text{ad}_L(x_n)$  and

$[x_s, x_n] = 0$ . Then  $x = x_s + x_n$  is called the *abstract Jordan decomposition* of the element  $x$ .

Let now  $L$  be a semisimple Lie algebra contained in  $\mathfrak{gl}(V)$  for some  $V$  finitely dimensional vector space over  $K$ . For every  $x \in L$  we have the abstract Jordan decomposition  $x = x_s + x_n$  and the usual Jordan decomposition  $x = s + n$ .

**Theorem 7.6.** *Let  $L \subseteq \mathfrak{gl}(V)$  be a semisimple Lie algebra, let  $x = s + n$  be the usual Jordan decomposition for  $x \in L$ . Then  $s, n \in L$ .*

*Proof.* See [1, 6.4]. □

**Corollary 7.7.** *Let  $L \subseteq \mathfrak{gl}(V)$  be a semisimple Lie algebra,  $x \in L$ . Then the usual Jordan decomposition for  $x$  coincides with the abstract Jordan decomposition.*

*Proof.* Let  $x = s + n$  be the usual Jordan decomposition. Then, using the above theorem,  $s, n \in L$  and  $s$  is semisimple,  $n$  is nilpotent and  $[s, n] = 0$ . Using exercise 7.2 we have that  $s$  and  $n$  are respectively ad-semisimple and ad-nilpotent. But this means exactly that  $x = s + n$  is the abstract Jordan decomposition. □

**Corollary 7.8.** *Let  $L$  be a semisimple Lie algebra,  $\rho: L \rightarrow \mathfrak{gl}(V)$  a finitely dimensional representation. If  $x = s + n$  is the abstract Jordan decomposition for  $x \in L$ , then  $\rho(x) = \rho(s) + \rho(n)$  is the Jordan decomposition for  $\rho(x)$  in  $\mathfrak{gl}(V)$ .*

*Proof.* Applying corollary 6.9, we have that  $\rho(L)$  is semisimple. Since  $L$  is spanned by the eigenvectors of  $\text{ad}_L(s)$ , one can easily check that this implies that  $\rho(L)$  is spanned by the eigenvectors of  $\text{ad}_{\rho(L)} \rho(s)$ , and so  $\rho(s)$  is semisimple. Similarly, since  $\text{ad}_L(n)$  is nilpotent, one can easily prove that  $\text{ad}_{\rho(L)} \rho(n)$  is nilpotent. Also, we have that  $[\rho s, \rho n] = \rho[s, n] = 0$ , hence  $\rho x = \rho s + \rho n$  is the abstract (and hence usual) Jordan decomposition of  $\rho x$ . □

## 8 Toral subalgebras

**Lemma 8.1.** *If  $L$  is a semisimple Lie algebra, then  $L$  contains a semisimple element.*

*Proof.* Let  $x \in L$  and consider the Jordan decomposition  $x = s + n$  in  $L$ , with  $s$  ad-semisimple and  $n$  ad-nilpotent. If every  $s$  is zero, then every element of  $L$  is ad-nilpotent, and thanks to Engel's theorem (theorem 4.5) this means that  $L$  is nilpotent, which is absurd. □

This lemma implies that inside any semisimple Lie algebra  $L$  there is always at least a semisimple line, namely  $Ks$ , for  $s$  a semisimple element of  $L$ .

**Definition.** Let  $L$  be a semisimple Lie algebra. A subalgebra  $T \subseteq L$  consisting of semisimple elements is called a *toral subalgebra*.

**Proposition 8.2.** *Let  $L$  be a semisimple Lie algebra. Any toral subalgebra  $T$  of  $L$  is abelian.*

*Proof.* In order to show that  $T$  is abelian, we need to show that  $\text{ad}_T(x) = 0$  for every  $x \in T$ . Since  $x$  is  $\text{ad}_T$ -semisimple, this amounts to show that  $\text{ad}_T(x)$  has no nonzero eigenvalues. Suppose that there exists  $y \in T$  such that  $\text{ad}_T(x)(y) = [x, y] = ay$  for  $0 \neq a \in K$ . Then  $\text{ad}_T(y)(x) = [y, x] = -ay$ . Since the element  $y$  is semisimple, we can decompose  $x = \sum_{\lambda} x_{\lambda}$  with  $x_{\lambda}$  eigenvectors for  $\text{ad}_T(y)$  of eigenvalue  $\lambda$ . Clearly  $y$  is an eigenvector of eigenvalue 0 for  $\text{ad}_T(y)$ . We can now write

$$-ay = \text{ad}_T(y)(x) = \sum_{\lambda \neq 0} \lambda x_{\lambda}.$$

On the left I have an eigenvector of zero eigenvalue for  $\text{ad}_T(y)$ , on the right I have a sum of eigenvectors of nonzero eigenvalues for  $\text{ad}_T(y)$ . This is absurd, and forces  $a = 0$ .  $\square$

