§§ Twisted differential operators

Def. X a smooth variety $/\mathbb{C}$
A sheaf of twisted differential operators on X is a sheaf $\mathcal{D}$ of $\mathcal{O}_X$-algebras on X equipped with a filtration $\{F_i\mathcal{D}\}$ s.t.

① $\mathcal{O}_X \hookrightarrow \mathcal{D}$ gives an iso $\mathcal{O}_X \simeq F_0\mathcal{D}$

② $\mathrm{Sym}_{\mathcal{O}_X}(F_1\mathcal{D}/F_0\mathcal{D}) \xrightarrow{\sim} \mathrm{gr}\,\mathcal{D}$ is an iso

③ $$\begin{aligned} F_1\mathcal{D}/F_0\mathcal{D} &\longrightarrow T_X \\ \xi &\longmapsto (f \mapsto [\xi, f]) \end{aligned}$$

is an iso.

For $\mathcal{L}$ a line bundle on X, one can define a sheaf $\mathcal{D}_X^{\mathcal{L}} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{L})$ of differential operators acting on $\mathcal{L}$

$F_i\mathcal{D}_X^{\mathcal{L}} = 0 \quad \forall\, i < 0$

$F_i\mathcal{D}_X^{\mathcal{L}} = \{P \in \mathcal{E}nd_{\mathbb{C}}(\mathcal{L}) : [P, f] \in F_{i-1}\mathcal{D}_X^{\mathcal{L}} \ \forall f \in \mathcal{O}_X\} \quad i \geq 0$

$\mathcal{D}_X^{\mathcal{L}} = \bigcup_0^{\infty} F_i\mathcal{D}_X^{\mathcal{L}}$ is a sheaf of TDO: ① $F_0\mathcal{D}_X^{\mathcal{L}} = \{\mathcal{O}_X\text{-linear } \mathcal{L} \to \mathcal{L}\} \simeq \mathcal{O}_X$
② & ③ : Check it locally, $\mathcal{L}|_U = \mathcal{O}_U$.

$\mathcal{D}_X^{\mathcal{L}} \simeq \mathcal{L} \otimes \mathcal{D}_X \otimes \mathcal{L}^\vee$

§1 TDO associated with integral weights

G a simply conn, semisimple alg gp $/\mathbb{C}$ $\qquad \mathfrak{g}$

max torus $T \leq B$ Borel subgp $\qquad \mathfrak{t}, \mathfrak{b}$

U unipotent radical of B $\qquad \mathfrak{n}$

$X = G/B$

$\lambda \in \mathfrak{t}^*$ integral if it lifts to a character of T

<u>Recall</u>: λ integral $\rightsquigarrow$ line bundle on X (G-equivariant) $\mathcal{O}(\lambda)$

$$\rightsquigarrow \mathcal{D}_X^{\mathcal{O}(\lambda)}$$

<u>Example</u>: $G = SL_2$, $\mathfrak{g} = \mathfrak{sl}_2$ with basis

$e = \begin{pmatrix} & 1 \\ & \end{pmatrix}$, $\quad h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $\quad f = \begin{pmatrix} & \\ 1 & \end{pmatrix}$

$X = \mathbb{P}^1$, $\quad U_0 = \mathbb{P}^1 - \{\infty\}$ with coord z

$\qquad\quad U_\infty = \mathbb{P}^1 - \{0\}$ $\qquad\qquad w$

$\Big\}$ related by $zw = 1$

The line bundles on $\mathbb{P}^1$ are $\mathcal{O}(n)$.

$$\begin{tikzcd} \mathbb{C}[z] \arrow[r, bend left, "\cdot z^{-n} = w^{n}"] & \mathbb{C}[w] \arrow[l, bend left, "\cdot z^{n} = w^{-n}"] \end{tikzcd}$$

$$\mathcal{D}_n := \mathcal{D}_{\mathbb{P}^1}^{\mathcal{O}(n)}$$

$$i_0 : \mathcal{D}_n|_{U_0} \xrightarrow{\sim} \mathcal{D}_{U_0} \quad , \quad i_\infty : \mathcal{D}_n|_{U_\infty} \xrightarrow{\sim} \mathcal{D}_{U_\infty}$$

The transition map is given by

$$i_\infty \circ i_0^{-1} : \mathcal{D}_{U_0 \cap U_\infty} \longrightarrow \mathcal{D}_{U_0 \cap U_\infty}$$
$$P \longmapsto z^{-n} P z^{n}$$

