Kashiwara equivalence, Bernstein's inequality & weak holonomicity (1)

§1 Kashiwara equivalence

Th^m (Kashiwara's equivalence) If $i : Y \to X$ is a closed embedding of smooth algebraic varieties then there is an equivalence of categories

$$\int_i^{\circ} = i_+ : \mathrm{Mod}_{q.c}(\mathcal{D}_Y) \longrightarrow \mathrm{Mod}_{q.c}^{Y}(\mathcal{D}_X)$$

q. coherent $\mathcal{D}$-modules on Y as $\mathcal{O}$-modules

supported on Y

This restricts to an equivalence between coherent objects as $\mathcal{D}$-modules.

Th^m (D Cap II) If $Y \to X$ is a closed embedding of smooth rigid analytic varieties over a (discretely valued) field K of characteristic 0 then there is an equivalence of categories

$$i_+ : \mathcal{C}_Y \to \mathcal{C}_X^{Y}$$

coadmissible $\widehat{\mathcal{D}}_Y$-modules locally

i_+ is given on right modules by

$$i_+M = M \hat{\otimes}_{\hat{\mathcal{D}}(Y)} \hat{\mathcal{D}}(X) / \mathcal{I}(Y)\hat{\mathcal{D}}(X) \quad \text{where } \mathcal{I}(Y) = \{f \in \mathcal{O}(X) \mid f|_Y = 0\}$$

& inverse of i_+ is given by

$$N \longmapsto N[\mathcal{I}] = \{n \in N \mid nf = 0 \ \forall f \in \mathcal{I}(Y)\}$$

It is defined on left modules by side-changing.

In the algebraic setting the key case is where X is affine & Y is a hypersurface defined by a local coordinate

In the $\hat{\mathcal{D}}$-setting the case X is affine & Y is a hypersurface defined by a local coordinate t is the crucial one

Thm For $M \in \mathcal{C}_X$ (X affine)

$M \in \mathcal{C}_X^Y \iff \forall m \in M(X), \lim_{n \to \infty} mt^n = 0$

& we can show for $M \in \mathcal{C}_X^Y$

$$M \simeq i_+(M[t])$$

$\uparrow$ $\{m \in M \mid mt = 0\}$

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§2 Bernstein's inequality

<u>Th^m</u> (Bernstein's inequality) If X is an irreducible smooth alg variety / $\mathbb{C}$ of dimension n then for every coherent $\mathcal{D}_X$-module M, $\dim Ch(M) \geq n$ $\quad [Ch(M) \subseteq T^*X]$

We also saw that if $U \subseteq X$ is an affine open

$$j(M(U)) + \dim Ch(M|_U) = 2n \text{ where}$$

$\cancel{j(M(U)) = \inf\{j \in \mathbb{N}_0 \mid Ext^i}$

$$j(M(U)) = \inf\{j \in \mathbb{N}_0 \mid Ext^j_{\mathcal{D}(U)}(M(U), \mathcal{D}(U)) \neq 0\}$$

Behind this definition is the notion of an Auslander–Gorenstein ring

Given a noetherian (non-commutative) ring U

the <u>grade</u> of a U-module M is

$$j(M) = \inf\{j \in \mathbb{N}_0 \mid \mathrm{Ext}^j_U(M, U) \neq 0\} \in \mathbb{N}_0 \cup \{\infty\}$$

A ~~f.g.~~ U-module M <u>satisfies Auslander's condition</u> if for every $i \geq 0$, $j(N) \geq i$ for every (right) submodule of $\mathrm{Ext}^i_U(M, U)$

U is <u>Auslander-Gorenstein</u> (AG) if every f.g. U-module satisfies Auslander's condition & U has finite injective dimension as a ~~left~~ U-module & as a right U-module

We say $\dim U = \mathrm{inj\,dim}\,{}_U U = \mathrm{inj\,dim}\, U_U$ (Zaks)

<u>Example</u> (1) $\mathcal{O}(U)$ for a smooth affine variety $/\mathbb{C}$ is AG.

c) (DCap III, Theorem 4.3)

Suppose A is a smooth K-affinoid algebra with affine formal model $\mathcal{A}$ & $\mathcal{L}$ is a smooth $(R, \mathcal{A})$-Lie algebra of rank r.

There is an integer $m \geq 0$ st $\widehat{U(\pi^n \mathcal{L})}_K$ is AG of dimension $\leq \dim A + r$ for each $n \geq m$.

In particular if $X = Sp(A)$ there is a way to present $\widehat{\mathcal{D}}(X) = \varprojlim \mathcal{D}_n(X)$ as a Fréchet-Stein algebra in such a way that each $\mathcal{D}_n(X)$ is AG of dimension $\leq 2 \dim X$

[assuming $\mathrm{Der}(\mathcal{O}(X))$ has a smooth Lie lattice]

<u>Def^n</u> We say a Fréchet-Stein alg U is <u>coadmissibly Auslander-Gorenstein</u> (c-AG) if there is an integer d & a presentation $U \cong \varprojlim U_n$ of U as a Fréchet-Stein algebra st each U_n is AG of dimension $\leq d$.