**Definition.** Let  $L$  be a Lie algebra and  $E$  be a subalgebra. Then the *centralizer of  $E$  in  $L$*  is

$$c_L(E) := \{x \in L : [x, e] = 0 \text{ for every } e \in E\}.$$

**Lemma 8.3.** *Let  $L$  be a semisimple Lie algebra. Any toral subalgebra  $T$  of  $L$  such that  $c_L(T) = T$  it is a maximal toral subalgebra.*

*Proof.* Suppose that there exists a toral subalgebra  $T'$  such that  $T \subseteq T' \subseteq L$ . Using proposition 8.2, we have that  $[T, T'] = 0$ , and so  $T' \subseteq c_L(T) = T$ . This means that  $T$  is a maximal toral subalgebra.  $\square$

Let now  $L$  be a semisimple Lie algebra and consider  $H \subseteq L$  a maximal toral subalgebra of  $L$ . We are going to study the action of  $\text{ad}_L(H)$  on  $L$ . Since the elements of  $\text{ad}_L(H)$  are commuting and diagonalizable, then they are simultaneously diagonalizable.

**Definition.** Let  $L$  be a semisimple Lie algebra and consider  $H \subseteq L$  a maximal toral subalgebra of  $L$ . For every  $\alpha \in H^*$ , where  $H^*$  is the dual of  $H$ , we call

$$L_{\alpha} := \{x \in L : [h, x] = \alpha(h)x \text{ for every } h \in H\}.$$

Using the above observations, we obtain a decomposition of  $L$

$$L = L_0 \oplus \left( \bigoplus_{\alpha \in H^* \setminus \{0\}} L_\alpha \right)$$

and we call  $\Phi$  the set of  $\alpha \in H^* \setminus \{0\}$  such that  $L_\alpha$  is nonzero.  $L_0$  is the *centralizer* of  $H$  inside  $L$ , and since  $H$  is abelian we have that  $L_0 \supseteq H$ .

**Proposition 8.4.** *Let  $L$  be a semisimple Lie algebra and consider  $H \subseteq L$  a maximal toral subalgebra of  $L$ .*

- (a) *If  $\alpha, \beta \in H^*$  then  $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ .*
- (b) *If  $x \in L_\alpha$  for  $H^* \ni \alpha \neq 0$ , then  $x$  is nilpotent.*

*Proof.* (a) Let  $h \in H$ ,  $x \in L_\alpha$  and  $y \in L_\beta$ . Then

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)[x, y] + \beta(h)[x, y] = (\alpha + \beta)(h)[x, y].$$

(b) Let  $\alpha \neq 0$  and  $\beta \in H^*$ . Take  $x \in L_\alpha$  and  $y \in L_\beta$ . Using the first point, we have that  $(\text{ad}_L(x))^k(y) \subseteq L_{\beta+k\alpha}$ . Since  $L$  is finitely dimensional, there exists a  $k$  such that  $L_{\beta+k\alpha} = 0$ . Since  $L$  is a finite sum of  $L_\beta$ 's, we conclude that  $x$  is nilpotent.  $\square$

**Proposition 8.5.** *Let  $L$  be a semisimple Lie algebra and consider  $H \subseteq L$  a maximal toral subalgebra of  $L$ . Then  $H = L_0$ .*

*Proof.* See [1, 8.2].  $\square$

**Corollary 8.6.** *A toral subalgebra  $H$  of a semisimple Lie algebra  $L$  is maximal if and only if  $c_L(H) = H$ .*

## 9 Representations of $\mathfrak{sl}_2(K)$

In this section we perform the classification of the representations of the simple Lie algebra  $\mathfrak{sl}_2(K)$ . First remember from example 3.4 that  $\mathfrak{sl}_2(K)$  is generated as a vector space by the elements  $e, f, h$ . The element  $h$  is semisimple (since it is diagonal) and  $Kh$  is a maximal toral subalgebra, since  $c_{\mathfrak{sl}_2(K)}(Kh) = Kh$ .