A global section in $\Gamma(\mathbb{P}^1, \mathcal{O}(n))$ is given by $p(z) \in \mathbb{C}[z]$ & $q(w) \in \mathbb{C}[w]$ s.t. $p(\frac{1}{w}) \cdot w^n = q(w)$

For $g \in SL_2$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$(g \cdot p)(z) = p\left(\frac{dz - b}{-cz + a}\right)(-cz + a)^n$$

$$(g \cdot q)(z) = q\left(\frac{-c + aw}{d - bw}\right)(d - bw)^n$$

e.g. $\varepsilon^2 = 0$

$$\underbrace{(1 + \varepsilon f)}_{\begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}} \cdot z^i = \left(\frac{z}{-\varepsilon z + 1}\right)^i \cdot (-\varepsilon z + 1)^n = z^i + \varepsilon(i - n) z^{i+1}$$

$\Rightarrow f$ acts by $z^2 \partial_z - nz$

$$\mathcal{U}(sl_2) \longrightarrow \Gamma(\mathbb{P}^1, \mathcal{D}_n)$$

$$\begin{aligned} e &\longmapsto -\partial_z &&= w^2\partial_w - nw \\ h &\longmapsto -2z\partial_z + n &&= 2w\partial_w - n \\ f &\longmapsto z^2\partial_z - nz &&= -\partial_w \end{aligned}$$

□

§ Interlude on $Z(\mathfrak{g})$.

W Weyl group

Define a dot action of W on $\mathfrak{t}^*$ by

$w \cdot \lambda = w(\lambda+\rho) - \rho \qquad \forall\, w \in W, \quad \lambda \in \mathfrak{t}^*$

Since $\mathfrak{t}$ is commutative, $\mathcal{U}(\mathfrak{t}) = S(\mathfrak{t})$

$\rightsquigarrow$ dot action on $\mathcal{U}(\mathfrak{t})$: $f \in \mathcal{U}(\mathfrak{t})$ a function on $\mathfrak{t}^*$

$(w \cdot f)(\lambda) = f(w^{-1} \cdot \lambda)$.

$Z(\mathfrak{g})$ centre of $\mathcal{U}(\mathfrak{g})$, $Z(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^G$

Harish-Chandra iso $Z(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{t})^{(W,\cdot)}$

$\mathrm{gr}\, Z(\mathfrak{g}) = S(\mathfrak{t})^W$

§3 TDO associated with general weights.

"basic affine space"

$$G \circlearrowright \tilde{X} = G/U \circlearrowleft T$$

$$\begin{tikzcd} \tilde{X} = G/U \arrow[d, "\pi"', "\text{a } T\text{-torsor}"] \\ X = G/B \end{tikzcd}$$

Define a sheaf

$$\tilde{\mathcal{D}} := (\pi_* \mathcal{D}_{\tilde{X}})^T$$

$$\begin{tikzcd} \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} \mathcal{U}(\mathfrak{t}) \arrow[r] \arrow[d] & \Gamma(X, \tilde{\mathcal{D}}) \\ \underbrace{\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{t})}_{=: \tilde{\mathcal{U}}} \arrow[ur] & \end{tikzcd}$$

$$\begin{tikzcd} Z(\mathfrak{g}) \arrow[r, hook] \arrow[d] & \mathcal{U}(\mathfrak{t}) \arrow[d] \\ \mathcal{U}(\mathfrak{g}) \arrow[r] & \Gamma(X, \tilde{\mathcal{D}}) \end{tikzcd}$$

For each $\lambda \in \mathfrak{t}^*$, define $\mathcal{D}_\lambda := \tilde{\mathcal{D}} \otimes_{S(\mathfrak{t})} \mathbb{C}_\lambda$

a sheaf on TDO on X

$$\rightsquigarrow \underbrace{\mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathbb{C}_\lambda}_{=: \mathcal{U}_\lambda} \longrightarrow \Gamma(X, \mathcal{D}_\lambda)$$

Thm. ① $H^i(X, \tilde{\mathcal{D}}) = 0 \quad \forall i > 0$

$\Gamma(X, \tilde{\mathcal{D}}) = \tilde{\mathcal{U}}$

② $H^i(X, \mathcal{D}_\lambda) = 0 \quad \forall i > 0$

$\Gamma(X, \mathcal{D}_\lambda) = \mathcal{U}_\lambda$

$$(b, x) \longmapsto x|_b \in (\mathfrak{b}/\mathfrak{n})^* = \mathfrak{t}^*$$

Pf: $(T^*\tilde{X})/T = \{(b, x) \in X \times \mathfrak{g}^* : x|_{\mathfrak{n}} = 0\} = \tilde{\mathfrak{g}}^* \longrightarrow \mathfrak{t}^*$