Can show in this case that every coadmissible U-module M satisfies Auslander's condition & $j_U(M) \leq d$

~~Indeed~~ Moreover $\mathrm{Ext}^i_U(M, U)$ is coadmissible for all i.

<u>Example</u> If X is a smooth affinoid variety the $\hat{\mathcal{D}}(X)$ is a AG of dimension $\leq 2 \dim X$.

(up to a technical condition) + tech. cond.

If X is a smooth connected affinoid variety of dim n & M is a coadmissible $\hat{\mathcal{D}}(X)$-module then

$$\dim M = 2n - j_{\hat{\mathcal{D}}(X)} M.$$

More generally

<u>Def.</u> If X is a smooth rigid analytic variety $/ K$ with a admissible cover $\mathcal{U}$ by affinoid subspaces (+ tech. cond) then for any coadmissible $\hat{\mathcal{D}}_X$-module $\mathcal{M}$

$$\dim \mathcal{M} = \sup \{ \dim (\mathcal{M}(Y)) \mid Y \in \mathcal{U} \}$$

Independent of the choice of $\mathcal{U}$

Thm (DCap III Theorem 6.3) Suppose X is a smooth quasi-compact equidimensional rigid analytic space, then every non zero coadmissible $\widehat{\mathcal{D}}_X$-module M satisfies

$$\dim M \geq \dim X$$

Sketch proof

Can easily reduce to the case where X is affinoid & find a closed embedding $i: X \hookrightarrow \mathbb{B}^n$ — unit polydisc of dimension n.

Moreover for any closed embedding $X \hookrightarrow Y$ & coadmissible $\widehat{\mathcal{D}}_X$-module M

$$\dim i_+ M = \dim M + \dim Y - \dim X$$

This reduces us to the case X is a polydisc. The proof of this case (from earlier work of [illegible]) exploits the fact there is a lot of symmetry in $\mathcal{T}(X)$ — form of Noether's normalisation together with the [illegible] of the affinoid version of [illegible]

§3 Weak holonomicity (DCap III)

We recall that an alg $\mathcal{D}_X$-module M is *holonomic* if $\dim M \le \dim X$ (ie $M = 0$ or $\dim M = \dim X$)

Def^n A ~~coadmissibity~~ coadmissible $\hat{\mathcal{D}}_X$-module M on a smooth equidimensional rigid analytic k-variety is *weakly holonomic* if $\dim M \le \dim X$

We write $\mathcal{C}_X^{wh} \subseteq \mathcal{C}_X$ to denote the full subcat of weakly holonomic objects

Prop^n Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is a s.e.s in $\mathcal{C}_X$ then

$$M_2 \in \mathcal{C}_X^{wh} \iff M_1 \,\&\, M_3 \in \mathcal{C}_X^{wh}$$

In particular $\mathcal{C}_X^{wh}$ is an abelian subcat of $\mathcal{C}_X$.

Prop^n Any integrable connection on X is weakly holonomic as a $\hat{\mathcal{D}}$-module

More generally if M is an coherent algebraic $\mathcal{D}$-module on X

of $\dim \leq \dim X$

then $\hat{\mathcal{D}}_X \otimes_{\mathcal{O}_X} M \in \mathcal{C}^{hol}_{X}$.

Prop^n If $i: Y \to X$ is a closed embedding of smooth rigid analytic varieties

$\hookrightarrow$ restricts to an equivalence

$$\mathcal{C}^{wh}_{\mathcal{D}_Y} \to \left(\mathcal{C}^{wh}_{\mathcal{D}_X}\right)^Y$$ supported on Y

Prop^n $\mathbb{D} = \mathcal{H}om_{\mathcal{O}_X}\left(\Omega_X, \mathcal{E}xt^{\dim X}_{\hat{\mathcal{D}}_X}\left(-, \hat{\mathcal{D}}_X\right)\right)$

$: \mathcal{C}_{\mathcal{D}_X} \to \mathcal{C}^{op}_{\mathcal{D}_X}$ sends weakly holonomic objects to weakly holonomic objects & $\mathbb{D}^2 \simeq id$

Th^m Suppose $j: U \to X$ is a Zariski open embedding of smooth rigid analytic spaces $/K$ & M is a integrable connection on X. Then

$$R^i j_* (M|_U) \in \mathcal{C}^{wh}_{\mathcal{D}_X} \text{ for } i \geq 0$$

Cor If $Z = X \setminus U$ & M as above

then $\underline{H}^i_Z(M) \in \mathcal{C}_X^{wh}$

local cohomology sheaves

However

Weakly holonomic modules need not have finite length & weakly holonomic modules need not pullback to coadmissible modules even under

$* \hookrightarrow \mathbb{D}$

Moreover the pushforward of a weakly holonomic (even an integrable connection) under an open embedding (even $\mathbb{D} \setminus \{0\} \hookrightarrow \mathbb{D}$) need not be coadmissible (Bode-Bitoun).