We consider then the adjoint action of  $Kh$  on  $\mathfrak{sl}_2(K)$ . From the relations  $[h, e] = 2e$  and  $[h, f] = -2f$  we see that  $e$  and  $f$  are eigenvectors for the action of  $Kh$ , hence we have the decomposition

$$\mathfrak{sl}_2(K) = \mathfrak{sl}_2(K)_0 \oplus \mathfrak{sl}_2(K)_2 \oplus \mathfrak{sl}_2(K)_{-2} = Kh \oplus Ke \oplus Kf.$$

The subalgebras  $Ke$  and  $Kf$  are maximal nilpotent subalgebras of  $\mathfrak{sl}_2(K)$ , and the subalgebras  $Ke \oplus Kh$  and  $Kf \oplus Kh$  are maximal solvable subalgebras of  $\mathfrak{sl}_2(K)$ .

We consider now a finitely dimensional representation

$$\rho : \mathfrak{sl}_2(K) \longrightarrow \mathfrak{gl}(V)$$

for some finitely dimensional  $K$ -vector space  $V$ . Since  $\mathfrak{sl}_2(K)$  is simple, using theorem 6.10 we know that  $\rho$  is completely reducible. We consider the action of  $h$  on  $V$ , and since  $\rho h$  is semisimple we have the decomposition

$$V = \bigoplus_{\lambda} V_{\lambda}$$

where  $\lambda$  varies among the eigenvalues for the action of  $h$ .

**Definition.** The spaces  $V_{\lambda}$  are called *weight spaces* and the vectors  $v \in V_{\lambda}$  are called *weight vectors*.

**Lemma 9.1.** *We have that  $eV_{\lambda} \subseteq V_{\lambda+2}$  and  $fV_{\lambda} \subseteq V_{\lambda-2}$ .*

*Proof.* Let  $v \in V_{\lambda}$ . Then  $hev = ehv + [h, e]v = \lambda ev + 2ev = (\lambda + 2)ev$ . Similarly for the action of  $f$ .  $\square$

**Definition.** A weight vector  $0 \neq v \in V_{\lambda}$  is called a maximal vector if  $ev = 0$ .

**Lemma 9.2.** *Let  $v_0 \in V_{\lambda}$  be a maximal vector. Define  $v_{-1} := 0$  and  $v_i := \frac{1}{i!} f^i v_0$  for  $i \geq 0$ . Then for every  $i \geq 0$  we have that*

$$(a) \quad hv_i = (\lambda - 2i)v_i.$$

$$(b) \quad ev_i = (\lambda - i + 1)v_{i-1}.$$

$$(c) \quad fv_i = (i + 1)v_{i+1}.$$

*Proof.* Just use induction, or see [1, 7.2].  $\square$

Since  $V$  is finitely dimensional, starting from a maximal vector  $v_0$  and using the sequence of the lemma, we find an  $m \in \mathbb{N}$  such that  $v_{m-1} \neq 0$  and  $v_m = 0$ . Thanks to the lemma, it is clear that the subspace of  $V$  generated by  $v_0, v_1, \dots, v_{m-1}$  is  $\mathfrak{sl}_2(K)$ -stable, hence if  $\rho$  is irreducible, then  $V$  must be equal to that subspace.

Moreover, using property (b) of the lemma, we have that  $0 = v_m = ev_m = (\lambda - m + 1)v_{m-1}$ , hence  $m = \lambda + 1$ . This means that we always have that  $\lambda \in \mathbb{N}$ . If  $V$  is an irreducible representation, we have that the dimension  $m$  of  $V$  is related to the weight of the maximal vector  $v_0$ : chosen the dimension of the representation, we have chosen also the maximal weight, hence the whole representation. This means that we have a one to one correspondence

$$\{\text{Irreducible representations of } \mathfrak{sl}_2(K)\} \longrightarrow \mathbb{N}.$$

If  $V$  is an irreducible representation of dimension  $m$ , then  $V$  is generated by the vectors in the sequence of the lemma, with  $v_0$  vector of weight  $\lambda = m - 1$ . From point (a) we also know that every  $v_i$  is a weight vector of weight  $\lambda - 2i$ . This means that all the weights have the same parity and  $v_{m-1}$  has weight  $-\lambda$ .

## References

- [1] James E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, 1972.