$T^*X = \{(b, x) \in X \times \mathfrak{g}^* : x|_{\mathfrak{b}} = 0\} = \tilde{\mathcal{N}}$

$$\begin{tikzcd}
 & \tilde{\mathfrak{g}}^* \arrow[r] \arrow[d] & \mathfrak{t}^* \arrow[d] \\
\tilde{\mathcal{N}} \arrow[ur] \arrow[d] & \mathfrak{g}^* \arrow[r, "\text{Chevalley map}"'] & \mathfrak{t}^*/W \\
\mathcal{N} \arrow[ur] & &
\end{tikzcd}$$

$\mathcal{O}(\mathfrak{t}^*/W) = S(\mathfrak{t})^W = S(\mathfrak{g})^G \hookrightarrow \mathcal{O}(\mathfrak{g}^*)$

Facts: $\mathrm{gr}\, \tilde{\mathcal{D}} = (p_*)\mathcal{O}_{\tilde{\mathfrak{g}}^*}$, $p : \tilde{\mathfrak{g}}^* \to X$

$\mathrm{gr}\, \mathcal{D}_\lambda = (q_*)\mathcal{O}_{\tilde{\mathcal{N}}}$, $q : \tilde{\mathcal{N}} \to X$

$\tilde{\mathfrak{g}}^* = G \times_B (\mathfrak{g}/\mathfrak{n})^* \longrightarrow X$, $\tilde{\mathcal{N}} = G \times_B (\mathfrak{g}/\mathfrak{b})^* \longrightarrow X$

Cohom. vanishing for $\mathcal{O}_{\tilde{\mathcal{N}}}$ follows from $H^i(T^*X, \mathcal{O}_{T^*X}) = 0 \quad \forall i > 0$

Cohom. vanishing for $\mathcal{O}_{\tilde{\mathfrak{g}}^*}$ follows from ~~that of~~ $\mathrm{gr}\, \mathcal{O}_{\tilde{\mathfrak{g}}^*} = \mathcal{U}(\mathfrak{t}) \otimes \mathcal{O}_{\tilde{\mathcal{N}}}$

$\Rightarrow$ cohom. vanishing ~~for~~ $\tilde{\mathcal{D}}$ & $\mathcal{D}_\lambda$ ✓.

WTS $\tilde{\mathcal{U}} = \mathcal{U}(\mathfrak{g}) \otimes_{Z(\mathfrak{g})} \mathcal{U}(\mathfrak{t}) \longrightarrow \Gamma(X, \tilde{\mathcal{D}})$ is an iso.

Suffices to show $\mathrm{gr}\, \tilde{\mathcal{U}} \longrightarrow \mathrm{gr}\, \Gamma(X, \tilde{\mathcal{D}})$ is an iso.

$$\begin{tikzcd}
S(\mathfrak{g}) \otimes_{S(\mathfrak{t})^W} S(\mathfrak{t}) \arrow[r, equal] \arrow[d, "\wr"'] & \mathcal{O}(\mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*) \arrow[d, "\wr"'] \\
\mathrm{gr}\, \tilde{\mathcal{U}} \arrow[r, "\sim"] & \mathrm{gr}\, \Gamma(X, \tilde{\mathcal{D}}) = \mathcal{O}(\tilde{\mathfrak{g}}^*)
\end{tikzcd}$$

induced by $\tilde{\mathfrak{g}}^* \to \mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ proper birational

<u>Fact</u>: $\mathfrak{g}^* \times_{\mathfrak{t}^*/W} \mathfrak{t}^*$ is a normal variety

- the singular locus has codim ≥ 2
- Cohen – Macaulay

$\Big\}$ $\xRightarrow[\text{criterion}]{\text{Serre's}}$ normal

$\Rightarrow \tilde{\mathcal{U}} \xrightarrow{\sim} \Gamma(X, \tilde{\mathcal{D}})$, $\quad \mathcal{U}_\lambda \xrightarrow{\sim} \Gamma(X, \mathcal{D}_\lambda)$. $\square$.